ELEMENTS OF LINEAR SPACES

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PREFACE

There seems to be a need for an elementary undergraduate course to bridge the gap between freshman mathematics and modern abstract algebra. There are books which attempt an elementary treatment of vector spaces, linear transformations, matrices, etc., but they usually avoid an approach through geometry. Many intelligent students of mathematics feel that their graduate study represents a change of major field, as the methods of generalization are completely new to them.

In this book, we present the elementary material on linear spaces, restricting ourselves to real spaces of dimension not more than three in the first five chapters. Thereafter, the ideas are generalized to complex-n-dimensional spaces, and, in chapter 8, we study the geometry of conic sections and quadric surfaces using the methods developed.

A ninth chapter covers the application of the same techniques to problems drawn from various branches of geometry. Following this, two chapters deal with the subject of algebraic structures, especially vector spaces and transformations, from the abstract point of view, including projections and the decomposition of Hermitian operators. A final chapter then treats some of the more accessible recent results on singular values and their use for estimating proper values of general transformations.

We believe both students and instructors are intelligent, and would like to supply details of proof or technique in many places, and have kept this idea in mind in writing the book. We feel also that the book can be used as a quick review for more advanced students.

Problems marked with an asterisk are of greater difficulty or generality, and occasionally require additional background.

The authors would appreciate suggestions and criticism of this text, which may be useful in future revisions.

A. R. A-M., A. L. F.

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PART I

1. REAL EUCLIDEAN SPACE

1.1 **Scalars and vectors:** We call any real number a *scalar.* A line segment **AB** with an orientation (sense or direction) from **A** to **B** is called the *vector* AB. **A** is called the beginning point and **B** is the end point of AB. By $|AB|$, the *norm* of AB, we mean the length of the line segment **AB**. In this book we consider only vectors having a fixed beginning **O**, called the origin. By a vector V we mean OV. In the special case where **V** is the origin we define the vector V to be the zero vector O. The zero vector may be assumed to have any direction.

1.2 **Sums and scalar multiples of vectors:** Let A and B be two vectors. Then $A + B$ is defined to be the vector R, where **R** is the fourth vertex of the parallelogram whose other vertices are **O**, **A**, and **B** (Fig. 1). In case the two vectors are collinear, the parallelogram degenerates to a line segment (Fig. 2). To add three vectors, we define $A + B + C$ to be R where **R** is that vertex of the parallelepiped determined by the vectors A, B, and C which is not on any edge containing **A**, **B**, or **C** (Fig. 3). The degenerate cases may be considered as before. We observe without proof that addition of three vectors is equivalent to adding one of them to the sum of the other two. Thus we may extend the definition to the sum of any number of vectors, and we observe that this sum is independent of the order of addition, that is, the addition is commutative and associative. By the difference $A - B$ we mean the vector C such that $B + C = A$.

If n is a scalar and A a vector, by nA we mean a vector B such that **B** is on the line **OA**, if n is positive, **B** and **A** are on the same side of **O**, if n is negative, they are on opposite sides of **O**, and when $n = 0$, $B = O$, and finally, $|B| = |n|\,|A|$. For any two vectors A and B on the same line there is a unique scalar x such that $A = xB$, if $B \neq O$. Comparing this with the degenerate cases of addition, we observe that $(x + y)A = xA + yB$, and $x(yA) = (xy)A$.

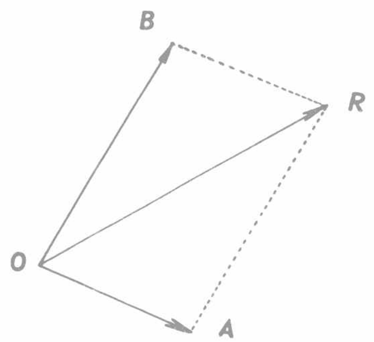

Fig. 1

Fig. 2

1.3 **Linear independence:** Two vectors U and V are called *linearly independent* if the points **O**, **U**, and **V** are not collinear, otherwise they are called *linearly dependent*.

Three vectors U, V, and W are called linearly independent if the points **O**, **U**, **V**, and **W** are not coplanar, otherwise they are linearly dependent. Any four or more vectors are linearly dependent.

1.4 **Theorem:** Let U and V be linearly independent. Given any vector A in the plane **UOV**, two scalars x and y are uniquely defined such that $A = xU + yV$. Conversely given the scalars x and y there is a unique vector A in the plane **UOV** such that $A = xU + yV$.

Proof: The lines through **A**, parallel to U and V respectively, intersect the lines **OV** and **OU** at **C** and **B** (Fig. 4). By 1.2 we have x and y such that $B = xU$ and $C = yV$, and $A = B + C$. Conversely given x and y we get B and C such that $B = xU$ and $C = yV$ and construct A to be $B + C$.

1.5 **Theorem:** Let U, V, and W be linearly independent. Then

(1) none of U, V, and W is in the plane of the other two,
(2) given any vector A, scalars x, y, and z are uniquely defined so that $A = xU + yV + zW$,
(3) given three scalars x, y, and z, there is a unique vector A such that $A = xU + yV + zW$.

Proof: The statement (1) follows trivially from the definition 1.3. The geometric construction of a unique parallelepiped with three sides on the lines **OU**, **OV**, and **OW** with the directions U, V, and W and the diagonal **OA**, and 1.2, make the proof of (2) clear.

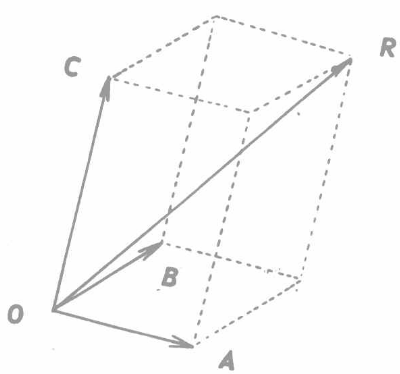

Fig. 3

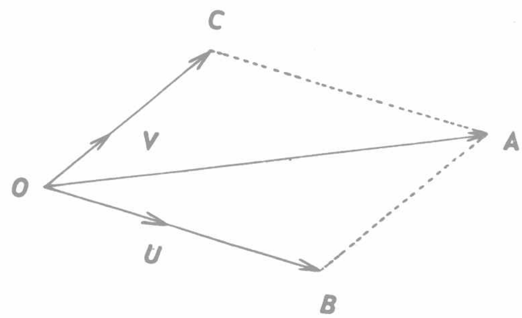

Fig. 4

By 1.2 there are points **B**, **C**, and **D** such that $B = xU$, $C = yV$, $D = zW$, and adding $B + C + D$ gives a unique sum A. This proves (3) (Fig. 5).

Given any set of n vectors $\{U_1, U_2, \ldots, U_n\}$, a vector V is called a *linear combination* of $U_1, U_2, \ldots, U_n$ if there are scalars $a_1, a_2, \ldots, a_n$ such that

$$V = a_1 U_1 + a_2 U_2 + \ldots + a_n U_n .$$

1.6 **Theorem:** A set of vectors is linearly dependent if and only if some one of them can be written as a linear combination of the others.

Proof: For the case of two vectors, if A and B are linearly dependent, then they are collinear, and either $A = xB$ or $B = 0.A$. Also if $A = xB$, then A and B are collinear [see 1.2]. For three vectors A, B, and C, if two are linearly dependent, the proof is as before. If no two are linearly dependent but the three are linearly dependent, then since C is in the plane of A and B, $C = xA + yB$. Also if $C = xA + yB$, then C is in the plane of A and B [see 1.4]. For the case of four or more vectors, again if there are three linearly independent vectors in the set, any other is a linear combination of those three [see 1.5]. Finally any four vectors are linearly dependent [see 1.3].

1.7 **Base (Co-ordinate system):** On any line l let a point **O** as the origin and a vector U, with $|U| \neq 0$, be chosen. Clearly by 1.2, to each vector A on the line l corresponds a scalar x such that $A = xU$ and to each scalar x corresponds a vector A on the line l such that $A = xU$. We call $\{U\}$ a *base* for the line l (Fig. 6), and x the corresponding *component* of the vector A.

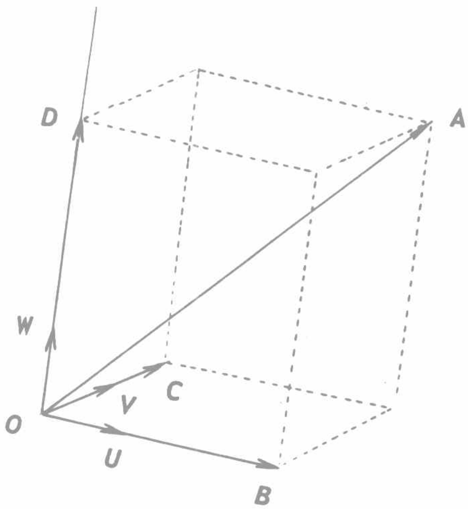

Fig. 5

Fig. 6

For a plane, consider two vectors U and V which are linearly independent. By 1.4 for any A there are scalars x and y such that $A = xU + yV$, and given any x and y there is an A such that $A = xU + yV$. We call the set $\{U,V\}$ a *base* for the plane **OUV**, and we order the set, calling U the *first* and V the *second* element of the base. In this way A corresponds to a unique *ordered pair* (x,y) and (x,y) corresponds to a unique vector A (Fig. 4). Again the scalars x and y will be called the *components* of A with respect to the base $\{U,V\}$.

For the whole space we choose U, V, and W linearly independent. By 1.4 for any A there are unique scalars x, y, and z such that $A = xU + yV + zW$ and, conversely, given x, y, and z there is a unique A such that $A = xU+yV+zW$. Exactly as above for the case of the plane, we can establish a correspondence between

A and the ***ordered triple*** (x,y,z) which is called a *one-to-one* correspondence. The set $\{U,V,W\}$ is called a *base* for the space (Fig. 5), and the scalars x,y, and z the *components* of A with respect to that base.

The choice of vectors for a base on a line or a plane or the space is arbitrary except for the number of elements. Because of this we can introduce the idea of *dimension*. A line is one-dimensional, a plane is of two dimensions, and the space has three dimensions, corresponding to the number of elements in the base. We may represent a vector by the ordered set of its components with respect to a base. These numbers are also called *coordinates* of the end point of the vector. We may therefore denote a vector by the set of its components with respect to a base.

If $\{U,V,W\}$ is a base for the space, we observe that

$$U = 1.U + 0.V + 0.W,$$

hence the components of U with respect to this base are $(1,0,0)$. We write $U = (1,0,0)$. Similarly,

$$V = 0.U + 1.V + 0.W,$$

$$W = 0.U + 0.V + 1.W,$$

hence $V = (0,1,0)$ and $W = (0,0,1)$. The reader will see readily that in a plane with base $\{U,V\}$, $U = (1,0)$ and $V = (0,1)$.

1.8 **Theorem:** Let a base $\{U,V,W\}$ be chosen in the space. Let the vectors $A = x_1U + y_1V + z_1W$ and $B = x_2U + y_2V + z_2W$ be denoted by (x_1,y_1,z_1) and (x_2,y_2,z_2) respectively. Then

I $(x_1,y_1,z_1) + (x_2,y_2,z_2) = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$,

II $a(x_1,y_1,z_1) = (ax_1,ay_1,az_1)$, a is a scalar.

Proof: I.

$$(x_1,y_1,z_1) + (x_2,y_2,z_2) = x_1U + y_1V + z_1W + x_2U + y_2V + z_2W =$$

$$(x_1 + x_2)\,U + (y_1 + y_2)\,V + (z_1 + z_2)\,W = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2) .$$

Part II. follows similarly [see 1.2]. The reader may supply a geometric proof.

Illustration 1: With respect to some base, let $A = (1,-1,2)$, and $B = (2,0,-1)$. Find $3A - 2B$.

By 1.8, $3A - 2B = 3(1,-1,2) - 2(2,0,-1) = (3,-3,6) + (-4,0,2) = (-1,-3,8)$.

Illustration 2: Find the coordinates of the midpoint of the line segment joining the end points of the vectors $A = (x_1,y_1,z_1)$ and $B = (x_2,y_2,z_2)$.

Since the diagonals of a parallelogram bisect each other, the midpoint of the line segment is also the midpoint of the other diagonal $R = A + B$ [see Fig. 1]. Thus the vector $\frac{1}{2}R$ ends at the desired point. Since

$$\frac{1}{2}R = \frac{1}{2}(A + B) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) ,$$

these numbers are the coordinates of the midpoint.

Illustration 3: Determine whether the vectors $(2,3,0)$, $(1,0,2)$, and $(6,-1,0)$ are linearly dependent or independent.

By 1.6, we observe that $(2,3,0)$ and $(1,0,2)$ are linearly independent since there is no scalar x for which

$$(2,3,0) = x(1,0,2), \text{ that is, } \quad 2 = x, \quad 3 = 0(x), \quad 0 = 2x.$$

We wish to determine, therefore, if there exist scalars x and y such that

$$(6,-1,0) = x(2,3,0) + y(1,0,2) ,$$

that is, such that

$$\begin{cases} 2x + y = 6 \\ 3x + 0y = -1 \\ 0x + 2y = 0 \end{cases} .$$

A solution of two of these equations does not satisfy the third, hence no such scalars exist. The three vectors are therefore linearly independent.

Illustration 4: Let U, V, and W be the vectors in the space ending at the points with coordinates (2,3,0), (1,0,2), and (6,-1,0). Find the components of the vector A ending at (-1,7,2) with respect to the base $\{U, V, W\}$.

We have already shown (Illustration 3) that the vectors U, V, and W are linearly independent. Thus we must find scalars x, y, z such that

$$A = xU + yV + zW, \qquad \text{that is,}$$

$$\begin{cases} 2x + y + 6z = -1 \\ 3x \quad\quad - z = 7 \\ \quad 2y \quad\quad = 2 \end{cases} .$$

This system of equations gives the solution $x = 2$, $y = 1$, $z = -1$, hence

$$A = 2U + V - W \quad .$$

1.9 **Inner product of two vectors:** Let U and V be two vectors. Let the angle between U and V be α. Then the *inner product* of U and V, denoted by (U, V), is defined to be $|U| \cdot |V| \cos \alpha$. It is clear that

$$(U, V) = (V, U) \quad \text{and} \quad |U|^2 = (U, U),$$

and if U and V are perpendicular

$$(U, V) = 0 \quad .$$

We use the word *orthogonal* as synonymous with perpendicular.

1.10 **Projection of a vector on an axis:** By an *axis* we mean a straight line through the origin **O** with a direction, as in analytic geometry. The projection of V on the axis **Ox** is the algebraic (signed) length of **OA**, where **A** is the foot of the perpendicular through **V** to **Ox** (Fig. 7). If V makes an angle α with the

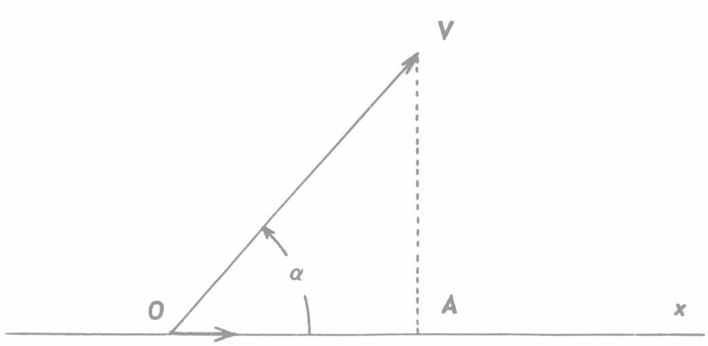

Fig. 7

positive direction of **Ox**, clearly

$$\text{projection of } V \text{ on } \mathbf{Ox} = |V| \cos\alpha .$$

1.11 **Theorem:** Let **Ox** be an axis and U and V two vectors, and let $R = U + V$. Then the projection of R on **Ox** is the sum of the projections of U and of V.

Proof: By constructing three planes through **U**, **V**, and **R** perpendicular to **Ox** we get **A**, **B**, and **C**, the projections of the points **U**, **V**, and **R** respectively (Fig. 8). Comparing the three projections, the result is clear.

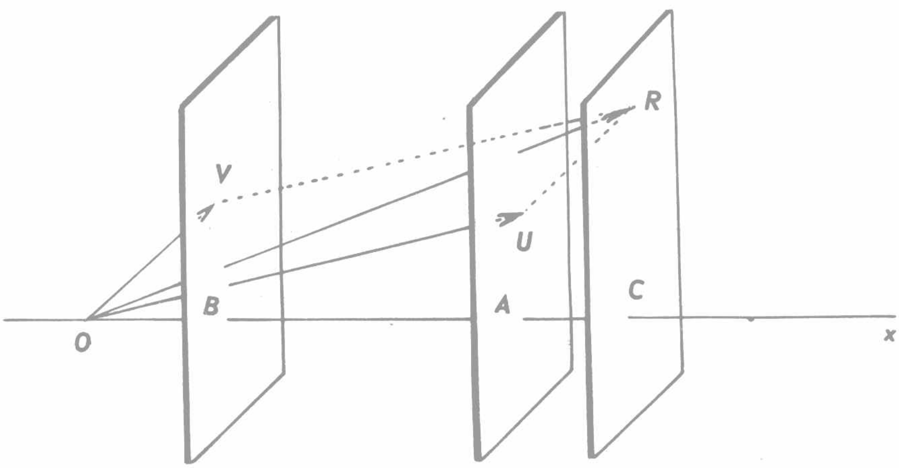

Fig. 8

1.12 **Theorem:** The inner product is distributive with respect to vector addition, i.e., given U, V, and W we have $(U, [V + W]) = (U, V) + (U, W)$.

Proof: Let the angle between U and V be α, the angle between U and W be β, and the angle between $R = V + W$ and U be γ (Fig. 9). Then by 1.10 we have

$$|V| \cos\alpha = \text{projection of } V \text{ on } \mathbf{OU},$$

$$|W| \cos\beta = \text{projection of } W \text{ on } \mathbf{OU},$$

$$|R| \cos\gamma = \text{projection of } R \text{ on } \mathbf{OU}.$$

By 1.11 we have

$$|R| \cos\gamma = |V| \cos\alpha + |W| \cos\beta.$$

Multiplying this by $|U|$ we get

$$|U|\,|V| \cos\alpha + |U|\,|W| \cos\beta = |U|\,|R| \cos\gamma, \quad \text{or}$$

$$(U, R) = (U, [V + W]) = (U, V) + (U, W) .$$

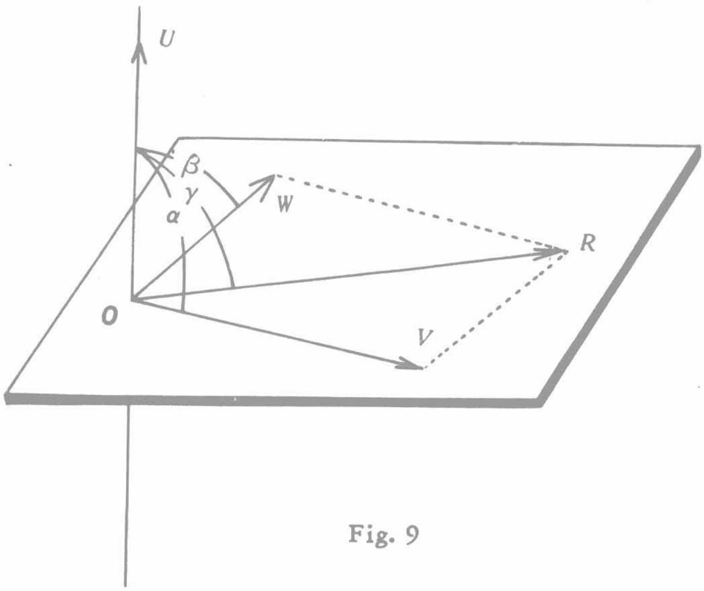

Fig. 9

1.13 **Theorem:** Let U and V be two vectors and c be a scalar. Then

$$(cU, V) = (U, cV) = c(U, V) \; .$$

Proof: The two cases $c > 0$ and $c < 0$ should be considered separately. The proof is left to the reader as an exercise.

1.14 **Orthonormal base:** If the elements of a base are mutually perpendicular, and the norm of each element is one, then we call the base *orthonormal*. That is, if $\{U, V, W\}$ is orthonormal, then

$$(U, U) = (V, V) = (W, W) = 1 \; , \quad \text{and}$$

$$(U, V) = (U, W) = (V, W) = 0 \; .$$

We note that this corresponds to the familiar standard choice of a coordinate system in analytic geometry. To each vector A corresponds an ordered triple (x, y, z) which is the set of ordinary coordinates of the point **A**. We call them also components of A. Here we see that

$$x = (A, U), \; y = (A, V), \text{ and } z = (A, W).$$

Therefore

$$A = (A, U)\, U + (A, V)\, V + (A, W)\, W \; .$$

1.15 **Norm of a vector and angle between two vectors in terms of components:** Let A be a vector. Clearly $|A| = (A, A)^{1/2}$. If α is the angle between A and B, then by 1.9,

$$\cos \alpha = \frac{(A, B)}{|A| \cdot |B|} \; .$$

Let $A = x_1 U + y_1 V + z_1 W$ and $B = x_2 U + y_2 V + z_2 W$, where $\{U, V, W\}$ is an orthonormal base in the space. Then

$$|A| = [(x_1 U + y_1 V + z_1 W, \; x_1 U + y_1 V + z_1 W)]^{1/2} = [x_1^2 + y_1^2 + z_1^2]^{1/2} \; .$$

This is proved by distributivity of inner product with respect to vector addition [see 1.12], and the fact that the base $\{U, V, W\}$ is orthonormal. Similarly it can be proved that

(1) $(A,B) = x_1x_2 + y_1y_2 + z_1z_2$, and

(2) $$\cos\alpha = \frac{x_1x_2 + y_1y_2 + z_1z_2}{(x_1^2 + y_1^2 + z_1^2)^{1/2} \cdot (x_2^2 + y_2^2 + z_2^2)^{1/2}}.$$

The reader should supply the proofs of (1) and (2) as an exercise. Note that if the base is not orthonormal what was said in 1.15 will not be true.

Illustration 1: Let (2,3,-1) and (2,-1,2) be components of vectors A and B with respect to an orthonormal base. Find (A,B) and the cosine of the angle between A and B.

$$(A,B) = (2)(2) + (3)(-1) + (-1)(2) = -1 ,$$

$$\cos = -\frac{-1}{\sqrt{4+9+1} \cdot \sqrt{4+1+4}} = \frac{-1}{3\sqrt{14}}.$$

Illustration 2: Find a vector which is perpendicular to the vectors in illustration 1, and has norm $5\sqrt{5}$.

Let the desired vector be $(x, y, z) = C$. Then

$(A,C) = (B,C) = 0,$ $(C,C) = 125.$ Thus

$$\begin{cases} 2x + 3y - z = 0 \\ 2x - y + 2z = 0 \\ x^2 + y^2 + z^2 = 125 . \end{cases}$$

Solving this system, we find the two solutions

(5,-6,-8) and (-5,6,8,) .

Illustration 3: Find the distance between the points $\mathbf{A} = (x_1,y_1,z_1)$ and $\mathbf{B} = (x_2,y_2,z_2)$.
The distance is equal to the norm of the vector $B - A$ [see 1.2].

$$|B - A|^2 = (B - A, B - A).$$

But

$$B - A = (x_2 - x_1, y_2 - y_1, z_2 - z_1).$$

hence by 1.15,

$$|B - A| = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2} .$$

1.16 **Orthonormalization of a base:** We can construct an orthonormal base out of any base $\{U, V, W\}$ as follows:

Take $U_1 = \dfrac{1}{|U|} U$,

$$V_1 = \frac{1}{|V - (V, U_1)U_1|}[V - (V, U_1)U_1], \text{ (Fig. 10), and}$$

$$W_1 = \frac{1}{|W - [(W, U_1)U_1 + (W, V_1)V_1]|}\{W - [(W, U_1)U_1 + (W, V_1)V_1]\}, \text{ (Fig. 11).}$$

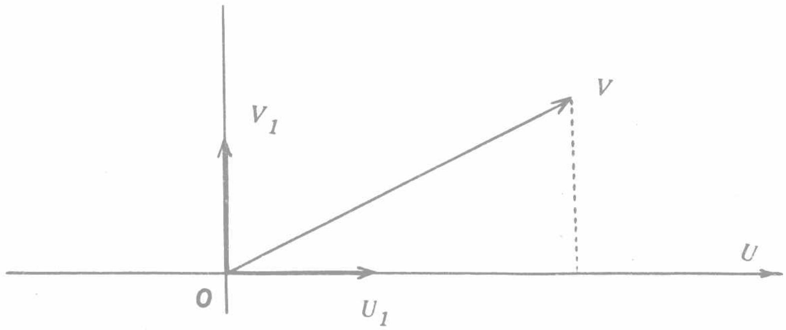

Fig. 10

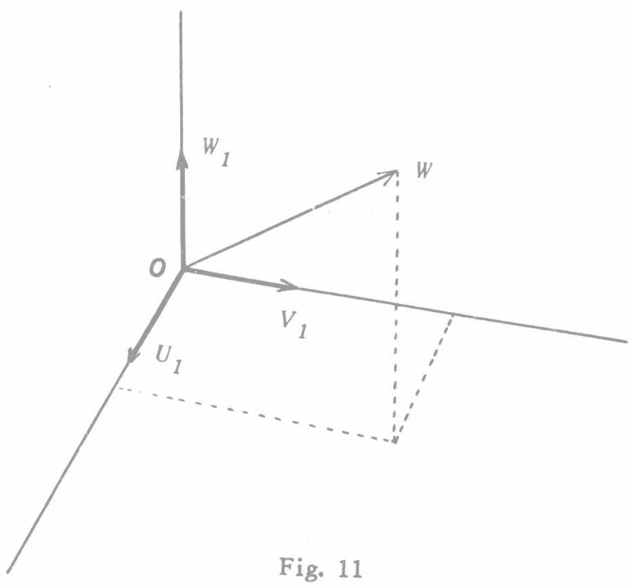

Fig. 11

The reader should show that $|U_1| = |V_1| = |W_1| = 1$, and

$$(U_1,V_1) = (U_1,W_1) = (V_1,W_1) = 0 \ .$$

1.17 **Subspaces:** A line through the origin is called a *one-dimensional linear subspace* of the whole space or of a plane containing it. A plane through the origin is called a *two-dimensional subspace* of the whole space. Such a line contains all scalar multiples of any non-zero vector whose end is on the line. A plane through the origin consists of all linear combination of any two linearly independent vectors in the plane.

1.18 **Straight line:** Given two points **A** and **B** in the space, we consider the vector $V = B - A$, (Fig. 12), and we call the components l, m, and n of V with respect to a given base the *direction numbers* of the straight line through **A** and **B**. The set (l,m,n) is an ordered set, and we call l, m, and n respectively the x, y, and z direction numbers of the line through **A** and **B**. If V is of unit length, and the base is orthonormal, we call the components of V *direction cosines* of the line through **A** and **B**, and we denote them by λ, μ, and ν. Clearly

$$\lambda^2 + \mu^2 + \nu^2 = 1 .$$

For any other point **W** on the line **AB**, $W - A$ is collinear with $B - A$, hence

$$W - A = t(B - A) \qquad [\text{see } 1.2].$$

Thus if $A = (x_1,y_1,z_1)$, $B = (x_2,y_2,z_2)$, and $W = (x,y,z)$, we have, by 1.8,

$$(1) \qquad \begin{cases} x - x_1 = t(x_2 - x_1) \\ y - y_1 = t(y_2 - y_1) \\ z - z_1 = t(z_2 - z_1) \end{cases}$$

as a set of parametric equations of the line **AB**. We can also suppose that

$$l = x_2 - x_1, \quad m = y_2 - y_1, \quad \text{and} \quad n = z_2 - z_1,$$

and we write (1) as

$$(2) \qquad \begin{cases} x = x_1 + tl \\ y = y_1 + tm \\ z = z_1 + tn . \end{cases}$$

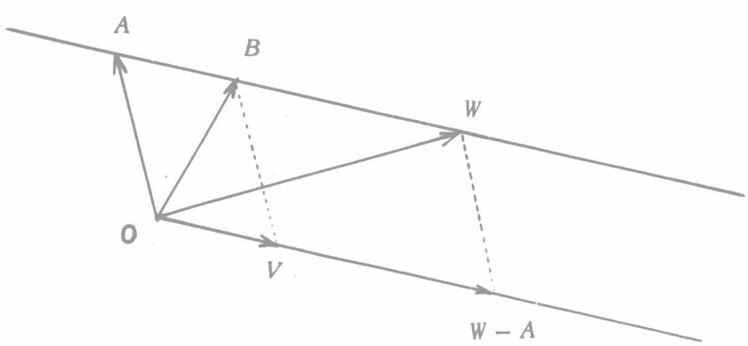

Fig. 12

In vector form, we have

$$W = A + t(B - A) .$$

If l, m, and n are all different from zero we can write (2) as

$$t = \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} .$$

It is clear that the form of the equations of a line is independent of the orthonormality of the base.