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Geometric Analysis of Hyperbolic Differential Equations: An Introduction

S. Alinhac

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Preface

The field of nonlinear hyperbolic equations or systems has seen a tremendous development since the beginning of the 1980s. We are concentrating here on multidimensional situations, and on quasilinear equations or systems, that is, when the coefficients of the principal part depend on the unknown function itself. The pioneering works by F. John, D. Christodoulou, L. Hörmander, S. Klainerman, A. Majda and many others have been devoted mainly to the questions of blowup, lifespan, shocks, global existence, etc. Some overview of the classical results can be found in the books of Majda [42] and Hörmander [24]. On the other hand, Christodoulou and Klainerman [18] proved in around 1990 the stability of Minkowski space, a striking mathematical result about the Cauchy problem for the Einstein equations. After that, many works have dealt with diagonal systems of quasilinear wave equations, since this is what Einstein equations reduce to when written in the so-called harmonic coordinates. The main feature of this particular case is that the (scalar) principal part of the system is a wave operator associated to a unique Lorentzian metric on the underlying space-time. This is in strong contrast with the more complicated case of general symmetric quasilinear systems: the compressible isentropic Euler equations, for instance, can be viewed as a quasilinear wave equation coupled to a vector field; the system of nonlinear elasticity involves two different wave equations, etc.

I consider here only the case of quasilinear wave equations. We observe two main domains of interest: the study of *global* smooth solutions, and the study of *low regularity* solutions, both domains being connected. The striking feature is the *unity* of the techniques and ideas used in the works on these domains: The emphasis is always on good directions and good components, these components being components of tensors relative to some special frames, the *null frames*. Hence the observed unity comes from the fact that most concepts, such as metrics, connexions, curvature, etc., are borrowed from Lorentzian geometry. This is, of course, related to mathematical work by Penrose and collaborators in the domain of general relativity, where null frames have been used extensively (see, for instance, Penrose and Rindler [43]).

Since the work of Christodoulou and Klainerman cited above, many mathematical papers on the subject of quasilinear wave equations or Einstein equations use the language of Lorentzian geometry and deal with energy–momentum tensors, deformation tensors, etc. However, there seem to be some difficulties: Riemannian geometry books do not include the specific Lorentzian tools such as null frames; most relativity books do not include a description of the relevant mathematical techniques. Let us, however, draw attention to the books of Hawking and Ellis [23] and Rendall [44]; which include substantial mathematics.

I believe that the use of Lorentzian tools in the mathematical study of nonlinear hyperbolic systems is going to intensify further, even in the aspects of the field not directly related with general relativity. This is what we call "geometric analysis of hyperbolic equations." It is true that there are examples of nonlinear wave equations which are perturbations of the standard wave equation by small nonlinear terms, where it is enough to consider only the geometry of the standard wave equation, that is, the Minkowski metric: These examples are striking, but the possibility of this simplification seems to be related to the fact that one is considering only *small* solutions; for large solutions, we believe that it will be necessary to take into account the geometry of the linearized operator, that is, a Lorentzian metric depending on the solution itself.

This book is meant for people wanting to access the mathematical literature on the subject of quasilinear wave equations or Einstein equations. Its goal is twofold:

(i) To give to analysts in the field of partial differential equations (PDEs) a self-contained and elementary access to the necessary tools of Lorentzian geometry,
(ii) To explain the fundamental ideas connected with the use of null frames.

This book can be read by students or researchers with an elementary background in distribution theory and linear PDEs, specifically hyperbolic PDEs. No knowledge of differential geometry is required. Though the largest part of the text is about geometric concepts, this book is not a book about Lorentzian geometry: it introduces the geometric tools required to understand the modern PDE literature only as and when they are needed. The author not being a geometer, I deliberately chose to give naive and self-contained proofs to all statements, which can be viewed as “do it yourself” exercises for the reader, without using sophisticated “well-known” facts. I hope that I will be forgiven for that.

Finally, I would like to thank S. Klainerman and F. Labourie for many helpful conversations.

Contents

1
Introduction

The prototype of all hyperbolic equations is the wave equation, or d'Alembertian

$$\Box \equiv \partial_t^2 - \Delta_x,\ \Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$$

in $\mathbf{R}^4_{x,t}$. In order to introduce the concepts and questions of this book, we first review briefly some decay properties of the solutions of $\Box\phi = 0$. We refer the reader to [9] for all formula and proofs.

1. We consider in $\mathbf{R}^4_{x,t}$ the Cauchy problem for the standard wave equation

$$\Box\phi = (\partial_t^2 - \Delta_x)\phi = 0,\ \phi(x,0) = \phi_0(x),\ (\partial_t\phi)(x,0) = \phi_1(x).$$

a. Suppose for simplicity $\phi_0, \phi_1 \in C_0^\infty$, $\phi_i(x) = 0$ for $r = |x| \geq M$: as a consequence of the classical solution formula, the function ϕ can be represented for $r \geq 1$ as

$$\phi(x,t) = \frac{1}{r}F\left(r - t, \omega, \frac{1}{r}\right),\ r = |x|,\ \omega = \frac{x}{r},\ \sigma = r - t,\ z = \frac{1}{r},$$

for some C^∞ function $F(\sigma, \omega, z)$. Since the propagation speed is 1, F vanishes for $r \geq t + M$, that is, $\sigma = r - t \geq M$. By the strong Huygens principle, the solution also vanishes for $r \leq t - M$, that is, $\sigma = r - t \leq -M$. Thus F and ϕ are supported in the strip $|r - t| \leq M$ close to the light cone $\{r = t\}$. Setting $\partial_r = \sum \omega^i \partial_i$, we introduce the two fields

$$L = \partial_t + \partial_r,\ \underline{L} = \partial_t - \partial_r,$$

and define the rotation fields $R = x \wedge \partial$,

$$R_1 = x^2\partial_3 - x^3\partial_2,\ R_2 = x^3\partial_1 - x^1\partial_3,\ R_3 = x^1\partial_2 - x^2\partial_1.$$

Note that $R_i(r) = 0$, and $\sum \omega^i R_i = 0$. Using the representation formula, we observe that

$$L\phi = \frac{-F}{r^2} - \frac{\partial_z F}{r^3} = O(r^{-2}),\ r \to +\infty.$$

Similarly, since $\partial_i \omega^j = (\delta_i^j - \omega^i \omega^j)/r$,

$$\frac{R}{r}\phi = O(r^{-2}),\ r \to +\infty,$$

while, for instance, $\underline{L}\phi$ has only magnitude r^{-1}. Hence the special derivatives $L\phi$, $(R/r)\phi$ behave better at infinity than the other components of $\nabla\phi$. We call them the "good derivatives" of ϕ.

b. We explain now how to obtain the same decay result for the good derivatives using an "energy method," which is an alternative approach to the preceding decay results that does not use an explicit representation for ϕ. We define the hyperbolic rotations $H = t\partial + x\partial_t$,

$$H_1 = t\partial_1 + x^1\partial_t,\ H_2 = t\partial_2 + x^2\partial_t,\ H_3 = t\partial_3 + x^3\partial_t,$$

and call Lorentz fields Z all the fields

$$\partial_\alpha,\ S = t\partial_t + \sum x^i \partial_i = t\partial_t + r\partial_r,\ R = x \wedge \partial,\ H = t\partial + x\partial_t.$$

These fields are known to commute with $\Box$, except for the scaling field S which satisfies $[\Box, S] = 2\Box$. In the situation in **a**, commuting the fields Z with $\Box$ we obtain $\Box Z\phi = 0$; using the standard energy inequality for the wave equation, we obtain the bound

$$\sum ||(\nabla Z\phi)(\cdot, t)||_{L^2_x} \leq C.$$

Now, the following easy formula establishes a connexion between the special derivatives L, R/r and the Lorentz fields:

$$(r+t)L = S + \sum \omega_i H_i,\ (t-r)\underline{L} = S - \sum \omega_i H_i,\ \frac{R}{r} = t^{-1}\omega \wedge H.$$

Note also that for any smooth function supported in $|r - t| \leq M$, we have the Poincaré inequality

$$||w(\cdot, t)||_{L^2} \leq C||(\partial_r w)(\cdot, t)||_{L^2}.$$

Using these formulas, we get for the special derivatives of $L\phi$, $(R/r)\phi$

$$||(\nabla L\phi)(\cdot, t)||_{L^2} = O(t^{-1}),\ \left\| \nabla \frac{R}{r}\phi(\cdot, t) \right\|_{L^2} = O(t^{-1}),\ t \to +\infty.$$

Taking into account the support of ϕ and using again the Poincaré inequality, we even obtain

$$||(L\phi)(\cdot,t)||_{L^2} = O(t^{-1}), \quad \left\|\frac{R}{r}\phi(\cdot,t)\right\|_{L^2} = O(t^{-1}), \; t \to +\infty.$$

Note the contrast with the information given by the standard energy inequality, which yields only the boundedness of these quantities.

It is, in fact, possible to recover the pointwise estimates from **a** using the preceding L^2-estimates. For this, we first commute a product Z^k of k of the Lorentz fields with the wave equation, thus obtaining $\Box Z^k\phi = 0$. Then we use the Klainerman inequality, which is valid for any smooth function v sufficiently decaying at infinity:

$$|v(x,t)|(1+t+r)(1+|t-r|)^{\frac{1}{2}} \le C \sum_{k\le 2} ||Z^k v(\cdot,t)||_{L^2}.$$

We thus obtain again the pointwise bounds that we had from the explicit representation formula

$$L\phi = O(t^{-2}), \quad \frac{R}{r}\phi = O(t^{-2}).$$

Note, however, that this "energy method" is likely to work in variable coefficients situations (or nonlinear situations), where we do not know the representation formula.

If the data are not compactly supported but are sufficiently decaying as $|x| \to +\infty$, this energy method still works, but the "interior" behavior of the solution (that is, away from the light cone $\{t = r\}$) is not as good as before.

c. In **b**, we commuted products of Lorentz fields with $\Box$ and then used the *standard* energy inequality. There is, however, still another type of "energy approach" that displays better behavior of the special derivatives $L\phi$, $(R/r)\phi$. This approach does not involve Lorentz fields, but instead requires a different type of energy inequality. We give two examples of this.

First, one can prove the following improvement of the standard energy inequality: for all $\epsilon > 0$, there is some constant $C_\epsilon > 0$ such that, assuming $\Box\phi = 0$,

$$E_\phi(T)^{\frac{1}{2}} + \left\{ \int_{0\le t\le T} \langle r-t\rangle^{-1-\epsilon} \left[(L\phi)^2 + \left|\frac{R}{r}\phi\right|^2 \right] dxdt \right\}^{\frac{1}{2}} \le C_\epsilon E_\phi(0)^{\frac{1}{2}}.$$

Here, E_ϕ is the standard energy

$$E_\phi(t) = \tfrac{1}{2}\int [(\partial_t\phi)^2 + |\nabla_x\phi|^2](x,t)dx.$$

This inequality is easily obtained in the same way as the usual energy inequality, using the multiplier ∂_t and a weight e^a, where $a = a(r - t)$ is appropriately chosen (see [9], for instance). This inequality is only useful in a region where $|r - t|$ is smaller than t, that is, close to the light cone. In the region $|r - t| \leq C$ for instance, the L_x^2 norm of the special derivatives $L\phi$, $(R/r)\phi$ is not just bounded, it is an L^2 function of t. We can thus identify the "good derivatives" of ϕ directly from the energy inequality, without commuting any fields with the equation.

The second example of an inequality displaying the good derivatives is the conformal energy inequality which gives, for $\Box\phi = 0$,

$$\tilde{E}_\phi(t)^{\frac{1}{2}} \leq C\tilde{E}_\phi(0)^{\frac{1}{2}},$$

where the conformal energy $\tilde{E}$ is

$$\tilde{E}_\phi(t) = \frac{1}{2}\int [(S\phi)^2 + |R\phi|^2 + |H\phi|^2 + \phi^2](x, t)dx.$$

This inequality is obtained in the usual way using the timelike multiplier K_0:

$$K_0 = (r^2 + t^2)\partial_t + 2rt\partial_r.$$

Using the identities $(r + t)L = S + \sum \omega_i H_i$, $R/r = t^{-1}\omega \wedge H$ from **b**, the bound of the quantities $||(Z\phi)(\cdot, t)||_{L^2}$ provided by the inequality yields the bounds

$$||(L\phi)(\cdot, t)||_{L^2} = O(t^{-1}), \quad \left\| \frac{R}{r}\phi(\cdot, t)\right\|_{L^2} = O(t^{-1}).$$

Once again, we can identify the good derivatives of ϕ directly from the conformal energy inequality.

2. Consider now, at each point away from $r = 0$, the null frame

$$e_1,\ e_2,\ e_3 = \underline{L} = \partial_t - \partial_r,\ e_4 = L = \partial_t + \partial_r,$$

where, at each point (x_0, t_0), $(e_1,\ e_2)$ form an orthonormal basis of the tangent space to the sphere

$$\{(x, t),\ t = t_0,\ |x| = |x_0|\}.$$

Using spherical coordinates

$$x^1 = r\sin\theta\cos\phi,\ x^2 = r\sin\theta\sin\phi,\ x^3 = r\cos\theta,$$

we can take (away from the poles)

$$e_1 = r^{-1}\partial_\theta,\ e_2 = (r\sin\theta)^{-1}\partial_\phi.$$

The fields e_1, e_2 are related to the rotation fields by the formulas

$$e_1 = -(\sin\phi)\frac{R_1}{r} + (\cos\phi)\frac{R_2}{r}, \; e_2 = (\sin\theta)^{-1}\frac{R_3}{r}.$$

Hence the "special derivatives" of ϕ on which we insisted above are just, equivalently, the components of $d\phi$ on e_1, e_2, and L, that is, some of the components of $d\phi$ in a null frame, the only bad derivative being $\underline{L}\phi$.

To understand the name "null frame," it is best to introduce on $\mathbf{R}^4$ the scalar product of special relativity. For two vectors $X = (X^0, X^1, X^2, X^3)$ and $Y = (Y^0, Y^1, Y^2, Y^3)$, we set

$$\langle X, Y \rangle = -X^0Y^0 + \sum_{1 \le i \le 3} X^iY^i.$$

We can then easily check the fundamental properties which define a null frame:

$$(e_1, e_2) \perp (e_3, e_4), \; \langle L, L \rangle = 0, \; \langle \underline{L}, \underline{L} \rangle = 0, \langle L, \underline{L} \rangle = -2.$$

The "gradient" $\tilde{\nabla} f$ of a function f in the sense of this scalar product is defined by

$$\forall Y, \; \langle \tilde{\nabla} f, Y \rangle = df(Y) = Y(f).$$

This gives immediately

$$\tilde{\nabla} f = (-\partial_t f, \partial_1 f, \partial_2 f, \partial_3 f).$$

For instance,

$$\tilde{\nabla}(t - r) = -\left(1, \frac{x}{r}\right) = -L.$$

Since L is "null," we also have, with $u = t - r$,

$$\langle \tilde{\nabla} u, \tilde{\nabla} u \rangle = 0,$$

and we say that u is an optical function. Note that the null frame $(e_1, e_2, \underline{L}, L)$ is associated to the functions u and t in the sense that:

(i) the surfaces $\{t = t_0, \; u = u_0\}$ are the usual spheres,
(ii) $L = -\tilde{\nabla} u$ and $(\underline{L}, L)$ are the two null vectors in the orthogonal space to these spheres.

This shows us how null frames and optical functions are related. Of course, the function $\underline{u} = t + r$ is also an optical function, and $\underline{L} = -\tilde{\nabla}\underline{u}$. Note that the level surfaces of u are outgoing light cones, while level surfaces of $\underline{u}$ are incoming light cones; also, the good derivatives (e_1, e_2, L) span, at each point, the tangent space to the outgoing cone through this point.

Let us mention to finish the relations between the fields S, K_0 (that we have already encountered) and u, L, $\underline{u}$, $\underline{L}$,

$$S = \frac{1}{2}(u\underline{L} + \underline{u}L),\ K_0 = \frac{1}{2}(u^2\underline{L} + \underline{u}^2 L).$$

3. The aim of this book is to explain how one can extend the previously discussed concepts and results to a general framework. More precisely, suppose we have, instead of the "flat" Minkowski metric $|X|^2 = \langle X, X\rangle$, a more general metric g:

$$g = \sum g_{\alpha\beta} dx^\alpha dx^\beta,\ g(X, Y) \equiv \langle X, Y\rangle = \sum g_{\alpha\beta} X^\alpha Y^\beta.$$

We assume, of course, that this metric has the signature $-, +, +, +$ just like the Minkowski metric. We define the wave equation $\Box$ associated with this metric by

$$\Box_g \phi \equiv \Box\phi = |g|^{-\frac{1}{2}} \sum \partial_\alpha (g^{\alpha\beta} |g|^{\frac{1}{2}} \partial_\beta \phi),$$

where $|g|$ is the determinant of the matrix $(g_{\alpha\beta})$ and $(g^{\alpha\beta})$ its inverse matrix. However, we sometimes write $\Box = \partial_t^2 - \Delta$ for the standard wave operator (instead of $-\partial_t^2 + \Delta$). Our interest centres on these wave equations, and also on the associated Maxwell and Bianchi equations. From the considerations above for the "flat" case of the Minkowski metric, the following natural questions arise: for solutions ϕ of $\Box_g \phi = 0$,

(i) Are there "good derivatives" of ϕ (in the sense of a better decay at infinity), analogous to $L\phi$, $(R/r)\phi$?
(ii) How should a null frame that captures these "good derivatives" be chosen?
(iii) What is the relation between null frames and optical functions?
(iv) Can one prove energy inequalities where the good derivatives are singled out, as in **1.c**?
(v) Are there good substitute for the Lorentz fields Z?
(vi) Can one commute these substitutes with $\Box$ to obtain pointwise bounds for the solutions, as in **1.b**?

The plan of the book follows from what we said before, about introducing the necessary geometric machinery only as and when it is needed.

- In chapter 2, we discuss the notions of metric, optical functions, and null frames, and give simple examples found in the literature.
- The differential geometry aspects appear in chapter 3, where the metric connexion is introduced, as a necessary tool to deal with frames; we then define the frame coefficients and compute them for simple examples.

- Chapter 4 is dedicated to the specific machinery used to prove energy inequalities: the energy–momentum tensor, the deformation tensor, etc. The idea is to do the computations in such a way that the energy and the additional "interior terms" can be easily expressed in the frame in which we are working.
- The question of how to chose a good frame and thus identify the good components of tensors is adressed in chapter 5, where we discuss extensions of the standard energy inequality and of the conformal energy inequality.
- The way to find substitutes for the standard Lorentz fields and to commute them with $\Box$ is explained in chapter 6.
- The curvature tensor is introduced only in chapter 7, where we explain how to control optical functions and their associated null frames. We establish there the transport equations and elliptic systems (on (nonstandard) 2-spheres) which govern the frame coefficients.
- Finally, the last two chapters are devoted to discussing a number of applications of the ideas presented in the previous chapters to nonlinear problems. Though it seems impossible to give complete proofs of very difficult results, we try to outline the constructions of frames, the inequalities used, etc., in the hope of providing a guide for further reading.