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Serge Lang

Fundamentals of Differential Geometry

微分几何基础

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Foreword

The present book aims to give a fairly comprehensive account of the fundamentals of differential manifolds and differential geometry. The size of the book influenced where to stop, and there would be enough material for a second volume (this is not a threat).

At the most basic level, the book gives an introduction to the basic concepts which are used in differential topology, differential geometry, and differential equations. In differential topology, one studies for instance homotopy classes of maps and the possibility of finding suitable differentiable maps in them (immersions, embeddings, isomorphisms, etc.). One may also use differentiable structures on topological manifolds to determine the topological structure of the manifold (for example, à la Smale [Sm 67]). In differential geometry, one puts an additional structure on the differentiable manifold (a vector field, a spray, a 2-form, a Riemannian metric, ad lib.) and studies properties connected especially with these objects. Formally, one may say that one studies properties invariant under the group of differentiable automorphisms which preserve the additional structure. In differential equations, one studies vector fields and their integral curves, singular points, stable and unstable manifolds, etc. A certain number of concepts are essential for all three, and are so basic and elementary that it is worthwhile to collect them together so that more advanced expositions can be given without having to start from the very beginnings.

Those interested in a brief introduction could run through Chapters II, III, IV, V, VII, and most of Part III on volume forms, Stokes' theorem, and integration. They may also assume all manifolds finite dimensional.

Charts and local coordinates. A chart on a manifold is classically a representation of an open set of the manifold in some euclidean space.

Using a chart does not necessarily imply using coordinates. Charts will be used systematically. It will be observed equally systematically that finite dimensionality is hereby not used.

It is possible to lay down *at no extra cost* the foundations (and much more beyond) for manifolds modeled on Banach or Hilbert spaces rather than finite dimensional spaces. In fact, it turns out that the exposition gains considerably from the systematic elimination of the indiscriminate use of local coordinates $x_1, \dots, x_n$ and $dx_1, \dots, dx_n$. These are replaced by what they stand for, namely isomorphisms of open subsets of the manifold on open subsets of Banach spaces (local charts), and a local analysis of the situation which is more powerful and equally easy to use formally. In most cases, the finite dimensional proof extends at once to an invariant infinite dimensional proof. Furthermore, in studying differential forms, one needs to know only the definition of multilinear continuous maps. An abuse of multilinear algebra in standard treatises arises from an unnecessary double dualization and an abusive use of the tensor product.

I don't propose, of course, to do away with local coordinates. They are useful for computations, and are also especially useful when integrating differential forms, because the $dx_1 \wedge \cdots \wedge dx_n$. corresponds to the $dx_1 \cdots dx_n$ of Lebesgue measure, in oriented charts. Thus we often give the local coordinate formulation for such applications. Much of the literature is still covered by local coordinates, and I therefore hope that the neophyte will thus be helped in getting acquainted with the literature. I also hope to convince the expert that nothing is lost, and much is gained, by expressing one's geometric thoughts without hiding them under an irrelevant formalism.

I am aware of a widespread apprehensive reaction the moment some geometers or students see the words "Banach space" or "Hilbert manifold". As a possible palliative, I suggest reading the material assuming from the start that Banach space means finite dimensional space over the reals, and Hilbert manifold or Riemannian manifold means a finite dimensional manifold with a metric, with the local constant model being ordinary euclidean space. These assumptions will not make any proof shorter.

One major function of finding proofs valid in the infinite dimensional case is to provide proofs which are especially natural and simple in the finite dimensional case. Even for those who want to deal only with finite dimensional manifolds, I urge them to consider the proofs given in this book. In many cases, proofs based on coordinate free local representations in charts are clearer than proofs which are replete with the claws of a rather unpleasant prying insect such as Γ^i_{jkl}. Indeed, the bilinear map associated with a spray (which is the quadratic map corresponding to a symmetric connection) satisfies quite a nice local formalism in charts. I think the local representation of the curvature tensor as in Proposition 1.2 of Chapter IX shows the efficiency of this formalism and its superiority over local coordinates. Readers may also find it instructive to compare the proof of Proposition 2.6 of Chapter IX concerning the rate of growth of Jacobi fields with more classical ones involving coordinates as in [He 78], pp. 71–73.

Applications in Infinite Dimension

It is profitable to deal with infinite dimensional manifolds, modeled on a Banach space in general, a self-dual Banach space for pseudo Riemannian geometry, and a Hilbert space for Riemannian geometry. In the standard pseudo Riemannian and Riemannian theory, readers will note that the differential theory works in these infinite dimensional cases, with the Hopf–Rinow theorem as the single exception, but not the Cartan–Hadamard theorem and its corollaries. Only when one comes to dealing with volumes and integration does finite dimensionality play a major role. Even if via the physicists with their Feynman integration one eventually develops a coherent analogous theory in the infinite dimensional case, there will still be something special about the finite dimensional case.

The failure of Hopf–Rinow in the infinite dimensional case is due to a phenomenon of positive curvature. The validity of Cartan–Hadamard in the case of negative curvature is a very significant fact, and it is only recently being realized as providing a setting for major applications. It is a general phenomenon that spaces parametrizing certain structures are actually infinite dimensional Cartan–Hadamard spaces, in many contexts, e.g. Teichmuller spaces, spaces of Riemannian metrics, spaces of Kähler metrics, spaces of connections, spaces associated with certain partial differential equations, ad lib. Cf. for instance the application to the KdV equation in [ScTZ 96], and the comments at the end of Chapter XI, §3 concerning other applications.

Actually, the use of infinite dimensional manifolds in connection with Teichmuller spaces dates back some time, because as shown by Bers, these spaces can be embedded as submanifolds of a complex Banach space. Cf. [Ga 87], [Vi 73]. Viewing these as Cartan–Hadamard manifolds comes from newer insights.

For further comments on some recent aspects of the use of infinite dimension, including references to Klingenberg's book [Kl 83/95], see the introduction to Chapter XIII.

Of course, there are other older applications of the infinite dimensional case. Some of them are to the calculus of variations and to physics, for instance as in Abraham–Marsden [AbM 78]. It may also happen that one does not need formally the infinite dimensional setting, but that it is useful to keep in mind to motivate the methods and approach taken in various directions. For instance, by the device of using curves, one can reduce what is a priori an infinite dimensional question to ordinary calculus in finite dimensional space, as in the standard variation formulas given in Chapter XI, §1.

Similarly, the proper domain for the geodesic part of Morse theory is the loop space (or the space of certain paths), viewed as an infinite dimensional manifold, but a substantial part of the theory can be developed without formally introducing this manifold. The reduction to the finite dimensional case is of course a very interesting aspect of the situation, from which one can deduce deep results concerning the finite dimensional manifold itself, but it stops short of a complete analysis of the loop space. (Cf. Boot [Bo 60], Milnor [Mi 63].) See also the papers of Palais [Pa 63] and Smale [Sm 64].

In addition, given two finite dimensional manifolds X, Y it is fruitful to give the set of differentiable maps from X to Y an infinite dimensional manifold structure, as was started by Eells [Ee 58], [Ee 59], [Ee 61], [EeS 64], and [Ee 66]. By so doing, one transcends the purely formal translation of finite dimensional results getting essentially new ones, which would in turn affect the finite dimensional case. For other connections with differential geometry, see [El 67].

Foundations for the geometry of manifolds of mappings are given in Abraham's notes of Smale's lectures [Ab 60] and Palais's monograph [Pa 68].

For more recent applications to critical point theory and submanifold geometry, see [PaT 88].

In the direction of differential equations, the extension of the stable and unstable manifold theorem to the Banach case, already mentioned as a possibility in earlier versions of *Differential Manifolds*, was proved quite elegantly by Irwin [Ir 70], following the idea of Pugh and Robbin for dealing with local flows using the implicit mapping theorem in Banach spaces. I have included the Pugh–Robbin proof, but refer to Irwin's paper for the stable manifold theorem which belongs at the very beginning of the theory of ordinary differential equations. The Pugh–Robbin proof can also be adjusted to hold for vector fields of class H^p (Sobolev spaces), of importance in partial differential equations, as shown by Ebin and Marsden [EbM 70].

It is a standard remark that the C^∞-functions on an open subset of a euclidean space do not form a Banach space. They form a Fréchet space (denumerably many norms instead of one). On the other hand, the implicit function theorem and the local existence theorem for differential equations are not true in the more general case. In order to recover similar results, a much more sophisticated theory is needed, which is only beginning to be developed. (Cf. Nash's paper on Riemannian metrics [Na 56], and subsequent contributions of Schwartz [Sc 60] and Moser [Mo 61].) In particular, some additional structure must be added (smoothing operators). Cf. also my Bourbaki seminar talk on the subject [La 61]. This goes beyond the scope of this book, and presents an active topic for research.

On the other hand, for some applications, one may complete the C^∞-space under a suitable Hilbert space norm, deal with the resulting Hilbert manifold, and then use an appropriate regularity theorem to show that solutions of the equation under study actually are C^∞.

I have emphasized differential aspects of differential manifolds rather than topological ones. I am especially interested in laying down basic material which may lead to various types of applications which have arisen since the sixties, vastly expanding the perspective on differential geometry and analysis. For instance, I expect the books [BGV 92] and [Gi 95] to be only the first of many to present the accumulated vision from the seventies and eighties, after the work of Atiyah, Bismut, Bott, Gilkey, McKean, Patodi, Singer, and many others.

Negative Curvature

Most texts emphasize positive curvature at the expense of negative curvature. I have tried to redress this imbalance. In algebraic geometry, it is well recognized that negative curvature amounts more or less to "general type". For instance, curves of genus 0 are special, curves of genus 1 are semispecial, and curves of genus $\geqq 2$ are of general type. Thus I have devoted an entire chapter to the fundamental example of a space of negative curvature. Actually, I prefer to work with the Riemann tensor. I use "curvature" simply as a code word which is easily recognizable by people in the field. Furthermore, I include a complete account of the equivalence between seminegative curvature, the metric increasing property of the exponential map, and the Bruhat–Tits semiparallelogram law. Third, I emphasize the Cartan–Hadamard further by giving a version for the normal bundle of a totally geodesic submanifold. I am indebted to Wu for valuable mathematical and historical comments on this topic.

There are several current directions whereby spaces of negative curvature are the fundamental building blocks of some theories. They are quotients of Cartan–Hadamard spaces. I myself got interested in differential geometry because of the joint work with Jorgenson, which naturally led us to such spaces for the construction and theory of certain zeta functions. Quite generally, we were led to consider spaces which admit a stratification such that each stratum is a quotient of a Cartan–Hadamard space (especially a symmetric space) by a discrete group. That such stratifications exist very widely is a fact not generally taken into account. For instance, it is a theorem of Griffiths that given an algebraic variety over the complex numbers, there exists a proper Zariski closed subset whose complement is a quotient of a complex bounded domain, so in this way, every algebraic variety admits a stratification as above, even with constant negative curvature. Thurston's approach to 3-manifolds could be viewed from our perspective also. The general problem then arises how zeta functions, spectral invariants, homotopy and homology invariants, ad lib. behave with respect to stratifications, whether additively or otherwise. In the Jorgenson–Lang program, we associate a zeta function to each stratum, and the zeta functions of lower strata are the principal fudge factors in the functional equation of the zeta function associated to the main stratum. The spectral expansion of the heat kernel amounts to a theta relation, and we get the zeta function by taking the Gauss transform of the theta relation.

From a quite different perspective, certain natural "moduli" spaces for structures on finite dimensional manifolds have a very strong tendency to be Cartan–Hadamard spaces, for instance the space of Riemannian metrics, spaces of Kahler metrics, spaces of connections, etc. which deserve to be incorporated in a general theory.

In any case, I find the exclusive historical emphasis at the foundational level on positive curvature, spheres, projective spaces, grassmanians, at the expense of quotients of Cartan–Hadamard spaces, to be misleading as to the way manifolds are built up. Time will tell, but I don't think we'll have to wait very long before a radical change of view point becomes prevalent.

New Haven, 1998 SERGE LANG

Acknowledgments

I have greatly profited from several sources in writing this book. These sources include some from the 1960s, and some more recent ones.

First, I originally profited from Dieudonné's *Foundations of Modern Analysis*, which started to emphasize the Banach point of view.

Second, I originally profited from Bourbaki's *Fascicule de résultats* [Bou 69] for the foundations of differentiable manifolds. This provides a good guide as to what should be included. I have not followed it entirely, as I have omitted some topics and added others, but on the whole, I found it quite useful. I have put the emphasis on the differentiable point of view, as distinguished from the analytic. However, to offset this a little, I included two analytic applications of Stokes' formula, the Cauchy theorem in several variables, and the residue theorem.

Third, Milnor's notes [Mi 58], [Mi 59], [Mi 61] proved invaluable. They were of course directed toward differential topology, but of necessity had to cover ad hoc the foundations of differentiable manifolds (or, at least, part of them). In particular, I have used his treatment of the operations on vector bundles (Chapter III, §4) and his elegant exposition of the uniqueness of tubular neighborhoods (Chapter IV, §6, and Chapter VII, §4).

Fourth, I am very much indebted to Palais for collaborating on Chapter IV, and giving me his exposition of sprays (Chapter IV, §3). As he showed me, these can be used to construct tubular neighborhoods. Palais also showed me how one can recover sprays and geodesics on a Riemannian manifold by making direct use of the canonical 2-form and the metric (Chapter VII, §7). This is a considerable improvement on past expositions.

In the direction of differential geometry, I found Berger–Gauduchon–Mazet [BGM 71] extremely valuable, especially in the way they lead to the study of the Laplacian and the heat equation. This book has been very influential, for instance for [GHL 87/93], which I have also found useful.

I also found useful Klingenberg's book [Kl 83/95], see especially chapter XIII. I am very thankful to Karcher and Wu for instructing me on several matters, including estimates for Jacobi fields. I have also benefited from Helgason's book [Hel 84], which contains some material of interest independently of Lie groups, concerning the Laplacian. I am especially indebted to Wu's invaluable guidance in dealing with the trace of the second fundamental form and its application to the Laplacian, giving rise to a new exposition of some theorems of Helgason in Chapters XIV and XV.

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