Advanced Courses in Mathematics CRM Barcelona

Giuseppe Da Prato

Kolmogorov Equations for Stochastic PDEs

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Preface

This book is devoted to some basic stochastic partial differential equations, in particular reaction-diffusion equations, Burgers and Navier–Stokes equations perturbed by noise.

Particular attention is paid to the corresponding Kolmogorov equations which are elliptic or parabolic equations with infinitely many variables.

The aim of the book is to present the basic elements of stochastic PDEs in a simple and self-contained way in order to cover the program of one year PhD course both in Mathematics and in Physics.

The needed prerequisites are some basic knowledge of probability, functional analysis (including fundamental properties of Gaussian measures) and partial differential equations.

This book is an expansion of a course given by the author in 1997 at the "Center de Recerca Matemàtica" in Barcelona (see [30]), which I thank for the warm hospitality.

I wish also to thank B. Goldys for reading the manuscript and making several useful comments.

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Pisa, October 2004 Giuseppe Da Prato

Contents

Chapter 1

Introduction and Preliminaries

1.1 Introduction

We are here concerned with a stochastic differential equation in a separable Hilbert space H,

$$\begin{cases} dX(t,x) = (AX(t,x) + F(X(t,x)))dt + B\,dW(t), & t > 0,\ x \in H, \\ X(0,x) = x, & x \in H. \end{cases} \tag{1.1}$$

Here $A\colon D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in H, B is a bounded operator from another Hilbert space U and H, $F\colon D(F) \subset H \to H$ is a nonlinear mapping and $W(t)$, $t \geq 0$, is a cylindrical Wiener process in U defined in some probability space $(\Omega, \mathscr{F}, \mathbb{P})$, see Chapter 2 for a precise definition.

In applications equation (1.1) describes the evolution of an infinite dimensional dynamical system perturbed by noise (the system being considered "isolated" when $F = 0$).

In this book we shall consider several stochastic partial differential equations which can be written in the form (1.1). In each case we shall first prove existence and uniqueness of a *mild* solution. A mild solution of equation (1.1) is a mean square continuous stochastic process, adapted to $W(t)$, such that $X(t,x) \in D(F)$ for any $t \geq 0$ and

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}F(X(s,x))ds + W_A(t), \quad t \geq 0, \tag{1.2}$$

where $W_A(t)$ is the *stochastic convolution* defined by

$$W_A(t) = \int_0^t e^{(t-s)A}BdW(s), \quad t \geq 0. \tag{1.3}$$

Moreover, we shall study several properties of the *transition semigroup* P_t defined by [(1)]

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in B_b(H),\ t \geq 0,\ x \in H, \tag{1.4}$$

as Feller and strong Feller properties and irreducibility. We recall that P_t is *Feller* if $P_t\varphi$ is continuous for all $t \geq 0$ and any continuous and bounded function φ, *strong Feller* if $P_t\varphi$ is continuous for all $t > 0$ and all $\varphi \in B_b(H)$. Moreover, P_t is *irreducible* if $P_t 1_I(x) > 0$ for all $x \in H$ and all open sets I, where 1_I is the characteristic function of I [(2)].

To study asymptotic properties of the transition semigroup P_t an important tool is provided by *invariant measures.* A Borel probability measure ν in H is said to be *invariant* for P_t if

$$\int_H P_t\varphi d\nu = \int_H \varphi d\nu \tag{1.5}$$

for all continuous and bounded functions $\varphi\colon H \to \mathbb{R}$.

If P_t is irreducible, then any invariant measure ν is *full*, that is we have $\nu(B(x,r)) > 0$ for any ball $B(x,r)$ of center $x \in H$ and radius r. In fact from (1.5) it follows that

$$\nu(B(x,r)) = \int_H P_t 1_I(x)\nu(dx) > 0.$$

If P_t is at the same time irreducible and strong Feller, then there is at most one invariant measure in view of the Doob theorem, see Theorem 1.12 [(3)].

We shall prove, under suitable assumptions, existence (and in some cases uniqueness) of an invariant measure ν. As it is well known, this allows us to extend uniquely P_t to a strongly continuous semigroup of contractions in $L^2(H,\nu)$ (still denoted P_t). We shall denote by K_2 its infinitesimal generator.

Particular attention will be paid to describing the relationship between K_2 and the concrete differential operator K_0 defined by

$$K_0\varphi(x) = \frac{1}{2}\ \text{Tr}\ [CD^2\varphi(x)] + \langle Ax + F(x), D\varphi(x)\rangle, \quad \varphi \in \mathscr{E}_A(H), \tag{1.6}$$

where Tr denotes the trace, $C = BB^*$ (B^* being the adjoint of B), and D denotes the derivative with respect to x. Moreover, $\mathscr{E}_A(H)$ is the linear span of all real and imaginary parts of *exponential functions* φ_h,

$$\varphi_h(x) := e^{i\langle h,x\rangle}, \quad x \in H, \quad h \in D(A^*),$$

where A^* is the adjoint of A. It is easy to see that the space $\mathscr{E}_A(H)$ is dense in $L^2(H,\nu)$. The reason for taking $h \in D(A^*)$ is that this fact is necessary in order

(1) $B_b(H)$ is the space of all bounded and Borel real functions in H.

(2) $1_I(x) = 1$ if $x \in I$, $1_I(x) = 0$ if $x \notin I$.

(3) Another powerful method to prove the uniqueness of an invariant measure is based on coupling, see [54], [74], [73], [68], [40].

that $K_0\varphi_h$ be meaningful. In fact, if $\varphi_h(x) = e^{i\langle h,x\rangle}$ we have

$$K_0\varphi_h(x) = -\left(\frac{1}{2}\,|C^{1/2}h|^2 + i\langle x, A^*h\rangle + i\langle F(x), h\rangle\right)\varphi_h(x), \quad x \in H.$$

So, $K_0\varphi_h$ belongs to $L^2(H,\nu)$ provided

$$x \mapsto \langle x, A^*h\rangle \text{ and } x \mapsto \langle F(x), h\rangle \in L^2(H,\nu). \tag{1.7}$$

It is not difficult, by using the Itô formula, to show that K_2 is an extension of K_0. More difficult (in some cases still an open problem) is to show that K_2 is the closure of K_0 or, equivalently, that $\mathscr{E}_A(H)$ is a core for K_2. When this is the case, one can prove existence and uniqueness of a *strong* solution (in the sense of Friedrichs) of the Kolmogorov equation

$$\lambda\varphi - K_0\varphi = f, \tag{1.8}$$

where $\lambda > 0$ and $f \in L^2(H,\nu)$ are given. This means that for any $\lambda > 0$ and any $f \in L^2(H,\nu)$, there exists a sequence $\{\varphi_n\} \subset \mathscr{E}_A(H)$ such that

$$\lim_{n\to\infty} \varphi_n \to \varphi, \quad \lim_{n\to\infty} (\lambda\varphi_n - K_0\varphi_n) \to f \quad \text{in } L^2(H,\nu).$$

This result has several important consequences. In particular the following integration by parts formula (called in French the " ídentité du carré du champs") holds,

$$\int_H K_2\varphi\,\varphi\,d\nu = -\frac{1}{2}\int_H |B^*D\varphi|^2 d\nu, \quad \varphi \in D(K_2). \tag{1.9}$$

Let us give an idea of the proof. Since we know that $\mathscr{E}_A(H)$ is a core for K_2, it is enough to prove (1.9) for $\varphi \in \mathscr{E}_A(H)$. In this case one can check, by a straightforward computation, the identity

$$K_0(\varphi^2) = 2K_0\varphi\,\varphi + |B^*D\varphi|^2.$$

Now, since ν is invariant, we have that $\int_H K_0(\varphi^2)d\nu = 0$, and so (1.9) follows.

Identity (1.9) implies that if $\varphi \in D(K_2)$, then $B^*D\varphi$ is well defined, so that one can study perturbation operators of the form

$$\varphi \to K\varphi + \langle G, B^*D\varphi\rangle,$$

with $G\colon H \to H$ bounded Borel. Some other interesting consequences of (1.9), such as Poincaré and log-Sobolev inequalities, will be presented later when we study specific equations.

We shall first consider the important special case when $F = 0$ (corresponding in the applications to the absence of interactions). In this case we shall write (1.1) as

$$\begin{cases} dZ(t,x) = AZ(t,x)dt + B\,dW(t), & t > 0,\ x \in H, \\ Z(0,x) = x, & x \in H. \end{cases} \tag{1.10}$$

The solution $Z(t,x)$ is called the Ornstein–Uhlenbeck process. The corresponding transition semigroup will be denoted by

$$R_t\varphi(x) = \mathbb{E}[\varphi(Z(t,x))], \quad \varphi \in B_b(H).$$

If the operator A is of negative type [(4)] it is not difficult to show that there exists a unique invariant measure μ for R_t. More precisely, μ is the Gaussian measure with mean 0 and covariance operator

$$Q = \int_0^\infty e^{tA}BB^*e^{tA^*}\,dt.$$

We shall denote by L_2 the infinitesimal generator of the extension of R_t to $L^2(H,\mu)$ and shall prove that L_2 is the closure of the Kolmogorov operator

$$L_0\varphi(x) = \frac{1}{2}\,\mathrm{Tr}\,[CD^2\varphi(x)] + \langle x, A^*D\varphi(x)\rangle, \quad \varphi \in \mathscr{E}_A(H). \tag{1.11}$$

However, it is useful to study the semigroup R_t also in other spaces as in the space $C_b(H)$ of all uniformly continuous and bounded real functions in H. Here the semigroup is not strongly continuous but its infinitesimal generator L can be defined, see [17], as the unique closed operator L in $C_b(H)$ such that

$$(\lambda - L)^{-1}f(x) = \int_0^\infty e^{-\lambda t}R_tf(x)dt, \quad x \in H,\ \lambda > 0,\ f \in C_b(H). \tag{1.12}$$

Then we shall consider the case when F is Lipschitz continuous. The results proved here will be useful to study by approximation equations with irregular coefficients.

Finally, we shall try to prove an explicit formula relating the invariant measures μ and ν of equations (1.1) and (1.10) respectively. More precisely, we shall show (under suitable assumptions), following the recent result in [35], that

$$\int_H f d\mu = \int_H f d\nu + \int_H \langle F, DL^{-1}f\rangle d\nu, \quad f \in B_b(H), \tag{1.13}$$

where L is the Ornstein–Uhlenbeck generator defined by (1.12). From (1.13) it follows that ν is absolutely continuous with respect to μ.

This book has an elementary character. For the sake of simplicity, we have only considered equations with additive noise and we have only studied Kolmogorov equations coming from some stochastic partial differential equations such as *reaction-diffusion* equations, *Burgers* equation and *2D-Navier–Stokes equations.*

[(4)] That is if there exists $M > 0$ and $\omega < 0$ such that $\|e^{tA}\| \le Me^{-\omega t}$ for all $t \ge 0$.

The same method could be applied to other equations such as the wave equation [6], [26], [27], [87], [88], the Cahn–Hilliard equation [31] and the Stefan problem [7].

We mention that Kolmogorov equations can also be studied by purely analytical methods, see the monograph [51] and references therein. This method is important when one is not able to solve (1.1), see [34], [43], [44], [39].

Also in concrete equations we have not presented the more general results of the literature, which in some cases are very technical but we have used simple situations as examples.

We end this chapter by giving some preliminaries and recalling some results which will be used in what follows.

1.2 Preliminaries

In this book H represents a separable Hilbert space (inner product $\langle \cdot, \cdot \rangle$, norm $|\cdot|$) and $L(H)$ the Banach algebra of all linear continuous operators from H into H endowed with the norm

$$\|T\| = \sup\{|Tx|;\ x \in H,\ |x| = 1\}, \quad T \in L(H).$$

For any $T \in L(H)$, T^* is the adjoint operator of T. Moreover,

$$\Sigma(H) = \{T \in L(H) :\ T = T^*\}$$

and

$$L^+(H) = \{T \in \Sigma(H) :\ \langle Tx, x \rangle \geq 0, \quad x, y \in H\}.$$

1.2.1 Some functional spaces

In this section H and U represent separable Hilbert spaces.

- $B_b(H;U)$ is the Banach space of all bounded and Borel mappings $\varphi\colon H \to U$, endowed with the norm

$$\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|, \quad \varphi \in C_b(H).$$

- $C_b(H;U)$ is the closed subspace of $B_b(H;U)$ consisting of all uniformly continuous and bounded mappings from H into U. If $U = \mathbb{R}$ we set $B_b(H;U) = B_b(H)$ and $C_b(H;U) = C_b(H)$.

- $C_b^1(H)$ is the space of all uniformly continuous and bounded functions $\varphi\colon H \to \mathbb{R}$ which are Fréchet differentiable on H with uniformly continuous and bounded derivative $D\varphi$. We set

$$\|\varphi\|_1 = \|\varphi\|_0 + \sup_{x \in H} |D\varphi(x)|, \quad \varphi \in C_b^1(H).$$

If $\varphi \in C_b^1(H)$ and $x \in H$, we shall identify $D\varphi(x)$ with the unique element h of H such that

$$D\varphi(x)y = \langle h, y\rangle, \quad y \in H.$$

- $C_b^2(H)$ is the subspace of $C_b^1(H)$ of all functions $\varphi\colon H \to \mathbb{R}$ which are twice Fréchet differentiable on H with uniformly continuous and bounded second derivative $D^2\varphi$. We set

$$\|\varphi\|_2 := \|\varphi\|_1 + \sup_{x\in H} \|D^2\varphi(x)\|, \quad \varphi \in C_b^2(H).$$

If $\varphi \in C_b^2(H)$ and $x \in H$, we shall identify $D^2\varphi(x)$ with the unique linear operator $T \in L(H)$ such that

$$D\varphi(x)(y, z) = \langle Ty, z\rangle, \quad y, z \in H.$$

For any $k \in \mathbb{N}$, $C_b^k(H)$ is defined in a similar way. We set finally

$$C_b^\infty(H) = \bigcap_{k=1}^{\infty} C_b^k(H).$$

- $C_b^{0,1}(H)$ is the subspace of $C_b(H)$ of all Lipschitz continuous functions. $C_b^{0,1}(H)$ is a Banach space with the norm

$$\|\varphi\|_1 := \|\varphi\|_0 + \sup\left\{\frac{|\varphi(x) - \varphi(y)|}{|x-y|},\ x, y \in H,\ x \neq y\right\}, \quad \varphi \in C_b^{0,1}(H).$$

- $C_b^{1,1}(H)$ is the space of all functions $\varphi \in C_b^1(H)$ such that $D\varphi$ is Lipschitz continuous. $C_b^{1,1}(H)$ is a Banach space with the norm

$$\|\varphi\|_{1,1} = \|f\|_1 + \sup\left\{\frac{|D\varphi(x) - D\varphi(y)|}{|x-y|},\ x, y \in H,\ x \neq y\right\}, \quad \varphi \in C_b^{1,1}(H).$$

We recall that $C_b^2(H)$ is not dense in $C_b(H)$, see [89]. The following result was proved in [75].

Theorem 1.1. *$C_b^{1,1}(H)$ is dense in $C_b(H)$.*

We finally consider functions having (at most) *quadratic growth*. We denote by $C_{b,2}(H)$ the space of all functions $\varphi\colon H \to \mathbb{R}$ such that the mapping

$$H \to \mathbb{R},\ x \to \frac{\varphi(x)}{1+|x|^2}$$

belongs to $C_b(H)$. $C_{b,2}(H)$, endowed with the norm

$$\|\varphi\|_{b,2} = \sup_{x\in H} \frac{|\varphi(x)|}{1+|x|^2},$$

is a Banach space.

Moreover we shall denote by $C^1_{b,2}(H)$, the space of all continuously differentiable mappings $\varphi : H \to \mathbb{R}$ of $C_{b,2}(H)$such that

$$[\varphi]_{1,2} := \sup_{x\in H} \frac{|D\varphi(x)|}{1+|x|^2} < +\infty.$$

1.2.2 Exponential functions

We are here concerned with the set $\mathscr{E}(H)$ of all *exponential functions*, defined as the span of all real and imaginary parts of functions,

$$\varphi_h(x) : = e^{i\langle x,h\rangle},\ x,h \in H.$$

$\mathscr{E}(H)$ is an agebra with the usual operations.

The following approximation result of continuous functions by exponential functions will be useful in what follows. It is easy to see that the closure of $\mathscr{E}(H)$ in $C_b(H)$ does not coincide with $C_b(H)$ [(5)]. So we shall prove only a pointwise approximation, see [51].

Proposition 1.2. *For all $\varphi \in C_b(H)$, there exists a two index sequence $\{\varphi_{n_1,n_2}\} \subset \mathscr{E}(H)$ such that*

(i) $\|\varphi_{n_1,n_2}\|_0 \le \|\varphi\|_0$,

(ii) $\lim_{n_1\to\infty} \lim_{n_2\to\infty} \varphi_{n_1,n_2}(x) = \varphi(x), \quad x \in H.$

Notice that we cannot replace $\{\varphi_{n_1,n_2}\}$ with a sequence by a diagonal extraction procedure due to the pointwise character of the convergence.

Proof. We first assume that $H = \mathbb{R}^d$ with $d \in \mathbb{N}$. Then for any $n \in \mathbb{N}$ there exists $\psi_n \in C_b(\mathbb{R}^d)$ with the properties:

(i) ψ_n is periodic with period n in all its coordinates.

(ii) $\psi_n(x) = \varphi(x)$ for all $x \in [-n + 1/2, n - 1/2]^d$.

(iii) $\|\psi_n\|_0 \le \|\varphi\|_0$.

Clearly $\psi_n(x) \to \varphi(x)$ for all $x \in \mathbb{R}^d$. Moreover, by using Fourier series, we can find a sequence $\{\varphi_n\}$ in $\mathscr{E}(H)$, close to $\{\psi_n\}$ and fulfilling (i) and (ii).

Let now H be infinite dimensional, $\{e_k\}$ a complete orthonormal system in H, and for any $m \in \mathbb{N}$ let P_m be the projector on the linear space spanned by $\{e_1, \dots, e_m\}$,

$$P_m x = \sum_{j=1}^m \langle x, e_j\rangle e_j, \quad x \in H.$$

(5) It is the space of all almost periodic functions in H.

Given $\varphi \in C_b(H)$ and $n_1 \in \mathbb{N}$, let us consider the function

$$H \to \mathbb{R}, \quad x \to \varphi(P_{n_1}x).$$

By the first part of the proof, for each $n_1 \in \mathbb{N}$, there exists a sequence $\{\varphi_{n_1,n_2}\} \subset \mathscr{E}(H)$ such that $\lim_{n_2\to\infty} \varphi_{n_1,n_2}(x) = \varphi(P_{n_1}x)$ for all $x \in H$, and $\|\varphi_{n_1,n_2}\|_0 \le \|\varphi\|_0$. Therefore

$$\lim_{n_1\to\infty}\lim_{n_2\to\infty} \varphi_{n_1,n_2}(x) = \varphi(x),$$

for all $x \in H$. □

Proposition 1.3. *For all $\varphi \in C_{b,2}(H)$ there exists a two index sequence $\{\varphi_{n_1,n_2}\} \subset \mathscr{E}(H)$ such that:*

(i) $\|\varphi_{n_1,n_2}\|_{b,2} \le \|\varphi\|_{b,2}$.

(ii) $\lim_{n\to\infty} \varphi_{n_1,n_2}(x) = \varphi(x), \quad x \in H.$

Proof. Let first $H = \mathbb{R}^d$ and set

$$\psi(x) = \frac{\varphi(x)}{1+|x|^2}, \quad x \in H.$$

By Proposition 1.2 there exists a sequence $\{\psi_n\} \subset \mathscr{E}(H)$ such that

(i) $\|\psi_n\|_0 \le \|\psi\|_0 = \|\varphi\|_{b,2}$,

(ii) $\lim_{n\to\infty} \psi_n(x) = \psi(x)$, $x \in H$.

Setting

$$\varphi_n(x) = 1 + \sum_{i=1}^d (n\sin(x_i/n))^2, \quad x \in \mathbb{R}^d,$$

we have $\varphi_n \in \mathscr{E}_A(H)$, $\|\varphi_n\|_{b,2} \le \|\varphi\|_{b,2}$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$, $x \in H$. If H is infinite dimensional we proceed as in the second part of the proof of Proposition 1.2. □

1.2.3 Gaussian measures

Let $L_1(H)$ be the Banach space of all trace class operators in H endowed with the norm

$$\|T\|_1 = \operatorname{Tr}\sqrt{TT^*}, \quad T \in L_1(H),$$

where Tr represents the trace. We set $L_1^+(H) = L_1(H) \cap L^+(H)$. We recall that a linear operator $Q \in L^+(H)$ is of *trace class* if and only if there exists a complete orthonormal system $\{e_k\}$ in H and a sequence of nonnegative numbers $\{\lambda_k\}$ such that

$$Qe_k = \lambda_k e_k, \quad k \in \mathbb{N},$$

and

$$\operatorname{Tr} Q := \sum_{k=1}^{\infty} \lambda_k < +\infty.$$

For any $a \in H$ and $Q \in L^+(H)$ we define the Gaussian probability measure $N_{a,Q}$ in H by identifying H with ℓ^2 (6), and setting

$$N_{a,Q} = \prod_{k=1}^{\infty} N_{a_k,\lambda_k}, \quad a_k = \langle a, e_k \rangle, \ k \in \mathbb{N}.$$

In this way the measure $N_{a,Q}$ is defined on the product space $\mathbb{R}^\infty$ of all real sequences, but it is concentrated on ℓ^2 (that is $\mu(\ell^2) = 1$) since, thanks to the monotone convergence theorem, we have

$$\int_{\mathbb{R}^\infty} |x|_{\ell^2}^2 N_{a,Q}(dx) = \sum_{k=1}^{\infty} \int_{\mathbb{R}} x_k^2 N_{a_k,\lambda_k}(dx_k) = \sum_{k=1}^{\infty} (\lambda_k + a_k^2) < +\infty.$$

If $a = 0$ we shall write $N_{a,Q} = N_Q$ for brevity. We shall always assume Ker $Q = \{0\}$ in what follows.

If H is n-dimensional and $\det Q > 0$, we have

$$N_{a,Q}(dx) = (2\pi)^{-n/2} (\det Q)^{-1/2} \, e^{-\frac{1}{2} \langle Q^{-1}(x-a), x-a \rangle} \, dx, \quad x \in H. \tag{1.14}$$

Let us list some useful identities. They are straightforward when H is n-dimensional and can be easily proved in the general case letting $n \to \infty$. For $\mu = N_{a,Q}$ we have

$$\int_H |x|^2 \mu(dx) = \operatorname{Tr} Q + |a|^2, \tag{1.15}$$

$$\int_H \langle x, h \rangle \mu(dx) = \langle a, h \rangle, \ \ h \in H, \tag{1.16}$$

$$\int_H \langle x-a, h \rangle \langle x-a, k \rangle \mu(dx) = \langle Qh, k \rangle, \ \ h, k \in H, \tag{1.17}$$

$$\int_H e^{i\langle x, h \rangle} \mu(dx) = e^{i\langle a, h \rangle} e^{-\frac{1}{2} \langle Qh, h \rangle}, \ \ h \in H. \tag{1.18}$$

The range $Q^{1/2}(H)$ of $Q^{1/2}$ is called the *Cameron–Martin* space of N_Q. If H is infinite dimensional, $Q^{1/2}(H)$ is dense in H but different from H and it is important to notice that

$$N_Q(Q^{1/2}(H)) = 0. \tag{1.19}$$

(6) ℓ^2 is the space of all sequences $\{x_k\}$ of real numbers such that $|x|_{\ell^2}^2 := \sum_{k=1}^{\infty} |x_k|^2 < +\infty$.