FOUR-PLACE TABLES OF TRANSCENDENTAL FUNCTIONS

by

W. FLÜGGE

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TRANSCENDENTAL FUNCTIONS

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PERGAMON PRESS LTD. LONDON

1954

Published in Great Britain by Pergamon Press Ltd.
242 Marylebone Road, London, N.W.1.

Published in U.S.A. by McGraw-Hill Book Co., Inc.,
330 West 42nd Street, New York 36, N.Y.

Printed by J. W. Arrowsmith Ltd., Bristol, England

PREFACE

THERE was a time when sines and cosines were reserved for the use of learned men; today they are in the hands of a wide public—all those who use mathematics for some practical purpose—and even high-school boys have to face them. During our lifetime the exponential function and the hyperbolic sines and cosines have emerged from mathematical textbooks and have found their way into the world of engineers. And now some more of these transcendental functions—created in the past by speculative minds—are on their way into practical usefulness.

After a slow start in the nineteenth century the last two decades have seen admirable work in the computing and tabulating of mathematical functions, most of it done in the United States and in England. Many big volumes full of multi-digit figures have been published, and they have become a treasure box of great value for those who need transcendental functions. But in spite of the rapid development of modern computing devices most computations are still done on the slide-rule, and for such kind of work the big original tables are somewhat unwieldy. The author felt that something was still missing for those who are old-fashioned enough to do their daily chores with the slide-rule, but modern enough to use mathematics beyond sines and cosines whenever that will help solve a problem.

For these men this little book has been compiled. It contains values, of slightly more than slide-rule accuracy, of those functions which are most likely to occur in a great many problems of physics and engineering. It also contains a collection of those formulas which are needed to handle the functions. They will suffice in many cases and may safely be used also by those who do not know all the mathematical theory that stands behind them.

Necessarily, the range of the tables had to be limited, and it will always be a controversial question where to stop. Some people may feel they could have done with less, but most users will some day need values for higher arguments. A compromise had to be made, and it was made so that beyond the end of the tables there are always *simple* means available to compute the missing values.

It is a matter of course that this book will not have " readers ". Nobody, except the proof-readers, will ever read it from cover to cover. But more than that: This book does not even want to be a book. It does not want to stand on your bookshelf, perhaps behind a glass door, well-kept, and dusted once a year. It wants to be a tool and to lie in the drawer where you keep your slide-rule, always at hand when its services may be required.

Although the preparation of these tables involved a good deal of computing, it would not have been possible to bring all this material together without ample use of existing table-literature. The author wishes in particular to express his thanks to the holders of copyright material who generously permitted its use for this work:

The British Association for the Advancement of Science and the Royal Society in London for permission to extract the values of the Bessel functions Y_0, Y_1 ($0 \leqq x \leqq 10$) and of the modified Bessel functions K_0, K_1 ($0 \leqq x \leqq 5$) from the sixth volume of the Mathematical Tables of this Association.

Professor G. Cassinis in Milan, Italy, for permission to extract values of the gamma-function from his table, mentioned on p. 118.

The author also owes a debt of gratitude to the various agencies of the United States Government who carried on the Mathematical Tables Project, producing tables of inestimable value.

Finally, the author has the pleasure of expressing his thanks to two of his colleagues, Professors J. N. Goodier and K. Klotter, and to his wife, Dr. I. Flügge-Lotz, who had a critical look at the manuscript and helped to improve it by their constructive criticism.

Stanford University, California
February 1953 W. F.

CONTENTS

Instructions for the Use of the Tables

1. *Notations*

For many of the tabulated functions different notations are in current use. The ones adopted here are as close as possible to the most common usage.

For the hyperbolic functions the English symbols have been capitalized, since this has proved to be very helpful in avoiding mistakes in formulas. This change is suggested for general adoption.

For the Bessel functions and all functions connected with them the notation of WATSON'S Treatise has been adopted.

In the case of FRESNEL'S integrals there are two pairs of functions which are commonly designated by the same symbols C and S. Here those functions have been tabulated which have a constant distance between consecutive maxima and therefore are best suited for tabulation. It is suggested that the alternate meaning of the symbols C and S be entirely discarded.

2. *Arrangement of Tables*

Except for the elliptic integrals, all tables are arranged in the same manner as the common logarithmic tables. The first column contains all but the last digit of the argument x, and an additional digit of x is given by the headings of the other ten columns.

When reading the value of a function, the following rules apply:

A cipher before the decimal point has usually been omitted.

Examples: sin 45.0° = 0.7071 appears on p. 15 as .7071; sin 3.40 = − 0.2555 appears on p. 19 as − .2555.

The columns 1 to 9 do not contain complete figures, and the first digits must be picked from the column 0. Usually they will be found on the same line or on a preceding line.

Examples: Tanh 0.72 = 0.6169, Tanh 1.30 = 0.8617, Tanh 1.31 = 0.8643, Tanh 3.05 = 0.9955 (p. 27), sin 3.43 = − 0.2844 (p. 19), J_0 (8.92) = − 0.07035 (p. 48).

Where this rule does not hold, an asterisk has been provided as a sign that special attention is necessary. Usually the asterisk indicates that the first digit (or digits) will be found on the next following line [sin 0.52 = 0.4969, but sin 0.53 = 0.5055 (p. 19), J_1 (2.83) = 0.3997 (p. 47)], but occasionally it may mean something else [cos 3.14 = − 1.0000 (p. 18)].

Where a power of 10 has been factored out, it stands only on that line where it is first needed and is repeated only if the sign of the function changes.

Examples: Cosh 8.72 = $10^3 \times 3.062$ = 3062 (p. 26), bei 9.00 = -10×2.471 = -24.71 (p. 71), Ei (-2.11) = $-10^{-1} \times 0.4204$ = -0.04204 (p. 126).

Where a new power of 10 appears. the preceding line must be read throughout with the preceding power of 10, including the figures with an asterisk.

Examples: exp 9.21 = $10^3 \times 9.997$, exp 9.22 = $10^3 \times 10.097$,
exp 9.23 = $10^3 \times 10.20$ (p. 34), exp ($-$ 9.21) = $10^{-3} \times 0.10003$,
exp ($-$ 9.22) = $10^{-3} \times 0.09904$ (p. 35).
It will be noticed that at the transition one value is given with 5 digits in order to assure a 4-digit interpolation everywhere.

3. *Accuracy of Tables*

Most of the tables in this book may be expected to be as accurate as the limited number of digits permits; i.e. the error is not in excess of 0.5 units of the last digit carried. Many of the tables have been obtained by subtabulation from 6-digit source material. In these tables rounding errors in the sixth digit may occasionally influence the result so far that an error of 0.54 units of the last digit may occur. The tables of the Fresnel integrals are probably somewhat less accurate, since only LOMMEL's table was available, which has a rather large interval and is not free from mistakes.

4. *Bibliography*

In the bibliography of the different sections only those tables have been mentioned which either give more digits or cover a wider range of arguments, and which were considered readily available in bookstores and libraries. For more detailed information the following books may be consulted:

A. FLETCHER, J. C. P. MILLER, L. ROSENHEAD: *An Index of Mathematical Tables.* New York and London, 1946;

Mathematical Tables and Other Aids to Computation. Published by the National Research Council since 1943.

Formulas

1. *Power Series*

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots \qquad \text{for any } x$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots \qquad \text{for any } x$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots \qquad \text{for } |x| < \frac{\pi}{2}$$

$$\cot x = x^{-1} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \dots \qquad \text{for } |x| < \pi$$

2. *Relations connecting the four functions*

$$\cos x = \sqrt{1 - \sin^2 x} = \frac{1}{\sqrt{1 + \tan^2 x}} = \frac{\cot x}{\sqrt{1 + \cot^2 x}}$$

$$\sqrt{1 - \cos^2 x} = \sin x = \frac{\tan x}{\sqrt{1 + \tan^2 x}} = \frac{1}{\sqrt{1 + \cot^2 x}}$$

$$\frac{\sqrt{1 - \cos^2 x}}{\cos x} = \frac{\sin x}{\sqrt{1 - \sin^2 x}} = \tan x = \frac{1}{\cot x}$$

$$\frac{\cos x}{\sqrt{1 - \cos^2 x}} = \frac{\sqrt{1 - \sin^2 x}}{\sin x} = \frac{1}{\tan x} = \cot x$$

3. *Relation to the exponential function*

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}) \qquad \sin x = \frac{1}{2i}(e^{ix} - e^{-ix})$$

$$e^{ix} \equiv \operatorname{cis} x = \cos x + i \sin x \qquad e^{-ix} = \cos x - i \sin x$$

4. *Imaginary argument*

$$\cos ix = \operatorname{Cosh} x \qquad \sin ix = i \operatorname{Sinh} x$$

$$\tan ix = i \operatorname{Tanh} x \qquad \cot ix = - i \operatorname{Coth} x$$

5. *Addition theorems*

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$\cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$$

Formulas

1. *Power series*

$$\mathrm{Cosh}\, x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \qquad \text{for any } x$$

$$\mathrm{Sinh}\, x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \qquad \text{for any } x$$

$$\mathrm{Tanh}\, x = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} - + \dots \qquad \text{for } |x| < \frac{\pi}{2}$$

$$\mathrm{Coth}\, x = x^{-1} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + - \dots \qquad \text{for } |x| < \pi$$

2. *Relations connecting the four functions*

$$\mathrm{Cosh}\, x = \sqrt{1 + \mathrm{Sinh}^2 x} = \frac{1}{\sqrt{1 - \mathrm{Tanh}^2 x}} = \frac{\mathrm{Coth}\, x}{\sqrt{\mathrm{Coth}^2 x - 1}}$$

$$\sqrt{\mathrm{Cosh}^2 x - 1} = \mathrm{Sinh}\, x = \frac{\mathrm{Tanh}\, x}{\sqrt{1 - \mathrm{Tanh}^2 x}} = \frac{1}{\sqrt{\mathrm{Coth}^2 x - 1}}$$

$$\frac{\sqrt{\mathrm{Cosh}^2 x - 1}}{\mathrm{Cosh}\, x} = \frac{\mathrm{Sinh}\, x}{\sqrt{1 + \mathrm{Sinh}^2 x}} = \mathrm{Tanh}\, x = \frac{1}{\mathrm{Coth}\, x}$$

$$\frac{\mathrm{Cosh}\, x}{\sqrt{\mathrm{Cosh}^2 x - 1}} = \frac{\sqrt{1 + \mathrm{Sinh}^2 x}}{\mathrm{Sinh}\, x} = \frac{1}{\mathrm{Tanh}\, x} = \mathrm{Coth}\, x$$

3. *Relation to the exponential function*

$\mathrm{Cosh}\, x = \frac{1}{2}(e^x + e^{-x})$ $\qquad$ $\mathrm{Sinh}\, x = \frac{1}{2}(e^x - e^{-x})$

$e^x \equiv \exp x = \mathrm{Cosh}\, x + \mathrm{Sinh}\, x$ $\qquad$ $e^{-x} = \mathrm{Cosh}\, x - \mathrm{Sinh}\, x$

4. *Imaginary argument*

$\mathrm{Cosh}\, ix = \cos x$ $\qquad$ $\mathrm{Sinh}\, ix = i \sin x$

$\mathrm{Tanh}\, ix = i \tan x$ $\qquad$ $\mathrm{Coth}\, ix = -i \cot x$

5. *Addition theorems*

$$\mathrm{Cosh}\,(x \pm y) = \mathrm{Cosh}\, x\, \mathrm{Cosh}\, y \pm \mathrm{Sinh}\, x\, \mathrm{Sinh}\, y$$

$$\mathrm{Sinh}\,(x \pm y) = \mathrm{Sinh}\, x\, \mathrm{Cosh}\, y \pm \mathrm{Cosh}\, x\, \mathrm{Sinh}\, y$$

$$\mathrm{Tanh}\,(x \pm y) = \frac{\mathrm{Tanh}\, x \pm \mathrm{Tanh}\, y}{1 \pm \mathrm{Tanh}\, x\, \mathrm{Tanh}\, y}$$

$$\mathrm{Coth}\,(x \pm y) = \frac{\mathrm{Coth}\, x\, \mathrm{Coth}\, y \pm 1}{\mathrm{Coth}\, y \pm \mathrm{Coth}\, x}$$

6. *Double argument and half argument*

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\sin 2x = 2\cos x \sin x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} \qquad \cot 2x = \frac{\cot^2 x - 1}{2\cot x}$$

$$\cos\frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}} \qquad \sin\frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}}$$

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x} = \frac{1 - \cos x}{\sin x}$$

7. *Sum and product of two functions*

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\sin x \pm \sin y = 2\sin\frac{x \pm y}{2}\cos\frac{x \mp y}{2}$$

$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}$$

$$\cot x \pm \cot y = \frac{\sin(y \pm x)}{\sin x \sin y}$$

$$2\cos x \cos y = \cos(x - y) + \cos(x + y)$$

$$2\sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2\cos x \sin y = -\sin(x - y) + \sin(x + y)$$

8. *Derivatives*

$$\frac{d\cos x}{dx} = -\sin x \qquad \frac{d\sin x}{dx} = \cos x$$

$$\frac{d\tan x}{dx} = \frac{1}{\cos^2 x} \qquad \frac{d\cot x}{dx} = -\frac{1}{\sin^2 x}$$

9. *Integrals*

$$\int\cos x\,dx = \sin x \qquad \int\sin x\,dx = -\cos x$$

$$\int\tan x\,dx = -\ln\cos x \qquad \int\cot x\,dx = \ln\sin x$$

$$\int x\cos x\,dx = x\sin x + \cos x$$

$$\int x\sin x\,dx = -x\cos x + \sin x$$

6. *Double argument and half argument*

$$\text{Cosh } 2x = \text{Cosh}^2 x + \text{Sinh}^2 x = 2 \text{ Cosh}^2 x - 1 = 1 + \text{Sinh}^2 x$$

$$\text{Sinh } 2x = 2 \text{ Cosh } x \text{ Sinh } x$$

$$\text{Tanh } 2x = \frac{2 \text{ Tanh } x}{1 + \text{Tanh}^2 x} \qquad \text{Coth } 2x = \frac{\text{Coth}^2 x - 1}{2 \text{ Coth } x}$$

$$\text{Cosh } \frac{x}{2} = \sqrt{\frac{\text{Cosh } x + 1}{2}} \qquad \text{Sinh } \frac{x}{2} = \sqrt{\frac{\text{Cosh } x - 1}{2}}$$

$$\text{Tanh } \frac{x}{2} = \frac{\text{Sinh } x}{\text{Cosh } x + 1} = \frac{\text{Cosh } x - 1}{\text{Sinh } x}$$

7. *Sum and product of two functions*

$$\text{Cosh } x + \text{Cosh } y = 2 \text{ Cosh } \frac{x + y}{2} \text{ Cosh } \frac{x - y}{2}$$

$$\text{Cosh } x - \text{Cosh } y = 2 \text{ Sinh } \frac{x + y}{2} \text{ Sinh } \frac{x - y}{2}$$

$$\text{Sinh } x \pm \text{Sinh } y = 2 \text{ Sinh } \frac{x \pm y}{2} \text{ Cosh } \frac{x \mp y}{2}$$

$$\text{Tanh } x \pm \text{Tanh } y = \frac{\text{Sinh } (x \pm y)}{\text{Cosh } x \text{ Cosh } y}$$

$$\text{Coth } x \pm \text{Coth } y = \frac{\text{Sinh } (y \pm x)}{\text{Sinh } x \text{ Sinh } y}$$

$$2 \text{ Cosh } x \text{ Cosh } y = \text{Cosh } (x + y) + \text{Cosh } (x - y)$$

$$2 \text{ Sinh } x \text{ Sinh } y = \text{Cosh } (x + y) - \text{Cosh } (x - y)$$

$$2 \text{ Cosh } x \text{ Sinh } y = \text{Sinh } (x + y) - \text{Sinh } (x - y)$$

8. *Derivatives*

$$\frac{d \text{ Cosh } x}{dx} = \text{Sinh } x \qquad \frac{d \text{ Sinh } x}{dx} = \text{Cosh } x$$

$$\frac{d \text{ Tanh } x}{dx} = \frac{1}{\text{Cosh}^2 x} \qquad \frac{d \text{ Coth } x}{dx} = -\frac{1}{\text{Sinh}^2 x}$$

9. *Integrals*

$$\int \text{Cosh } x \, dx = \text{Sinh } x \qquad \int \text{Sinh } x \, dx = \text{Cosh } x$$

$$\int \text{Tanh } x \, dx = \ln \text{Cosh } x \qquad \int \text{Coth } x \, dx = \ln \text{Sinh } x$$

$$\int x \text{ Cosh } x \, dx = x \text{ Sinh } x - \text{Cosh } x$$

$$\int x \text{ Sinh } x \, dx = x \text{ Cosh } x - \text{Sinh } x$$

9. *continued*

$\int x^2 \cos x \, dx = (x^2 - 2) \sin x + 2\, x \cos x$

$\int x^2 \sin x \, dx = (-\, x^2 + 2) \cos x + 2\, x \sin x$

$\left.\begin{aligned}\int x^{-1} \cos x \, dx &= \mathrm{Ci}\, x \\ \int x^{-1} \sin x \, dx &= \mathrm{Si}\, x\end{aligned}\right\}$ see p. 115.

$\int x^{-2} \cos x \, dx = -\, x^{-1} \cos x - \mathrm{Si}\, x$

$\int x^{-2} \sin x \, dx = -\, x^{-1} \sin x + \mathrm{Ci}\, x$

$\int \cos^2 x \, dx = \frac{1}{2} (x + \cos x \sin x)$

$\int \sin^2 x \, dx = \frac{1}{2} (x - \cos x \sin x)$

$\int \cos^3 x \, dx = \frac{1}{3} \sin x \, (2 + \cos^2 x)$

$\int \sin^3 x \, dx = -\, \frac{1}{3} \cos x \, (2 + \sin^2 x)$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

$$\int \sin^n x \, dx = -\, \frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \, .$$

$$\int \frac{dx}{\cos x} = \ln \tan \left(\frac{x}{2} + \frac{\pi}{4}\right) \qquad \int \frac{dx}{\sin x} = \ln \tan \frac{x}{2}$$

$$\int \frac{dx}{\cos^2 x} = \tan x \qquad \int \frac{dx}{\sin^2 x} = -\cot x$$

$$\int \frac{dx}{\cos x \sin x} = \ln \tan x$$

10. *Extreme values of the argument x*

For very small arguments use the power series (1). In the vicinity of the singularities of tan x compute cot $x = 1/\tan x$ from the table values and use then linear interpolation. For arguments $x > 10$ write the argument in degrees and make use of the periodicity of the trigonometric functions (see p. 13).

9. *continued*

$$\int x^2 \operatorname{Cosh} x \, dx = (x^2 + 2) \operatorname{Sinh} x - 2x \operatorname{Cosh} x$$

$$\int x^2 \operatorname{Sinh} x \, dx = (x^2 + 2) \operatorname{Cosh} x - 2x \operatorname{Sinh} x$$

$$\left.\begin{aligned}\int x^{-1} \operatorname{Cosh} x \, dx &= \tfrac{1}{2} (\overline{\operatorname{Ei}}\, x + \operatorname{Ei}(-x)) \\ \int x^{-1} \operatorname{Sinh} x \, dx &= \tfrac{1}{2} (\overline{\operatorname{Ei}}\, x - \operatorname{Ei}(-x))\end{aligned}\right\} \text{ see p. 115.}$$

$$\int x^{-2} \operatorname{Cosh} x \, dx = -x^{-1} \operatorname{Cosh} x + \tfrac{1}{2} (\overline{\operatorname{Ei}}\, x - \operatorname{Ei}(-x))$$

$$\int x^{-2} \operatorname{Sinh} x \, dx = -x^{-1} \operatorname{Sinh} x + \tfrac{1}{2} (\overline{\operatorname{Ei}}\, x + \operatorname{Ei}(-x))$$

$$\int \operatorname{Cosh}^2 x \, dx = \tfrac{1}{2} (\operatorname{Cosh} x \operatorname{Sinh} x + x)$$

$$\int \operatorname{Sinh}^2 x \, dx = \tfrac{1}{2} (\operatorname{Cosh} x \operatorname{Sinh} x - x)$$

$$\int \operatorname{Cosh}^3 x \, dx = \tfrac{1}{3} \operatorname{Sinh} x (\operatorname{Cosh}^2 x + 2)$$

$$\int \operatorname{Sinh}^3 x \, dx = \tfrac{1}{3} \operatorname{Cosh} x (\operatorname{Sinh}^2 x - 2)$$

$$\int \operatorname{Cosh}^n x \, dx = \frac{1}{n} \operatorname{Cosh}^{n-1} x \operatorname{Sinh} x + \frac{n-1}{n} \int \operatorname{Cosh}^{n-2} x \, dx$$

$$\int \operatorname{Sinh}^n x \, dx = \frac{1}{n} \operatorname{Sinh}^{n-1} x \operatorname{Cosh} x - \frac{n-1}{n} \int \operatorname{Sinh}^{n-2} x \, dx$$

$$\int \frac{dx}{\operatorname{Cosh} x} = 2 \arctan e^x \qquad \int \frac{dx}{\operatorname{Sinh} x} = \ln \operatorname{Tanh} \frac{x}{2}$$

$$\int \frac{dx}{\operatorname{Cosh}^2 x} = \operatorname{Tanh} x \qquad \int \frac{dx}{\operatorname{Sinh}^2 x} = -\operatorname{Coth} x$$

$$\int \frac{dx}{\operatorname{Cosh} x \operatorname{Sinh} x} = \ln \operatorname{Tanh} x$$

10. *Extreme values of the argument x*

For very small arguments use the power series (1). For arguments $x > 10$ use the approximations

$$\operatorname{Cosh} x \approx \operatorname{Sinh} x \approx \tfrac{1}{2} e^x, \qquad \operatorname{Tanh} x \approx 1.0000.$$

The exponential for $x > 10$ is found as indicated on p. 29.

11. *Bibliography for trigonometric and hyperbolic functions*

J. Peters: *Seven-Place Values of Trigonometric Functions.* Berlin, 1918, New York, 1942.

sin x, cos x with	7	dec.	for $x =$	0.000° (0.001°) 90.000°,
tan x	,, 7	dig.	,, $x =$	0.000° (0.001°) 88.000°,
	,, 6	,,	,, $x =$	88.000° (0.001°) 89.820°,
	,, 5–4	,,	,, $x =$	89.820° (0.001°) 90.000°.

Mathematical Tables Project: *Tables of Sines and Cosines for Radian Arguments.* 1940.

sin x, cos x with 8 dec. for x = 0.000 (0.001) 25.000.

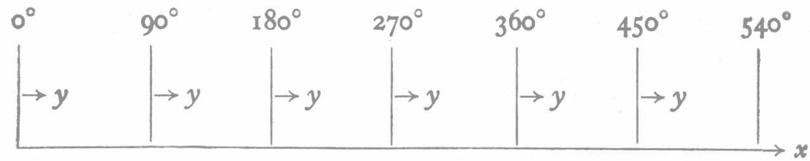
Mathematical Tables Project: *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments.* 1939.

sin x, cos x, Sinh x, Cosh x with 9 dec. for x = 0.0000 (0.0001) 1.9999,
,, 9 ,, ,, x = 0.0 (0.1) 10.0.

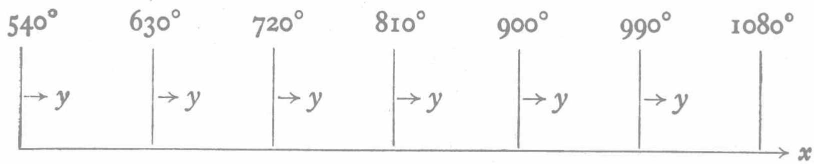
Mathematical Tables Project, National Bureau of Standards: *Table of Circular and Hyperbolic Tangents and Cotangents for Radian Argument.* New York, 1943.

tan x, cot x, Tanh x, Coth x with 8 dig. for x = 0.0000 (0.0001) 2.0000.

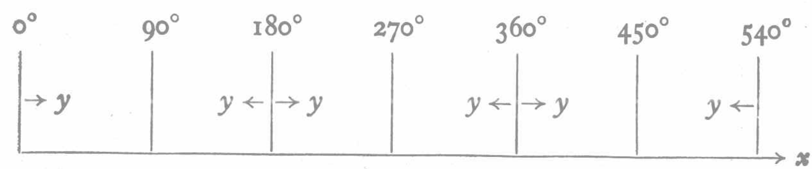
12. *Periodicity of Trigonometric Functions*



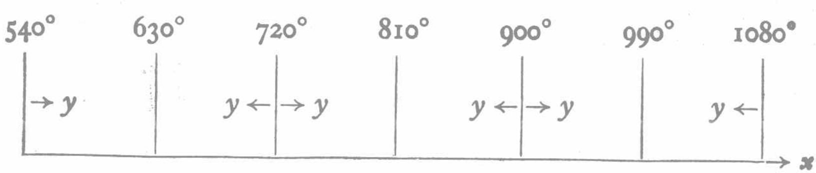
	0°–90°	90°–180°	180°–270°	270°–360°	360°–450°	450°–540°
$\sin x =$	$\sin y$	$\cos y$	$-\sin y$	$-\cos y$	$\sin y$	$\cos y$
$\cos x =$	$\cos y$	$-\sin y$	$-\cos y$	$\sin y$	$\cos y$	$-\sin y$
$\tan x =$	$\tan y$	$-\cot y$	$\tan y$	$-\cot y$	$\tan y$	$-\cot y$
$\cot x =$	$\cot y$	$-\tan y$	$\cot y$	$-\tan y$	$\cot y$	$-\tan y$



	540°–630°	630°–720°	720°–810°	810°–900°	900°–990°	990°–1080°
$\sin x =$	$-\sin y$	$-\cos y$	$\sin y$	$\cos y$	$-\sin y$	$-\cos y$
$\cos x =$	$-\cos y$	$\sin y$	$\cos y$	$-\sin y$	$-\cos y$	$\sin y$
$\tan x =$	$\tan y$	$-\cot y$	$\tan y$	$-\cot y$	$\tan y$	$-\cot y$
$\cot x =$	$\cot y$	$-\tan y$	$\cot y$	$-\tan y$	$\cot y$	$-\tan y$



	0°–90°	90°–180°	180°–270°	270°–360°	360°–450°	450°–540°
$\sin x =$	$\sin y$	$\sin y$	$-\sin y$	$-\sin y$	$\sin y$	$\sin y$
$\cos x =$	$\cos y$	$-\cos y$	$-\cos y$	$\cos y$	$\cos y$	$-\cos y$
$\tan x =$	$\tan y$	$-\tan y$	$\tan y$	$-\tan y$	$\tan y$	$-\tan y$
$\cot x =$	$\cot y$	$-\cot y$	$\cot y$	$-\cot y$	$\cot y$	$-\cot y$



	540°–630°	630°–720°	720°–810°	810°–900°	900°–990°	990°–1080°
$\sin x =$	$-\sin y$	$-\sin y$	$\sin y$	$\sin y$	$-\sin y$	$-\sin y$
$\cos x =$	$-\cos y$	$\cos y$	$\cos y$	$-\cos y$	$-\cos y$	$\cos y$
$\tan x =$	$\tan y$	$-\tan y$	$\tan y$	$-\tan y$	$\tan y$	$-\tan y$
$\cot x =$	$\cot y$	$-\cot y$	$\cot y$	$-\cot y$	$\cot y$	$-\cot y$

sin x

x	.0°	.1°	.2°	.3°	.4°	.5°	.6°	.7°	.8°	.9°		
0	.00000	0175	0349	0524	0698	0873	1047	1222	1396	1571	1745	89
1	1745	1920	2094	2269	2443	2618	2792	2967	3141	3316	3490	88
2	3490	3664	3839	4013	4188	4362	4536	4711	4885	5059	5234	87
3	5234	5408	5582	5756	5931	6105	6279	6453	6627	6802	6976	86
4	6976	7150	7324	7498	7672	7846	8020	8194	8368	8542	8716	85
5	8716	8889	9063	9237	9411	9585	9758	9932	*0106	*028	*045	84
6	.1045	063	080	097	115	132	149	167	184	201	219	83
7	219	236	253	271	288	305	323	340	357	374	392	82
8	392	409	426	444	461	478	495	513	530	547	564	81
9	564	582	599	616	633	650	668	685	702	719	736	80
10	736	754	771	788	805	822	840	857	874	891	908	79
11	908	925	942	959	977	994	*011	*028	*045	*062	*079	78
12	.2079	096	113	130	147	164	181	198	215	233	250	77
13	250	267	284	300	317	334	351	368	385	402	419	76
14	419	436	453	470	487	504	521	538	554	571	588	75
15	588	605	622	639	656	672	689	706	723	740	756	74
16	756	773	790	807	823	840	857	874	890	907	924	73
17	924	940	957	974	990	*007	*024	*040	*057	*074	*090	72
18	.3090	107	123	140	156	173	190	206	223	239	256	71
19	256	272	289	305	322	338	355	371	387	404	420	70
20	420	437	453	469	486	502	518	535	551	567	584	69
21	584	600	616	633	649	665	681	697	714	730	746	68
22	746	762	778	795	811	827	843	859	875	891	907	67
23	907	923	939	955	971	987	*003	*019	*035	*051	*067	66
24	.4067	083	099	115	131	147	163	179	195	210	226	65
25	226	242	258	274	289	305	321	337	352	368	384	64
26	384	399	415	431	446	462	478	493	509	524	540	63
27	540	555	571	586	602	617	633	648	664	679	695	62
28	695	710	726	741	756	772	787	802	818	833	848	61
29	848	863	879	894	909	924	939	955	970	985	*000	60
30	.5000	015	030	045	060	075	090	105	120	135	150	59
31	150	165	180	195	210	225	240	255	270	284	299	58
32	299	314	329	344	358	373	388	402	417	432	446	57
33	446	461	476	490	505	519	534	548	563	577	592	56
34	592	606	621	635	650	664	678	693	707	721	736	55
35	736	750	764	779	793	807	821	835	850	864	878	54
36	878	892	906	920	934	948	962	976	990	*004	*018	53
37	.6018	032	046	060	074	088	101	115	129	143	157	52
38	157	170	184	198	211	225	239	252	266	280	293	51
39	293	307	320	334	347	361	374	388	401	414	428	50
40	428	441	455	468	481	494	508	521	534	547	561	49
41	561	574	587	600	613	626	639	652	665	678	691	48
42	691	704	717	730	743	756	769	782	794	807	820	47
43	820	833	845	858	871	884	896	909	921	934	947	46
44	947	959	972	984	997	*009	*022	*034	*046	*059	*071	45
		.9°	.8°	.7°	.6°	.5°	.4°	.3°	.2°	.1°	.0°	y

cos y