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Dynamics of Internal Layers and Diffusive Interfaces

PAUL C. FIFE
University of Utah

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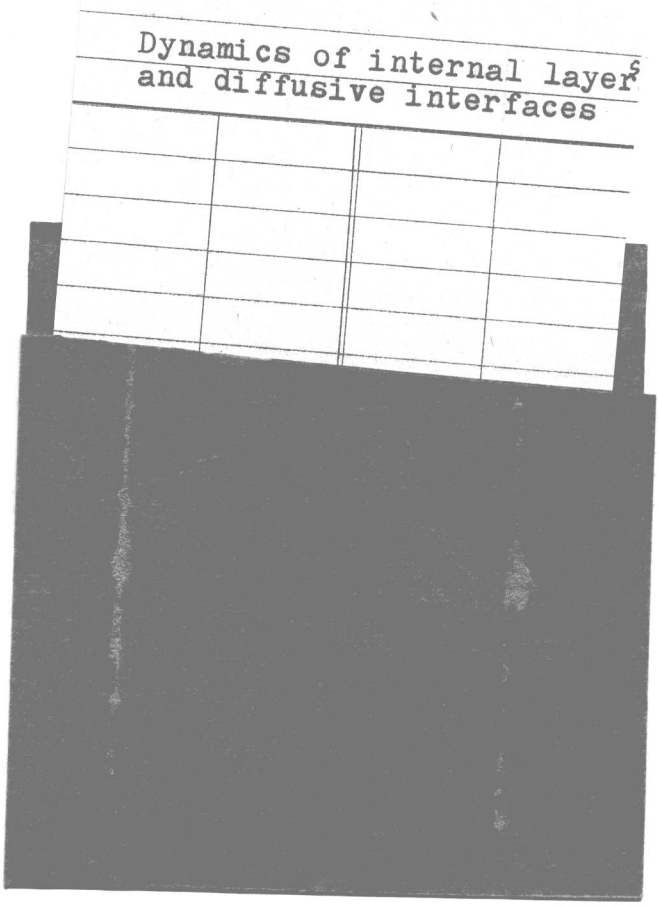

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Preface

Interfacial phenomena are commonplace in physics, chemistry, biology, and in various disciplines bridging these fields. They occur whenever a continuum is present that can exist in at least two different chemical or physical "states," and there is some mechanism that generates or enforces a spatial separation between these states. The separation boundary is then called an interface. In the examples studied here, the separation boundary and its internal structure result from the balance between two opposing tendencies: a diffusive effect that attempts to mix and smooth the properties of the material and a physical or chemical mechanism that works to drive it to one or the other pure state.

This latter is an "unmixing" tendency. In our cases, it is one of the following: (1) a chemical kinetic mechanism with two stable steady states or two attracting slow manifolds in concentration space; (2) a double-well potential that drives a substance into one of two possible phases, such as solid or liquid; (3) an imposed electric field that affects different kinds of ions in different ways; (4) a chemical reaction rate that is so sensitive to temperature that a temperature isocline can serve as an interface separating (a) the region in space where the reaction (and other reactions it triggers) has gone to completion, from (b) the other region, where the reaction is so slow due to the lower temperature that it can be neglected; or finally (5) a very complex biophysical process responsible for the triggering of physical change in biological tissue, followed by its recovery to its original state.

This fifth mechanism, together with the diffusive-type process alluded to above, is responsible for the propagation of signals along a nerve axon or cardiac tissue, and is quite commonly modeled by systems of the type studied in Chap. 4. The Belousov–Zhabotinskii and other excitable chemical reagents subject to mechanism (1) above are appropriately modeled the same way, and are also treated in Chap. 4.

The separation of ions due to the electric field mentioned in (3) is explained in Chap. 3. The thermal propagation of flames, caused by mechanism (4) plus the heat release of the reaction and the diffusion of heat and material species,

is the subject of Chap. 2. A phase change model with far-reaching consequences is considered in Chap. 1, §3, and in Chap. 1, §2, we explain an equation that has been used to study waves in population genetics, physiology, and nonlinear transmission lines, and is an essential ingredient for the analysis in Chap. 4. Separations due to a diffusive shock layer are briefly covered in Chap. 1, §4.

In view of all of this, it must be emphasized that this work is primarily mathematical. One of the reasons, in fact, for its existence is that a certain body of techniques and concepts forms the basis of the mathematical study of most of these types of interfaces. This common ground consists of the asymptotic study of internal layers, and is introduced in Chap. 1, §1.

There are important differences, however, among the applications treated here. In fact, the bulk of the monograph by far is devoted to problems and phenomena that are particular to the various contexts.

In all cases, the most interesting phenomena have to do with the motion of the interfaces. In the case of flame interfaces (called layers), the tendency for irregular motion leads to well-known stability considerations, as well as to an excellent example of the power of multiple scaling techniques for determining their motion near the onset of instability. The latter topic is only briefly mentioned in Chap. 2, §2, but the difficult linear stability analysis is covered in great detail there.

In the case of the chemical and biophysical interfaces studied in Chap. 4, their motion leads to such fascinating spatiotemporal structures as rotating spirals and expanding rings. Scroll rings, the three-dimensional analogues of spirals, are not covered, but conceptually are subject to the same considerations.

The basis of this monograph was a set of notes prepared for three minicourses which I gave in 1987: a CBMS Conference on Nonlinear Waves in Little Cottonwood Canyon, Utah, in May; a series of lectures at Peking University and the Institute of Mathematics, Beijing, also in May; and a Rocky Mountain Mathematical Consortium summer course in Laramie, Wyoming, in July. I am very grateful to the organizers of these courses: Peter Bates at Brigham Young University, Ye Qi-xiao and Hsiao Ling in Beijing, and Duane Porter at the University of Wyoming. The original notes were written while I was a Visiting Professor at Brigham Young University. I am also grateful to the many people who read parts of the notes and made corrections. Finally, thanks go to the National Science Foundation, which makes CBMS conferences such as the one in Utah possible.

Most of this material exclusive of Chap. 1, §1, Chap. 2, §§1,2, and Chap. 4, §4C, represents research supported by National Science Foundation grants DMS-8202056 and DMS-8703247, and Air Force Office of Scientific Research grant F4962086C0130.

Contents

CHAPTER 1

Internal Layers

1. Dynamics of internal layers: Asymptotics and matching.

Since most of the nonlinear waves considered in these lectures are interfaces in the form of internal layers, it is appropriate that we begin by discussing the latter in general terms. Asymptotic methods somewhat related to the ones given here appear in a number of books, such as [Ec].

Imagine some smooth state variable u, a function of space and time, taking values in R^m, and evolving according to some system of differential equations. For simplicity we will assume space to be two-dimensional. This dynamical process also depends, we suppose, on a small positive parameter ϵ, so u does as well: $u = u(x, t; \epsilon)$, $x = (x_1, x_2)$.

We will study several situations in which the dynamical process generates and preserves a moving internal layer, which we will call an interface, of width $O(\epsilon)$. The generation process is interesting and in most cases has been studied little; we overlook that part here and focus on the dynamics of a fully developed layer. This layer is located on a curve $\Gamma(t; \epsilon)$ in the plane (its exact relation to Γ will be made precise in each case). It will be convenient to track the movement of Γ by considering the evolution of the function $r(x, t; \epsilon)$ defined to be $\pm$ the distance from x to Γ. Thus

$$\Gamma(t; \epsilon) = \{r(x, t; \epsilon) = 0\}. \tag{1}$$

We suppose that Γ divides the plane into two parts: $\mathcal{D}_+$, where r is taken to be positive, and $\mathcal{D}_-$, where r is negative. Further assumptions are as follows.

The state function may be approximated by truncations of a formal power series

$$u(x, t) \simeq \Sigma \epsilon^n u_n(x, t) \tag{2}$$

in regions away from the interface, and of another formal power series near Γ. The higher the truncation, of course, the better the approximation. To be more specific, let $\Gamma_\delta(t; \epsilon) = \{|r| \leq \delta\}$ and $\mathcal{D}_\delta(t; \epsilon)$ be the complement of Γ_δ in the plane. Let $u^{(N)}$ denote the summation on the right of (2) up to terms of

order ϵ^N. Then for some $\delta(N, \epsilon)$ to be specified below, with $\delta \to 0$ as $\epsilon \to 0$, we assume

$$|u - u^{(N)}| = O(\epsilon^{N+1}) \quad \text{as } \epsilon \to 0$$

uniformly for $(x, t) \in \mathcal{D}_{\delta(N,\epsilon)}$. This is to be true for all N up to some integer N_0 that depends on the context. The same approximability relation is to hold for the corresponding derivatives of u and $u^{(N)}$ up to some order, again depending on the context.

The "outer" functions u_n may be discontinuous or otherwise nonsmooth on Γ, but are smooth in $\mathcal{D}_+$ uniformly up to Γ, and are also smooth in $\mathcal{D}_-$ uniformly up to Γ.

The representation of u near Γ is as follows. We consider a local orthogonal coordinate system $(r(x, t; \epsilon), s(x, t; \epsilon))$ in a neighborhood of $\Gamma(t; \epsilon)$. Here s, for x on Γ, represents arclength along Γ. We then introduce a stretched variable

$$\rho(x, t; \epsilon) \equiv \frac{r(x, t; \epsilon)}{\epsilon}, \tag{3}$$

and think of the same function u now expressed in terms of the coordinates ρ and s: $u(x, t; \epsilon) = U(\rho, s, t; \epsilon)$ in that neighborhood. Again, U and its derivatives are supposed to be approximable by truncations of a formal series

$$U \simeq \Sigma \epsilon^n U_n(\rho, s, t), \tag{4}$$

i.e., by polynomials in ϵ with coefficients depending only on ρ, s, and t. These are the "inner" approximations. To be more precise again, we suppose there is a function $K(\epsilon, N)$ with a property to be given later such that $K \to \infty$ as $\epsilon \to 0$, so that the approximation of U and its derivatives by the N-truncation of (4) has an error $\leq O(\epsilon^{N+1})$ uniformly for $|\rho| < K(\epsilon, N)$.

Denote the normal velocity $-(\partial r/\partial t)(s, t; \epsilon) = -r_t$ of the interface at the point s in the direction of positive r by the function $v(s, t; \epsilon)$. Assume that Γ depends uniformly smoothly on t and ϵ; it then follows that v and r can be approximated by truncated power series, uniformly in (s, t) in the case of v and uniformly in a neighborhood of Γ in the case of r. Note that knowledge of the function r determines s, in view of the orthogonality of the coordinate system and the fact that s measures arclength. The function s also may be approximated by a polynomial in ϵ.

We often will denote partial derivatives by ∂_t, etc. The symbol $\partial_{\rho\rho}$ will denote the second partial with respect to ρ.

The following relations between the outer functions u_i and the inner functions U_i are to hold. Their justification on the basis of the above assumptions for a particular choice of δ and K will be given below. They are called "matching conditions." In the following the outer functions will be

written as functions of r and s rather than x. As $\rho \to \pm\infty$,

$$U_0(\pm\infty, s, t) = u_0(0\pm, s, t), \tag{5a}$$

$$U_1(\rho, s, t) = u_1(0\pm, s, t) + \rho\partial_r u_0(0\pm, s, t) + o(1), \tag{5b}$$

$$U_2(\rho, s, t) = u_2 + \rho\partial_r u_1 + \tfrac{1}{2}\rho^2\partial_{rr}u_0 + o(1), \tag{5c}$$

etc. The arguments of the functions on the right of (5c) are the same as in (5b). Here the notation $\partial_r u_0(0\pm, s, t)$, for example, denotes the limit, as Γ is approached from $\mathscr{D}_\pm$ at the point s, of the normal derivative of the function u_0 in the direction of increasing r. Sometimes the symbol $\partial_r u_0(\Gamma\pm, t)$ will be used instead.

In each case studied here, the various coefficients U_n and u_n will be subject to determination from differential equations under suitable boundary conditions. The matching relations (5) will be instrumental in defining the desired solutions of these problems.

The various problems for the U_n and u_n will constitute formal reductions to various orders of refinement of our original evolution model for u, and in this sense will be alternate models for the natural phenomenon being described by that original problem.

One may object justifiably at this point to the fact that the approximability assumptions we have made in the inner and outer regions are very restrictive. How do we know a priori that this approach can be followed in any particular circumstance? Of course we do not, but experience indicates that it will be successful in many cases, including those studied here. Formal verification of these assumptions would be accomplished by being able to construct, reasonably and systematically, the various inner and outer functions. Rigorous verification would consist of proving that the approximations so constructed are indeed close to an exact solution of the original problem. This latter step is only rarely done; early examples of where it was done for stationary interfaces can be found in [FGr], [Fi74], [Fi76a], and [MTH]. Important recent results for quite general systems with internal layers were obtained by Lin [Lin]. The usual practice in the applied literature (at least implicitly) is to accept and interpret the problems for the various approximations as being the alternate models mentioned above.

Certain properties of the local coordinate system (r, s) are appropriately stated here, since they will be useful later on several occasions.

On Γ, we have

$$|\nabla r| \equiv 1 \quad \text{and} \quad \Delta r = \kappa, \tag{6}$$

where Δ and ∇ refer only to the spatial variable x, and κ is the curvature of Γ, counted as positive if Γ is concave as seen from $\mathscr{D}_-$. The first equation in (6), in fact, holds in the entire neighborhood where r is defined.

A standard calculation (making use of (6) and the orthogonality of the local system) shows that the change to local variables near Γ transforms the

Laplacian and time derivative as follows:

$$\Delta u = \partial_{rr}u + \Delta r\partial_r u + \partial_s u \Delta s + \partial_{ss}u\,|\nabla s|^2, \tag{7a}$$

$$\partial_t u \quad \text{becomes} \quad \partial_t u + r_t\partial_r u + s_t\partial_s u. \tag{7b}$$

We now return to the justification of the matching relations (5). Suppose t is fixed; hence Γ is also. Near Γ, we write u as a function of the local coordinates r and s: $u = u(r, s; \epsilon)$. We equate u and U in that region:

$$U(\rho, s, t; \epsilon) = u(\epsilon\rho, s, t; \epsilon). \tag{8}$$

Both the inner and outer representations (2) and (4) will be valid in the sense explained above in an intermediate region $\mathcal{J}$ in the (ϵ, ρ) plane defined by

$$\mathcal{J}: \quad \epsilon^{-1}\delta(\epsilon, N) \le \rho \le K(\epsilon, N), \tag{9}$$

provided this intermediate region is nonempty, which will be guaranteed in a moment by our construction of it. Note that for fixed x, r and ρ also depend on ϵ; however, we are not fixing x, and so that dependence need not be considered at this stage. Now (8) still holds except for an error term of order $O(\epsilon^{N+1})$, if we replace U on the left by the truncation $U^{(N)}$ and u on the right by $u^{(N)}$. A similar statement holds for the derivatives of (8) up to some order. We make this replacement. Suppressing dependence on s and t, we have

$$U_0(\rho) + \cdots + \epsilon^N U_N(\rho) = u_0(\epsilon\rho) + \cdots + \epsilon^N u_N(\epsilon\rho) + O(\epsilon^{N+1}) \quad \text{in } \mathcal{J}. \tag{10}$$

Now since the u_i were assumed to be smooth up to Γ, we can expand them in finite Taylor series in r around $r = 0$, and thereby get an expansion (for $\rho > 0$) such as

$$u_0(r, s, t) = u_0(\epsilon\rho, s, t) = u_0(0+, s, t) + \epsilon(\rho\partial_r u_0(0+, s, t) + u_1(0+, s, t)) + \cdots.$$

There results

$$\sum_{n=0}^{N} \epsilon^n U_n(\rho, s, t) = \sum_{n=0}^{N} \epsilon^n P_n(\rho, s, t) + \epsilon^{N+1} R_N + O(\epsilon^{N+1}), \tag{11}$$

where

$$P_n(\rho, s, t) \equiv \frac{1}{n!}\frac{\partial^n}{\partial\epsilon^n} u^{(N)}(\epsilon\rho, s, t; \epsilon)\,|_{\epsilon=0}, \tag{12}$$

and $R_N =$ (similar expression, with $n = N + 1$ and evaluated at some $\hat{\epsilon}$ rather than $\epsilon = 0$). (For $\rho > 0$, the expression on the right means the limit, as $\epsilon \downarrow 0$, of the derivative indicated. This is also true for $\rho < 0$.)

It is clear from (12) that for $\rho > 0$, P_n is a polynomial in ρ of degree n with coefficients depending (linearly) on the $u_k(0+, s, t)$, $\partial_r u_k(0+, s, t)$, $\partial_{rr}u_k(0+, s, t)$, etc. A similar statement holds when $\rho < 0$. Therefore, the functions P_n in general will be discontinuous at $\rho = 0$, and be polynomials on either side. To make this clear, on occasion we may write them as $P_n^{\pm}(\rho, s, t)$.

Moreover, $|R_N| \leq O(\rho^{N+1})$, so that

$$\sum_{n=0}^{N} \epsilon^n U_n(\rho, s, t) = \sum_{n=0}^{N} \epsilon^n P_n(\rho, s, t) + O(\epsilon^{N+1}\rho^{N+1}). \tag{13}$$

At this point we specify δ and K to be of the form

$$\delta = \epsilon^{\alpha}, \quad K = \epsilon^{-\beta}, \quad \beta > 1 - \alpha, \tag{14}$$

α and β being positive constants depending on N to be given below.

We will give the rest of the argument for the specific case $N = 2$; the extension to larger values of N will be immediate. Let $\mathcal{J}_\gamma$ denote the line $\epsilon = \rho^{-\gamma}$ in the (ϵ, ρ) plane, where γ is chosen so that $\mathcal{J}_\gamma \in \mathcal{J}$, in fact, so that

$$1 - \alpha < \frac{1}{\gamma} < \beta. \tag{15}$$

Because of the strict inequalities, for $\gamma' - \gamma$ small enough, $\mathcal{J}_{\gamma'} \in \mathcal{J}$ as well.

On $\mathcal{J}_\gamma$, (13) for $N = 2$ takes the form

$$(U_0 - P_0) + \rho^{-\gamma}(U_1 - P_1) + \rho^{-2\gamma}(U_2 - P_2) = O(\epsilon^3\rho^3) \quad \text{on } \mathcal{J}_\gamma. \tag{16}$$

Let the three terms on the left be denoted by V_0, V_1, and V_2. We may then write (16) in the form

$$\frac{V_0 + V_1 + V_2}{\epsilon^3\rho^3} \quad \text{is bounded on } \mathcal{J}_\gamma. \tag{17a}$$

Also let $\gamma' = \gamma + \eta$ for $\eta > 0$ sufficiently small so $\mathcal{J}_{\gamma'} \in \mathcal{J}$. Then the analogous relation is

$$\frac{V_0 + \rho^{-\eta}V_1 + \rho^{-2\eta}V_2}{\epsilon^3\rho^3\rho^{-3\eta}} \quad \text{is bounded on } \mathcal{J}_\gamma. \tag{17b}$$

In (17b) we may omit the factor $\rho^{-3\eta}$ in the denominator without changing the limit relation. We do this and subtract (17b) from (17a) to obtain

$$\frac{(1 - \rho^{-\eta})V_1(\rho) + (1 - \rho^{-2\eta})V_2(\rho)}{\epsilon^3\rho^3} \quad \text{is bounded on } \mathcal{J}_\gamma.$$

As $\rho \to \infty$, the powers of ρ indicated on the left approach zero, so we obtain as a consequence

$$V_1 + V_2 = O(\epsilon^3\rho^3) \quad \text{on } \mathcal{J}_\gamma \quad \text{as } \rho \to \infty, \tag{18}$$

and from (17a),

$$V_0 = O(\epsilon^3\rho^3) \quad \text{on } \mathcal{J}_\gamma \quad \text{as } \rho \to \infty. \tag{19}$$

Since the left side of (19) in fact does not depend on ϵ, we obtain

$$U_0(\rho) \to P_0 \quad \text{as } \rho \to \infty. \tag{20}$$

Now multiply (18) by $\rho^\gamma = \epsilon^{-1}$ to obtain

$$(U_1 - P_1) + \rho^{-\gamma}(U_2 - P_2) = O(\epsilon^2\rho^3) \quad \text{on } \mathcal{J}_\gamma \quad \text{as } \rho \to \infty.$$

Repeating the above argument, we get

$$U_1 - P_1 = O(\epsilon^2\rho^3) \quad \text{on } \mathcal{J}_\gamma \quad \text{as } \rho \to \infty. \tag{21}$$

Proceeding one more step, we get

$$U_2 - P_2 = O(\epsilon\rho^3) \quad \text{on } \mathcal{J}_\gamma \quad \text{as } \rho \to \infty. \tag{22}$$

It follows that if $0 < \beta < \frac{1}{3}$, then from (14) and (15) we have $\epsilon\rho^3 \to 0$ on $\mathcal{J}_\gamma$, and from (20), (21), (22),

$$U_n(\rho, s, t) = P_n^{\pm}(\rho, s, t) + o(1) \quad (\rho \to \pm\infty), \tag{23}$$

for $n = 0, 1$, and 2. (The superscript $\pm$ is inserted because clearly the same argument holds for $\rho < 0$.)

The same result holds for any N if we merely choose

$$0 < \beta < \frac{1}{N+1}.$$

And of course α must satisfy (15); this completes the specification of δ and K, and completes the assumptions under which we are operating.

In particular, it follows that the large ρ behavior of U^n must be that of a polynomial in ρ of degree $n-1$. The relations (5) give the specific results on this large ρ behavior for $n = 0$, 1, and 2. They were obtained by calculating P_0, P_1, and P_2 explicitly.

2. Example: Bistable fronts in an inhomogeneous medium.

Consider the single equation

$$\epsilon\partial_t u = \epsilon^2\Delta u + f(u, x), \tag{24}$$

where f is of "bistable type" in u for each x. This means that for each x, the equation $f(u, x) = 0$ can be solved for exactly three values of u as functions $h(x)$, and the condition $f_u(h(x), x) < 0$ is satisfied at the two extremal functions (see Fig. 1.1). The maximal and minimal such functions are denoted by $h_+(x)$

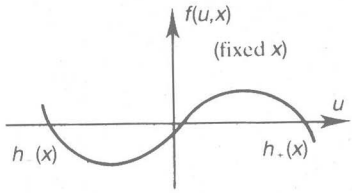

FIG. 1.1

and $h_-(x)$, respectively. For convenience, we suppose that

$$h_-(x) < 0 < h_+(x). \tag{25}$$

Note that in (24) the presence of the coefficient ϵ of the time derivative is of no consequence. Time can be scaled so that any desired coefficient appears there; we choose ϵ for convenience. This is not true with the coefficient ϵ^2 of the Laplacian; in fact, rescaling x also changes the function f. That coefficient simply means that the ratio of the characteristic space scale of the solution in regions where diffusion is important to the characteristic scale of the function f's spatial variation is small, of order ϵ. In other words, relative to the spatial scale associated with diffusion, the function f varies slowly.

The theory of such bistable equations is very well developed in the case when f does not depend on x (then, of course, space, as well as time, may be rescaled to make all ϵ's disappear) and space is one-dimensional. We will make good use of those results. So let us fix $x = x_0$, and consider the corresponding equation

$$\partial_t \psi = \partial_{xx} \psi + f(\psi, x_0). \tag{26}$$

It is known (see [Ka], [AW75], [AW78], [FM], [Fi79c], for example) that this equation has a globally stable traveling wave solution $\psi(x - \bar{v}t)$ satisfying $\psi(\pm\infty) = h_\pm(x_0)$ for exactly one value of $\bar{v}$. We will write it as $\psi(x - \bar{v}t, x_0)$ to indicate its dependence on x_0. Moreover, the solution ψ is unique, modulo shifts in the independent variable. Let us denote the velocity by the function

$$\bar{v} = V(x_0). \tag{27}$$

This function is also known to be smooth if f is smooth [FH].

Now let us return to the original problem (24), with x in R^2. The outer functions u_n are obtained by substituting the formal series (2) into (24) and equating the coefficients of the various powers of ϵ. This ensures that the residual error in (24) produced by using the series truncated to order N is formally of order ϵ^{N+1}, which is the highest order possible.

The lowest order outer problem in this case is

$$f(u_0, x) = 0. \tag{28}$$

As the solution of (28), we choose

$$u_0(x) = h_-(x), \quad x \in \mathcal{D}_-; \quad u_0(x) = h_+(x), \quad x \in \mathcal{D}_+. \tag{29}$$

The solution of the inner problem will smooth out the discontinuity on Γ. Recall that in §1, the curve Γ was not defined precisely. To remedy this imprecision in the present case, we define it to be the location where the inner function U equals 0. Thus

$$U(0, s, t) \equiv 0 \tag{30}$$

to all orders.

For the inner problems we represent the time and space derivatives in (24)

according to (7). Points on the moving interface $\Gamma(t)$ may be specified by coordinates (s, t); we denote them by the function $X(s, t)$. Points in a neighborhood of Γ, therefore, can be denoted by $X(s, t) + re_r$, where $e_r(s, t)$ denotes the unit vector normal to Γ at $X(s, t)$ in the direction of increasing r. We perform this transformation in (24) to obtain the following inner equation:

$$r_t\partial_\rho U + \epsilon\partial_t U + \epsilon s_t\partial_s U = \partial_{\rho\rho} U + f(U, X(s, t) + \epsilon\rho e_r) + \epsilon\Delta r\partial_\rho U + \epsilon^2(\partial_{ss} U\,|\nabla s|^2 + \partial_s U\Delta s). \tag{31}$$

Again, the inner functions U_n are obtained by substituting (4) into (31) and equating coefficients of the different powers of ϵ. To lowest order, we have (recalling $v_0 = -\partial_t r_0$)

$$\partial_{\rho\rho} U_0 + v_0\,\partial_\rho U_0 + f(U_0, X) = 0, \tag{32}$$

and the matching condition (5a) together with (29) requires

$$U_0(\pm\infty, s, t) = h_\pm(X(s, t)). \tag{33}$$

(This and (25) show that (30) may always be arranged.) Here and in (32), of course, $X = X(s, t)$ is any point on the interface. Now note that (32) is exactly the traveling wave equation for (26) with x_0 replaced by X, and in view of the uniqueness mentioned above, we must have

$$v_0 = V(X(s, t)), \tag{34}$$

$$U_0(\rho, s, t) = \psi(\rho, X(s, t)), \tag{35}$$

where ψ now denotes the solution of (26) with the independent variable shifted so that ψ vanishes at $\rho = 0$.

These facts suffice to obtain, to lowest order, the motion of Γ, because at each instant of time (34) gives the normal velocity of Γ at any point X on Γ. We therefore have the function $r_0(x, t)$ and hence $s_0(x, t)$. Along with that, we of course have the dominant order inner layer solution and the outer solution on both sides of Γ.

We now proceed to the next order to obtain u_1, U_1, and v_1. For u_1 we have

$$f_u(u_0, x)u_1 = \partial_t u_0 = 0,$$

so

$$u_1(x) \equiv 0. \tag{36}$$

For U_1 we have, from (31), the problem

$$\begin{aligned}\partial_{\rho\rho} U_1 + v_0\partial_\rho U_1 + f_u(U_0(\rho, s, t), X)U_1 = {} & (-v_1 - \kappa(s, t))\partial_\rho\psi(\rho, X) \\ & + \partial_t\psi(\rho, X(s, t)) + \partial_t s_0\partial_s\psi(\rho, X(s, t)) \\ & - \rho\nabla_x f(U_0(\rho, X), X)\cdot e_r,\end{aligned} \tag{37}$$

where in the last term, $\nabla_x f \cdot e_r = \partial_r f$ represents the directional derivative of f with respect to x in the direction of e_r.

The right side of (37) can be simplified, as we now show.

In (35), U_0 is given in terms of the function $\psi(x - vt, x_0)$, defined following (26), with the argument $x - vt$ replaced by ρ and x_0 by $X(s, t)$. Since ψ is defined for any spatial point x_0 occurring as the second argument, we may write

$$U_0(\rho, s, t) = \psi(\rho, x)\,|_{x=X(s,t)}.$$

More generally, consider any smooth function $F(\rho, x, t)$, in which the three arguments are considered to be independent. We examine the effect of the differential operator

$$\hat{D} \equiv \partial_t^{(s)} + (\partial_t s)\partial_s$$

acting on the function $F(\rho, X(s, t), t)$, where as before $X(s, t)$ is the point on Γ described by the arclength coordinate s at time t, and $\partial^{(s)}$ represents differentiation with s held constant. By the chain rule, we have

$$\hat{D}F(\rho, X(s, t), t) = \partial_3 F + \nabla_x F(\rho, X(s, t), t) \cdot [\partial_t X(s, t) + (\partial_t s)\partial_s X(s, t)], \tag{38}$$

where ∂_3 represents differentiation with respect to the third argument of F.

We temporarily denote the dependence of s on x and t by the function $s = S(x, t)$. Thus in particular

$$s = S(X(s, t), t),$$

and the variables s and t are independent in this relation, so we may differentiate with respect to t to obtain

$$0 = \nabla_x S(X, t) \cdot \partial_t X + \partial_t S(X, t). \tag{39}$$

Also note by elementary differential geometry that the vectors $\nabla_x S$ and $\partial_s X$ are (identical) unit vectors tangent to Γ; call that vector T. We therefore have from (39)

$$\begin{aligned} \partial_t X(s, t) + \partial_t s \partial_s X(s, t) &= \partial_t X - (\nabla_x S \cdot \partial_t X)\partial_s X \\ &= \partial_t X - (T \cdot \partial_t X)T = P\partial_t X, \end{aligned} \tag{40}$$

the symbol P denoting projection onto the vector e_r. From (38), (40), and the fact that the norm of the vector $P\partial_t X$ is equal to the normal velocity v, we obtain

$$\hat{D}F = \partial_3 F + \nabla_x F \cdot P\partial_t X = \partial_3 F + v\partial_r F. \tag{41}$$

We now apply (41) to (37) to obtain

$$\begin{aligned} &\partial_{\rho\rho} U_1 + v_0 \partial_\rho U_1 + f_u(U_0(\rho, s, t), X)U_1 \\ &= (-v_1 - \kappa(s, t))\partial_\rho \psi(\rho, X) + v_0 \partial_r \psi(\rho, x)|_{x=X(s,t)} - \rho \partial_r f(U_0(\rho, X), X). \end{aligned} \tag{42}$$

We write the left side of (42) in the form $\mathcal{L}U_1$, where $\mathcal{L}$ is the linear

differential operator indicated. It is easily checked that

$$\mathcal{L}p = 0,$$

where $p(\rho, X) \equiv \partial_\rho \psi(\rho, X)$. Thus p is an eigenfunction of $\mathcal{L}$ in $L_2(-\infty, \infty)$. We know that the eigenvalues of $\mathcal{L}$ are simple (see below). Let p^* be the eigenfunction corresponding to eigenvalue 0 of the adjoint operator $\mathcal{L}^*$. Then any equation $\mathcal{L}q = f(\rho)$, $f \in L_2$, is solvable in L_2 if and only if f is orthogonal to p^*. Equation (42) is of that form, with $q = U_1$. We therefore require orthogonality to $p^*(\rho)$:

$$(-v_1 - \kappa)A(s, t) + B(s, t) = 0, \tag{43}$$

where

$$A = \int p(\rho)p^*(\rho)\, d\rho,$$

$$B = \int p^*(\rho)[v_0 \partial_r \psi - \rho \partial_r f]\,|_X\, d\rho.$$

The simplicity of the eigenvalue ensures that $A \neq 0$, so that (43) may be written

$$v_1(X) = -\kappa(X) + g(X), \tag{44}$$

for every $X \in \Gamma(t)$, where $g(X(s, t)) = B(s, t)/A(s, t)$, and κ is the curvature of Γ_0 at the point X.

Combining (34) and (44), we obtain a more accurate expression for the normal velocity of Γ in the form

$$v_0(X) + \epsilon v_1(X) = V(X) - \epsilon\kappa(X) + \epsilon g(X). \tag{45}$$

Here, of course, V and g are functions of X that can be determined a priori, but κ depends on Γ itself and is determined as part of the solution. The right side of (45) shows two correction terms to the usual velocity V of a planar front in a homogeneous medium. The first results from the possible curvature of the front, and the second from the inhomogeneous nature of the medium.

It should be noted that this asymptotic result, to lowest order, in the case of one space dimension has recently been rigorously justified by Fife and Hsiao [FH]. The initial formation of the layer was also studied in that paper.

An interesting recent study of the dynamics of interfaces for gradient systems in a homogeneous medium was made by Rubenstein, Sternberg, and Keller [RSK]. They treated systems of the form (24) but with f replaced by the gradient $V_u(u)$ without x dependence.

We return to show that 0 is a simple eigenvalue of $\mathcal{L}$. If q is another eigenvector with that eigenvalue, then the Wronskian $W = pq' - qp'$ satisfies

$$W'' + v_0 W' = 0.$$

But $W \to 0$ at $\pm\infty$, so $W \equiv 0$; hence p and q are linearly dependent.