Ordinary Differential Equations

Second Edition

Philip Hartman

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Ordinary Differential Equations

Second Edition

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To the memory
of my parents

To the patience
of Sylvia, Judith, and Marilyn

Foreword to the Classics Edition

Those of us who grew up having this tome as our text or supplementary guide often referred to it with a certain reverence as being the ultimate authority. Some of my colleagues simply called it "The Bible." This is not a book that trots out the usual topics and examples in a formulaic way. Instead it is written by one of the world's leading practitioners, whose purpose appears to have been to give the serious student all the necessary tools for work at the forefront and to introduce research topics which, at the time of the book's original publication in 1964, were hot off the press. Thus, the book is comprehensive in fundamental theory but goes far beyond that in its presentation of several important and interesting topics as well as of some proofs which had never before appeared in print.

I am delighted that SIAM has undertaken to reprint Hartman's wonderful text, not only because of its historical significance but because it contains almost everything one would want in a modern advanced course in ordinary differential equations. The topological ideas and techniques of functional analysis that are so important today are clearly explained and applied. The basic theory of ordinary differential equations is given in sufficient generality that the book serves as a valuable reference, whereas many other texts supply only the standard simplified theorems.

What makes the text outstanding, in my mind, are the chapters on invariant manifolds, perturbations, and dichotomies. The idea of first establishing the existence of invariant manifolds for maps, and then applying the theory to solution operators for ordinary differential equations suitably modified outside a neighborhood of a critical point, has become a standard approach, now also followed for equations in infinite-dimensional spaces. One finds a complete proof of the Hartman–Grobman theorem on transforming a nonlinear to a linear flow in the neighborhood of a hyperbolic equilibrium. Theorems on smooth equivalence and on the smoothness of invariant manifolds are presented—these being important for perturbation and normal form theory. Poincaré sections are used to show persistence of hyperbolic periodic orbits under perturbations in the system—again, a technique that is now widely used.

All of this has found considerable use in the geometrical theory of differential equations and dynamical systems, such as in helping to resolve questions of structural stability. But it is also true that invariant manifold theory has become ubiquitous in both science and engineering. Typically the theory is used to deduce qualitative information about solutions to large, possibly infinite-dimensional, systems by reducing the essential dynamics to a small-dimensional system.

Readers will find many parts of this outstanding book as essential today as they were when the book was first published. So it is that in Philip Hartman's book we find the ever-useful classical theory and what could be considered the modern theory of ordinary differential equations and dynamical systems.

Peter Bates, Brigham Young University

Preface to the First Edition

This book is based on lecture notes of courses on ordinary differential equations which I have given from time to time for advanced undergraduates and graduate students in mathematics, physics, and engineering. It assumes a knowledge of matrix theory and, if not a thorough knowledge of, at least a certain maturity in the handling of functions of real variables.

I was never tempted to scatter asterisks liberally throughout this book and claim that it could serve as a sophomore–junior–senior textbook, for I believe that a course of this type should give way to basic courses in analysis, algebra, and topology.

This book contains more material than I ever covered in one year but not all of the topics which I treated in the many courses. The contents of these courses always included the subject matter basic to the theory of differential equations and its many applications to other disciplines (as, for example, differential geometry). A "basic course" is covered in Chapter I; §§ 1–3 of Chapter II; §§ 1–6 and 8 of Chapter III; Chapter IV except for the "Application" in § 3 and part (ix) in § 8; §§ 1–4 of Chapter V; §§ 1–7 of Chapter VII; §§ 1–3 of Chapter VIII; §§ 1–12 of Chapter X; §§ 1–4 of Chapter XI; and §§ 1–4 of Chapter XII.

Many topics are developed in depth beyond that found in standard textbooks. The subject matter in a chapter is arranged so that more difficult, less basic, material is usually put at the end of the chapter (and/or in an appendix). In general, the content of any chapter depends only on the material in that chapter and the portion of the "basic course" preceding it. For example, after completing the basic course, an instructor can discuss Chapter IX, or the remainder of the contents of Chapter XII, or Chapter XIV, etc. There are two exceptions: Chapter VI, Part I, as written, depends on Chapter V, §§ 5–12; Part III of Chapter XII is not essential but is a good introduction to Chapter XIII.

Exercises have been roughly graded into three types according to difficulty. Many of the exercises are of a routine nature to give the student an opportunity to review or test his understanding of the techniques just explained. For more difficult exercises, there are hints in the back of the book (in some cases, these hints simplify available proofs). Finally, references are given for the most difficult exercises; these serve to show extensions and further developments, and to introduce the student to the literature.

The theory of differential equations depends heavily on the "integration of differential inequalities" and this has been emphasized by collecting some of the main results on this topic in Chapter III and § 4 of Chapter IV. Much of the material treated in this book was selected to illustrate important techniques as well as results: the reduction of problems on differential equations to problems on "maps" (cf. Chapter VII, Appendix, and Chapter IX); the use of simple topological arguments (cf. Chapters VIII, § 1; X, §§ 2–7; and XIV, § 6); and the use of fixed point theorems and other basic facts in functional analysis (cf. Chapters XII and XIII).

I should like to acknowledge my deep indebtedness to the late Professor Aurel Wintner from whom and with whom I learned about differential equations, first as a student and later as a collaborator. My debt to him is at once personal, in view of my close collaboration with him, and impersonal, in view of his contributions to the resurgence of the theory of ordinary differential equations since the Second World War.

I wish to thank several students at Hopkins, in particular, N. Max, C. C. Pugh, and J. Wavrik, for checking parts of the manuscript. I also wish to express my appreciation to Miss Anna Lea Russell for the excellent typescript created from nearly illegible copy, numerous revisions, and changes in the revisions.

My work on this book was partially supported by the Air Force Office of Scientific Research.

PHILIP HARTMAN

Baltimore, Maryland
August, 1964

Preface to the Second Edition

This edition is essentially that of 1964 (John Wiley and Sons, Inc.) with some corrections and additions. The only major changes are in Chapter IX. A few pertinent additions have been made to the bibliography as a Supplement, but no effort has been made to bring it up to date.

I should like to thank Professors T. Butler, K.T. Chen, W.A. Coppel, L. Lorch, M.E. Muldoon, L. Nicolson and C. Olech for corrections and/or suggestions, and Mrs. Margaret A. Einstein for able secretarial assistance.

P.H. (10/73)

Errata

This book has been photographically reproduced from the edition published by Birkhäuser, Boston, Basel, Stuttgart, 1982. The following are corrections for typographical errors in that edition.

Page 262: In lines 6 and 2 from the bottom, replace $\notin K^+ - K^0$ by $\in K^-$.

Page 264: At the end of the line following (14.8), replace $\notin$ by $\in$.

Page 264: Replace the last five lines of page by the following:

(d) There is a constant $C \geq 1$ such that

$$||T_1^j \xi - T_1^j \eta|| \leq C^j ||\xi - \eta|| \quad \textit{for} \quad j \geq 0. \tag{14.10}$$

This is clear for $j = 1$ from the normalizations following (4.1), and the cases $j \geq 1$ follow by induction.

Page 265: Replace the part beginning at **Proof** by the following:

Proof. If $0 < r_2 \leq r_1, \xi \in K^+(r_2) - M^+$ and $\eta = T^{-k(\xi)}\xi$, then, by the property of $r_1, T^i\eta \in S(r_0)$ for $i = 0, \ldots, k(\xi)$. Put

$$\zeta(j) = T_1^j \eta - T^j \eta, \text{ where } \eta = T^{-k(\xi)}\xi.$$

We show by induction that r_2 can be chosen so that

$$T^i \eta \in S(r_1) \text{ and } \zeta(i) \in S(r_1) \tag{14.14}$$

for $0 \leq i \leq k(\xi)$. Assume (14.14) for $0 \leq i \leq j - 1 < k(\xi)$. Then $\zeta(j) = (T_1 T_1^{j-1}\eta - T_1 T^{j-1}\eta) + (T_1 T^{j-1}\eta - T T^{j-1}\eta)$, so that

$$\begin{aligned}
||\zeta(j)|| &\leq C||\zeta(j-1)|| + c_N ||T^{j-1}\eta||^N \text{ by } (14.10), (14.1) \\
&\leq c_N \sum_{i=0}^{j-1} C^{j-1-i} ||T^i \eta||^N \\
&\leq c_N \sum_{i=0}^{j-1} C^{j-1-i} 2^N a^{(k(\xi)-i)N} ||\xi||^N \text{ by } (14.7)
\end{aligned}$$

$$\leq 2^N c_N a^{(k(\xi)-j)N} \sum_{i=0}^{j-1} (Ca^N)^{j-i} ||\xi||^N.$$

Thus $Ca^\lambda < 1$ implies that

$$(14.15) \qquad ||\zeta(j)|| \leq C_N ||\xi||^N \quad \text{if} \quad C_N = \frac{2^N c_N}{1 - Ca^N}.$$

If we define $r_2 = r_2(\lambda)$ by $r_2 = \min[r_1, r_1/(C_\lambda + 2), 1]$, then $N \geq \lambda$ gives both $||\zeta(j)|| \leq C_\lambda r_2^\lambda \leq C_\lambda r_2 \leq r_1$ and

$$\begin{aligned} ||T_1^j T^{-k(\xi)}\xi|| &= ||\zeta(j) + T^{j-k(\xi)}\xi|| \\ &\leq C_\lambda r_2^\lambda + 2a^{k(\xi)-j}||\xi|| \leq (C_\lambda + 2)r_2 \leq r_1. \end{aligned}$$

This completes the proof of the induction for (14.14) and of assertion (14.12). The definition of $\zeta(j)$ and the choice $j = k(\xi)$ in (14.15) give (14.13).

Page 266: Delete the first two lines.

Page 267: Line 4 should read "*respectively, for suitably small* $r_2 = r_2(\lambda)$. Note that"

Page 267: Line 10 should read "In view of (e) and its proof, this makes the validity of (h) clear."

Page 271G: In $B_n\xi$ of the last line, replace ξ by η.

Page 271J: In the third line following (4.6), in $\{T_n\}$ replace T by τ.

Page 272: Add the following to the text:

The translators for the Russian edition noted errors in the original arguments of steps (d) and (e) in the proof of Theorem 12.2; see P. Hartman, *Obyknovennye Differencial'nye Uravneniya,* Izdatel'stvo "Mir," Moscow (1970), translated by I. H. Sabitov and Yu. B. Egorov, edited by V. M. Alexeev. Compare the corrections for pages 262–267 in the errata, pages xix–xx.

Page 501: In the ninth line from the bottom insert a 2 before Im.

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