NONLINEAR WAVES

Edited by

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PREFACE

The last two decades have produced major advances in the mathematical theory of nonlinear wave phenomena and their applications. In an effort to acquaint researchers in applied mathematics, physics, and engineering and to stimulate further research, an NSF-CBMS regional research conference on Nonlinear Waves and Integrable Systems was convened at East Carolina University in June, 1982. Many distinguished applied mathematicians and scientists from all over the world participated in the conference, and provided a digest of recent developments, open questions, and unsolved problems in this rapidly growing and important field.

As a follow-up project, this book has developed from manuscripts submitted by renowned applied mathematicians and scientists who have made important contributions to the subject of nonlinear waves. This publication brings together current developments in the theory and applications of nonlinear waves and solitons that are likely to determine fruitful directions for future advanced study and research.

The book has been divided into three parts. Part I, entitled Nonlinear Waves in Fluids, consists of seven chapters. Nonlinear Waves in Plasmas are the contents of Part II, which has five chapters. Part III contains six chapters on current results and extensions of the inverse scattering transform and of evolution equations. Included also is recent progress on statistical mechanics of the sine-Gordon field.

The opening chapter, by M.S. Longuet-Higgins, is devoted to recent progress in the analytical representation of overturning waves. Among the forms suggested for the fluid flow are, for the tip of the jet, a rotating Dirichlet hyperbola, and, for the tube, a "$\sqrt{3}$-ellipse" or a parametric cubic. All these have been expressed in a semi-Lagrangian form. The semi-Lagrangian form for the rotating hyperbola is derived by a new and simpler method, and certain integral invariants are obtained which have the dimensions of mass, angular momentum and energy. The relation of these to the previously known constants of integration is discussed, and directions for further generalizations are indicated. Also, a new class of polynomial solutions of the semi-Lagrangian boundary conditions is derived. These, or their generalizations, may be of use when combining different solutions so as to form a complete description of the overturning wave. In Chapter 2, R.S. Johnson describes how the classical problem of inviscid water waves is used as the vehicle for introducing various forms of the Korteweg-deVries (KdV) and nonlinear Schrödinger (NLS) equations. The appropriate equations in one and two dimensions are given with some discussion on the effect of shear and variable depth. It is shown that KdV and NLS equations match in a suitable limit of parameter space, and the various KdV solutions-notably similarity-are themselves matched to corresponding near fields. Some other equations based on more complicated physics are mentioned together with a brief comment on two-dimensional problems with shear or variable depth. In Chapter 3, R. Grimshaw discusses canonical equations for the evolution of long nonlinear solitary waves in slowly varying environments. These equations are of KdV type and include the effects of dissipation. The slowly varying solitary wave is constructed as an asymptotic solution of these equations by a multiscale perturbation expansion, and is shown to consist of a solitary wave with slowly varying amplitude and trailing shelf. The specific case of a solitary wave decaying due to dissipation is described in detail. Chapter 4, by M.C. Shen, is concerned with some approximate equations for the study of nonlinear water waves in a channel of variable cross section. He gives a system of shallow water equations for finite amplitude waves, and a KdV equation with variable coefficients for small amplitude waves. Some problems deserving more study are mentioned in this chapter. Chapter 5, by I.M. Moroz and J. Brindley, is concerned with the derivation of a system of evolution equations for slowly varying amplitude of a baroclinic wave packet. The self-induced transparency, sine-Gordon and nonlinear Schrödinger equations, all of which possess soliton solutions, each arise for different inviscid limits. The presence of viscosity, however, alters the form of the evolution equations and changes the character of the solutions from highly predictable soliton solutions to unpredictable chaotic solutions. When viscosity is weak, equations related to the Lorenz attractor equations obtain, while for strong viscosity the Ginzburg-Landau equation obtains. P.J. Bryant, in Chapter 6, discusses specific wave geometries which occur in deep water and are calculated by a numerical method based on Fourier transforms. Examples are presented of permanent waves and wave groups of permanent envelope in two and three dimensions without restriction on wave height. Although the method is applied here only to gravity waves in deep water, it may be generalized to further forms of nonlinear wave motion. Chapter 7, by Alex Craik, deals with linear, or direct, resonance of two waves, and weakly nonlinear three-wave resonance. Special attention is given to non-conservative three-wave systems, for which the mathematical theory is least developed. In addition, subharmonic resonance and further complications involving quadratic interaction of more than three waves are discussed.

PREFACE

In Chapter 8, N.E. Andreev, V.P. Silin, and P.V. Silin discuss various aspects of the stationary theory of the interaction of an electromagnetic field with moving plasmas, with special attention to the field self-restriction phenomena in supersonic plasma. The authors also suggest a direction for further research and study on the theory. In Chapter 9, P.C. Clemmow discusses finite-amplitude plane waves travelling with uniform speed through a cold homogeneous plasma in a Lorentz frame of reference. This problem can be reduced to solving a single nonlinear ordinary vector differential equation. Periodic solutions of this equation are investigated. It is found that some new results for propagation in a direction perpendicular to the ambient magnetostatic field go some way towards elucidating the conditions under which various types of wave can exist. H. Okuda presents the results of analytic theory as well as of numerical simulations on electrostatic ion cyclotron (EIC) waves in Chapter 10. In Chapter 11, P.K. Shukla presents an evaluative review on theories of solitons in plasma physics along with a discussion on some open questions and unsolved problems. Chapter 12, by R.J. Gribben, is concerned with uniformly-valid perturbations of uniform, monochromatic nonlinear, periodic wave solutions of the Vlasov and Poisson equations in one space dimension in the absence of a magnetic field. Also, a theory for the propagation of slowly varying nonlinear waves in a non-uniform plasma is presented. Appropriate basic uniform wave solutions are reviewed, some general consequences of the theory given, and current work described, including solutions obtained for particular cases, and directions in which further study might proceed.

In Chapter 13, A.S. Fokas describes some recent results and developments on the extension of the inverse scattering transform to solve nonlinear evolution equations in one time and two space dimensions. Based on the Schrödinger partial differential operator as a simple mathematical model, A. Degasperis studies linear evolution equations associated with isospectral evolutions of differential operators in Chapter 14. He also discusses how to solve the corresponding initial value problem using the spectral properties of the Schrödinger operator. Then the scattering operator expression is divided in the case of a linear evolution equation associated with a pure many-soliton solution. Some natural extensions and generalizations of these results are pointed out. In Chapter 15, Peter Schuur develops an inverse scattering formalism for the NxN matrix Schrödinger equation with arbitrary, in general non-Hermitian potential matrix, decaying sufficiently rapidly for $|x| \to \infty$. A general nth order spectral transform and a technique for inverting this transform are developed by P.J. Caudrey in Chapter 16. The use of the whole procedure is illustrated by the solution of a system of nonlinear Klein-Gordon equations. In Chapter 17, A. Thyagaraja gives an elaborate account of recurrence phenomena and the number of effective degrees of freedom in nonlinear wave motion. The relationships between recurrence phenomena and different motions of stability due to Lagrange, Poisson, and Lyapunov are described. The chapter concludes with a brief discussion of some unsolved problems relevant to applications. The final chapter, by R.K. Bullough, D.J. Pilling, and J. Timonen is devoted to the statistical mechanics of the sine-Gordon (s-G) field. Functional integrals for the classical and quantum partition functions Z for the s-G field $\phi(x,t)$ are calculated in different ways including methods which exploit the complete integrability of the classical s-G and its canonical transformation to a Hamiltonian involving action variables alone. The free Klein-Gordon field poses no problems. But discrepant results for the s-G kinks and anti-kinks are explained by the observation that the functional integrals for Z are defined best by discretization to a lattice of spacing a on finite support L. The s-G problem then becomes that of a sequence of problems involving a finite number of degrees of freedom; and for $L \to \infty$ and $a \to 0$ kinks and anti-kinks are dressed by coupled K-G modes. These dressings are calculated in different ways both quantally and in classical limit, and connections established with kinks and anti-kinks are largely resolved, but quantum WKB methods, for example, pose problems of their own.

I am grateful to the authors for their cooperation and contributions, and hope that this monograph brings together all of the most important, recent developments in the mathematical theory and physical applications of nonlinear waves and solitons in fluids and plasmas, besides describing all major current research on the inverse scattering transform. I want the reader to share in the excitement of present day research in this rapidly growing subject and to become stimulated to explore nonlinear phenomena.

I express my grateful thanks to Dr. Carroll A. Webber for his help in improving the readability of several papers. I am thankful to my wife for her constant encouragement during the preparation of the book. In conclusion, I wish to express my sincere thanks to the Cambridge University Press for publishing the monograph.

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PREFACE

PART I NONLINEAR WAVES IN FLUIDS

PART II NONLINEAR WAVES IN PLASMAS

PART III SOLITONS, INVERSE SCATTERING TRANSFORM, AND NONLINEAR WAVES IN PHYSICS

CHAPTER 1

TOWARDS THE ANALYTIC DESCRIPTION OF OVERTURNING WAVES

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1. INTRODUCTION.

Till recently, one notable hiatus in the theory of surface waves was the absence of any satisfactory analysis to describe an overturning wave. In this category we include both the well-known "plunging breaker" and also any standing or partially reflected wave which produces a symmetric or an asymmetric jet, with particle velocities sometimes much exceeding the linear phase-speed.

A first attempt to describe the jet from a two-dimensional standing wave was made by Longuet-Higgins (1972), who introduced the "Dirichlet hyperbola", a flow in which any cross-section of the free surface takes the form of a hyperbola with varying angle between the asymptotes. Numerical experiments by McIver and Peregrine (1981) have shown this solution to fit their calculations quite well. The solution was further analysed in a second paper (Longuet-Higgins, 1976) where a limiting form, the "Dirichlet parabola", was shown to be a member of a wider class of self-similar flows in two and three dimensions. Using a formalism introduced by John (1953) for irrotational flows in two dimensions, the author also showed the Dirichlet parabola to be one of a more general class of self-similar flows having a time-dependent free surface.

All the above flows were gravity-free, that is to say they did not involve g explicitly; they are essentially descriptions of a rapidly evolving flow seen in a frame of reference which itself is in free-fall.

A useful advance came with the development of a numerical time-stepping technique for unsteady gravity waves by Longuet-Higgins and Cokelet (1976, 1978). As later refined and modified by Vinje and Brevig (1981), McIver and Peregrine (1981) and others, this has given accurate and reproducible results for overturning waves, with which analytic expressions can be compared.

A further advance on the analytic front came with the introduction

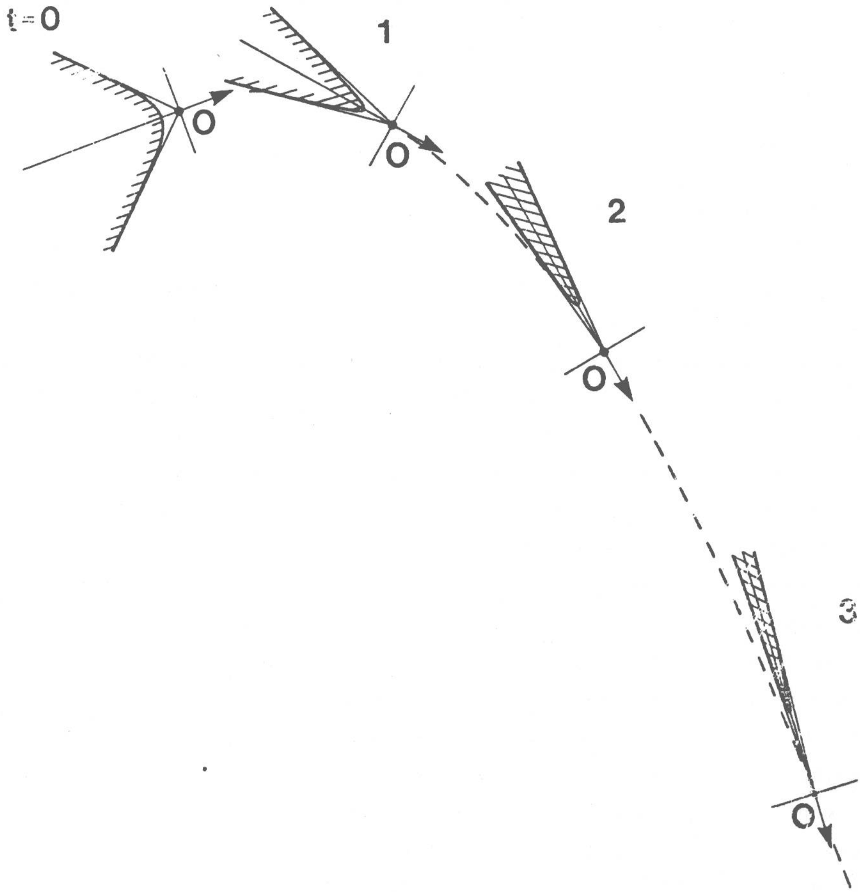

Figure 1 (after Longuet-Higgins 1980b). Example of the free surface in a rotating hyperbolic flow when ϖ = 0.30 (see equation (6.15)). The origin 0 is in a free-fall trajectory.

by Longuet-Higgins (1980a) of a general technique for describing free-surface flows, that is flows satisfying the two boundary conditions

$$p = 0 \quad \text{and} \quad Dp/Dt = 0 \tag{1.1}$$

at a free surface. Particular attention was paid to the parametric representation of the flow in a form

$$\chi = X(\varpi,t) \quad , \qquad z = Z(\varpi,t) \tag{1.2}$$

where both the complex coordinate $z = x + iy$ and the velocity potential χ are expressed as functions of the intermediate complex variable ω and the time t. This was a generalisation of the formalism of F. John (1953), in which ω was, however, assumed to be Lagrangian at the free surface, though not elsewhere in the fluid. For this reason John's formalism was called "semi-Lagrangian".

The more general formalism was put to immediate use in a second paper (Longuet-Higgins 1980b) in which the "Dirichlet hyperbola" of previous papers was generalised to include "rotating hyperbolic flow". Besides the time-variation of the asymptotes, the principal axes were allowed to rotate, as shown in Figure 1. This solution, in which the velocity potential χ was given in closed form, allowed for the first time a convincing possible description of the later stages of a plunging jet. The initial evolution of the jet, however, was not included. In a third paper (Longuet-Higgins 1981a) the author made use of the more general (non-Johnian) formalism to derive a plausible analytic description of the development of the flow, up to about the instant when the free surface first becomes vertical. This description introduced the approximate potential

$$\chi = \frac{2}{3} ig\, \varpi^3 + U\varpi^2 + 2A\varpi \tag{1.3}$$

where U is a constant, A is a linear function of the time and

$$z = \omega^2 \quad . \tag{1.4}$$

The first term on the right of (1.3) by itself represents Stokes's 120° corner-flow. The third term represents a finite-amplitude perturbation of the Stokes flow. The expression (1.3) gives a rather convincing representation of the initial development of the breaking wave (see Figure 2). However, the task of matching this flow to the later stages, including the time-dependent jet, remains still to be accomplished.

In another direction New (1981) found empirically that in some of his

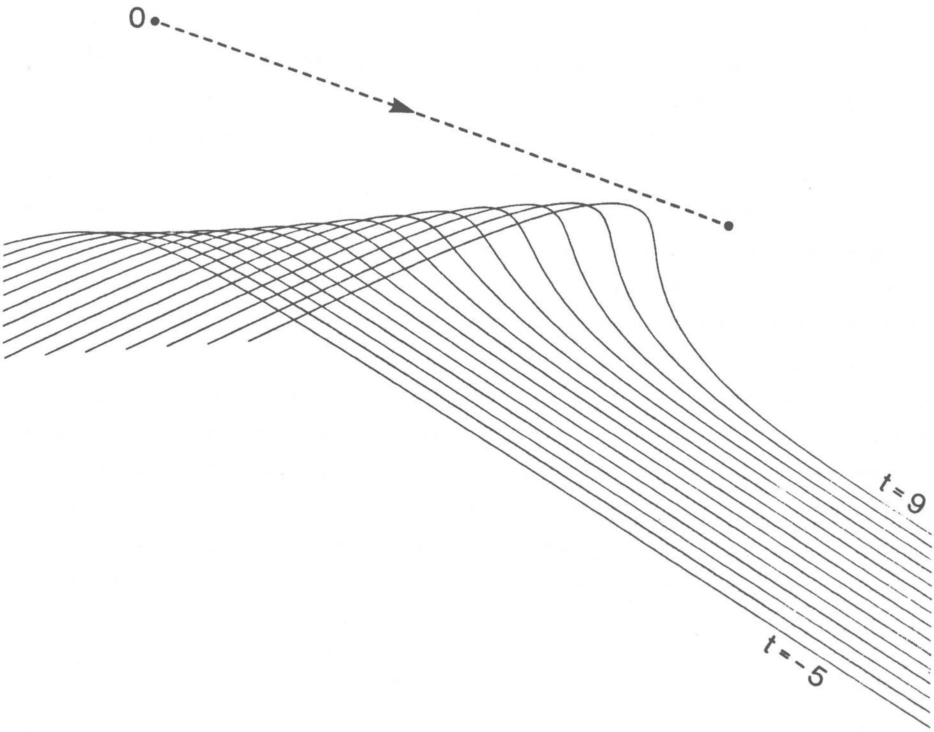

Figure 2 (from Longuet-Higgins 1981). Initial development of the overturning flow as given by equations (1.3) and (1.4) when g = 1, U = (-1,0.5), and A(t) is chosen so as to minimise $\int (Dp/Dt)^2 ds$ on p = 0. The origin O is in uniform motion; the solution includes gravity.

numerical calculations of breaking waves the forward face, or "tube", of the breaker was remarkably well fitted by part of the circumference of an ellipse, with axes in the ratio $\sqrt{3}:1$. Whereupon Longuet-Higgins (1981b, 1982) pointed out that the free surface was even better fitted (see Figure 3) by the cubic curve:

$$z = it\omega^3 + 3t^2\omega^2 - it^3\omega - \frac{1}{3}t^4 \tag{1.5}$$

which is a limiting case of one of the self-similar flows found previously (Longuet-Higgins 1976). Moreover the flow (1.5) contains another surface $p = 0$ which comes close to the rear surface of the wave, though the second boundary condition $Dp/Dt = 0$ is not satisfied on it. Nevertheless there was perhaps some possibility that by suitably perturbing the flow (1.5) and by matching it to a rotating hyperbolic flow near the tip of the jet a complete solution might be found. Since (1.5) is expressed in semi-Lagrangian form a next step would be to express the rotating hyperbolic flow in semi-Lagrangian form also.

This has in fact been done in a very recent paper (Longuet-Higgins 1983) where the rotating hyperbolic flow is shown to be expressible in the form

$$z = F(t)\cosh\omega + G(t)\sinh\omega \tag{1.6}$$

the functions F and G being related to the solutions of a Riccati equation. The corresponding particle trajectories have also been computed.

Meanwhile in still unpublished work New (1983) has succeeded in finding a flow, in semi-Lagrangian representation, which is outside his elliptical free surface, and he has shown that the velocity field resembles that in numerically calculated waves, over about half the circumference of the ellipse. Unlike the cubic (1.5), the elliptical model cannot of course describe the velocity discontinuity which must occur when the jet meets the forward face of the wave. Greenhow (1983) has made further progress in deriving a semi-Lagrangian expression, polynomial in ω, which for large values of t approximates the hyperbolic jet on the one hand and New's ellipse on the other. His expression also provides a "rear face" to the wave, but is still gravity-free.

The purpose of the present paper is twofold: first, to derive the semi-Lagrangian representation for the rotating hyperbolic flow in an alternative, and perhaps simpler, way than in Longuet-Higgins (1983). The present method has the advantage that it brings to light naturally some

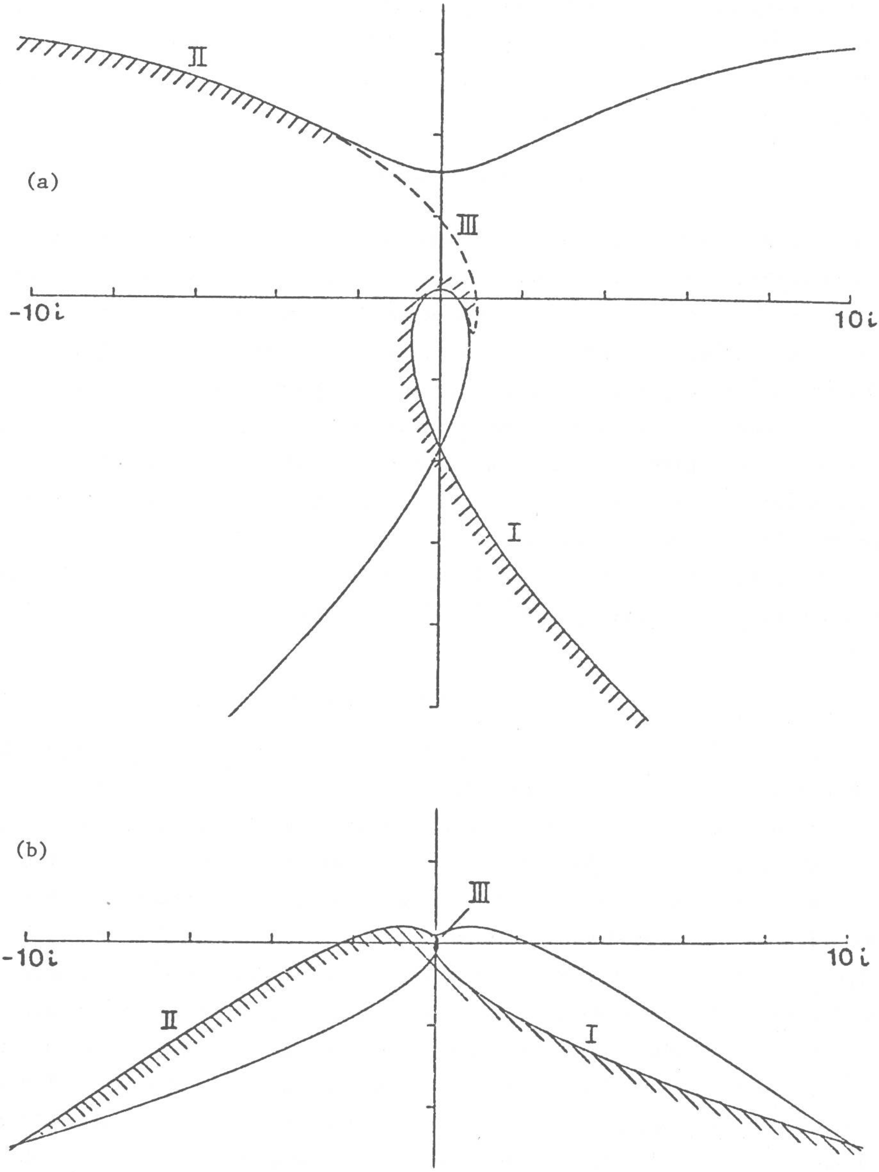

Figure 3 (after Longuet-Higgins 1982). Profile of the surfaces p = 0 in the cubic flow (1.4)(a) when t = 1.0; (b) when t = 0.5. On the curves I, both p and Dp/Dt vanish. On II only p vanishes. The broken curve III indicates a possible perturbation.

integral invariants κ, μ, and ν which in turn provide constraints on the functions F and G. It is shown how κ, μ, and ν are related to the constants of integration found in earlier papers. Moreover the method suggests some possible generalizations.

A second purpose is to give some exact polynomial solutions to one of the problems investigated by Greenhow (1983). The same methods may, in turn, be employed in other, more general, problems occuring in the same context.

2. SEMI-LAGRANGIAN COORDINATES.

In the semi-Lagrangian representation of irrotational, free-surface flows in two dimensions, the coordinate $z = x + iy$ is expressed as an analytic function of a complex parameter ω and the time t:

$$z = z(\omega,t) \tag{2.1}$$

such that on the free surface ω is real ($\omega = \omega^*$) and Lagrangian ($D\omega/Dt = 0$). The condition that the pressure be constant along this surface can then be expressed as

$$z_{tt} - g = irz_\omega \tag{2.2}$$

where g denotes gravity (the x-axis being vertically downwards) and $r(\omega,t)$ is some function that must be real when ω is real. If gravity is negligible, or if the motion is viewed in a free-fall reference frame, then (2.2) reduces to

$$z_{tt} = irz_\omega \tag{2.3}$$

In the interior of the fluid, the coordinate ω is generally not Lagrangian, and the velocity is given by $z_t(\omega^*,t)$, which of course equals $z_t(\omega,t)$ on the boundary. The vanishing of the derivative z_ω implies a singularity in the flow field, unless at the same point $[z_{\omega t}(\omega^*,t)]^*$ vanishes also, hence $z^*_{\omega t}(\omega,t) = 0$. In other words

$$z_\omega = 0 \quad \text{implies} \quad z^*_{\omega t} = 0 \tag{2.4}$$

everywhere in the interior.

When equations (2.3) and (2.4) are satisfied we can, if necessary, find a velocity potential $\chi(\omega,t)$ throughout the fluid by calculating

$$\chi(\omega,t) = \int z^*_t(\omega,t)\, z_\omega(\omega,t)\,d\omega \tag{2.5}$$

for then

$$\chi_z = \chi_\omega / z_\omega = z_t^* \tag{2.6}$$

as required.

3. ROTATING HYPERBOLIC FLOW.

As a very simple form of solution suppose that

$$z = a\omega - b\omega^{-1} \tag{3.1}$$

where a(t) and b(t) are some functions of the time, to be determined. This will satisfy equation (2.3) with $r = \omega R(t)$ provided

$$a_{tt}\omega - b_{tt}\omega^{-1} = iR(a\omega + b\omega^{-1}) \ . \tag{3.2}$$

and R is real. Also from equation (2.4) the two equations

$$\left.\begin{aligned} a + b\omega^{-2} &= 0 \\ a_t^* + b_t^*\omega^{-2} &= 0 \end{aligned}\right\} \tag{3.3}$$

are to be satisified simultaneously, if the corresponding point $z(\omega,t)$ is to lie within the fluid.

Starting from equations (3.2) and (3.3) we shall deduce a chain of results leading eventually to a differential equation for the unknown function R(t).

On equating coefficients of ω and ω^{-1} in equation (3.2) we have

$$\left.\begin{aligned} a_{tt} &= iRa \\ b_{tt} &= -iRb \end{aligned}\right\} \tag{3.4}$$

where R is not necessarily real. On eliminating R from these two equations we get

$$ab_{tt} + a_{tt}b = 0 \ . \tag{3.5}$$

Again, on eliminating ω from equations (3.3) we have

$$ab_t^* - a_t^* b = 0 \ . \tag{3.6}$$

From (3.6) and its complex conjugate there follows

$$(ab^* - a^*b)_t = 0 \ , \tag{3.7}$$

hence

$$ab^* - a^*b = \text{constant} = i\kappa \tag{3.8}$$