MATHEMATICS DICTIONARY

FOURTH EDITION

JAMES / JAMES

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PREFACE

Throughout the preparation of each edition of this dictionary, the guiding objective has been to make it useful for students, scientists, engineers, and others concerned with the meaning of mathematical terms. It is intended to be reasonably complete in the coverage of topics frequently included in precollege or undergraduate college mathematics courses. In addition, many other interesting and important mathematical concepts are included. Thus the dictionary is a valuable reference book for both amateur and professional mathematicians.

This edition continues revisions and enlargements as in previous editions, with particular emphasis on updating the coverage of probability and statistics. The major change in the present edition is the introduction of a large number of short biographical statements for persons whose contributions have been particularly important or whose names appear in the dictionary for other reasons. An important feature continued in this edition is the multilingual index in French, German, Russian, and Spanish. The English equivalents of mathematical terms in these languages enable the reader not only to learn the English meaning of a foreign-language mathematical term, but also to find its definition in the body of this book.

Main headings are printed in boldface capitals beginning at the left margin. Each main heading that is also a proper name is followed by the appropriate given names, birth and death dates, and a short biographical statement, to the extent that these have been determinable. As in previous editions, other main headings are followed by the part (or parts) of speech of the main headings—as determined by its definition and its uses in the subheadings that follow. Subheadings are printed in boldface type at the beginning of paragraphs. Citations to subheadings under other main headings give the main heading in small capitals, followed by a dash and then by the subheading (if giving the subheading seems useful) as: ANGLE—adjacent angle.

Although this is by no means a mere word dictionary, neither is it an encyclopedia. It is a correlated condensation of mathematical concepts, designed for time-saving reference work. Nevertheless, the general reader can come to an understanding of concepts in which he has not been schooled by looking up the unfamiliar terms in the definition at hand and following this procedure down to familiar concepts.

Comments on definitions as well as discussions of any phase of this dictionary are invited. Information concerning possible errors, omissions, and inadequacies will be particularly appreciated.

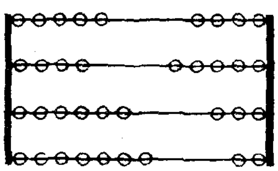
ROBERT C JAMES
EDWIN F BECKENBACH

CONTENTS

A

A.E. An abbreviation for *almost everywhere.* See MEASURE—measure zero.

AB'A-CUS, *n.* [*pl.* **abaci**]. A counting frame to aid in arithmetic computation; an instructive plaything for children, used as an aid in teaching place value; a primitive predecessor of the modern computing machine. One form consists of a rectangular frame carrying as many parallel wires as there are digits in the largest number to be dealt with. Each wire contains nine beads free to slide on it. A bead on the lowest wire counts unity, on the next higher wire 10, on the next higher 100, etc. Two beads slid to the right on the lowest wire, three on the next higher, five on the next and four on the next denote 4532.



ABEL, Niels Henrik (1802-1829). Norwegian algebraist and analyst. When about nineteen, he proved the general quintic equation in one variable cannot be solved by a finite number of algebraic operations (see RUFFINI). Made fundamental contributions to the theories of infinite series, transcendental functions, groups, and elliptic functions.

Abel's identity. The identity

$$\sum_{i=1}^{n} a_i u_i \equiv s_1(a_1-a_2)+s_2(a_2-a_3)+\cdots+s_{n-1}(a_{n-1}-a_n)+s_n a_n,$$

where

$$s_n = \sum_{i=1}^{n} u_i.$$

This is easily obtained from the evident identity:

$$\sum_{i=1}^{n} a_i u_i \equiv a_1 s_1 + a_2(s_2-s_1)+\cdots+a_n(s_n-s_{n-1}).$$

Abel's inequality. If $u_n \geq u_{n+1} > 0$ for all positive integers n, then $\left|\sum_{n=1}^{p} a_n u_n\right| \leq L u_1$, where L is the largest of the quantities: $|a_1|$, $|a_1+a_2|$, $|a_1+a_2+a_3|, \cdots, |a_1+a_2+\cdots+a_p|$. This inequality can be easily deduced from Abel's identity.

Abel's method of summation. The method of *summation* for which a series $\sum_0^\infty a_n$ is *summable* and has *sum* S if $\lim_{x\to 1-} \sum_0^\infty a_n x^n = S$ exists. A convergent series is summable by this method [see below, Abel's theorem on power series (2)]. Also called *Euler's method of summation.* See SUMMATION—summation of a divergent series.

Abel's problem. Suppose a particle is constrained (without friction) to move along a certain path in a vertical plane under the force of gravity. **Abel's problem** is to find the path for which the time of descent is a given function f of x, where the x-axis is the horizontal axis and the particle starts from rest. This reduces to the problem of finding a solution $s(x)$ of the *Volterra integral equation of the first kind* $f(x) = \int_0^x \frac{s(t)}{\sqrt{2g(x-t)}}\,dt$, where $s(x)$ is the length of the path. If f' is continuous, a solution is

$$s(x) = \frac{\sqrt{2g}}{\pi}\frac{d}{dx}\int_0^x \frac{f(t)}{(x-t)^{1/2}}\,dt.$$

Abel's tests for convergence. (1) If the series $\sum u_n$ converges and $\{a_n\}$ is a bounded monotonic sequence, then $\sum a_n u_n$ converges. (2) If $\left|\sum_{n=1}^{k} u_n\right|$ is equal to or less than a properly chosen constant for all k and $\{a_n\}$ is a positive, monotonic decreasing sequence which approaches zero as a limit, then $\sum a_n u_n$ converges. (3) If a series of complex numbers $\sum a_n$ is convergent, and the series $\sum (v_n - v_{n+1})$ is absolutely convergent, then $\sum a_n v_n$ is convergent. (4) If the series $\sum a_n(x)$ is uniformly convergent in an interval (a, b), $v_n(x)$ is positive and monotonic decreasing for any value of x in the interval, and there is a number k such that $v_0(x) \leq k$ for all x in the interval, then $\sum a_n(x)v_n(x)$ is uniformly convergent (this is **Abel's test for uniform convergence**).

Abel's theorem on power series. (1) If a power series, $a_0 + a_1 x + a_1 x^2 + \cdots + a_n x^n + \cdots$, converges for $x = c$, it converges absolutely for $|x| < |c|$. (2) If $\sum_0^\infty a_n$ is convergent, then $\lim_{t\to 1-} \sum_0^\infty a_n t^n = \sum_0^\infty a_n$, where the limit is the limit on the left at $+1$. An equivalent statement is that if $\sum_0^\infty a_n x^n$ converges when $x = R$, then S is continuous if $S(x)$ is defined as $\sum_0^\infty a_n x^n$ when x is in the closed interval with end points 0 and R. This theorem is designated in various ways, most explicitly by "Abel's theorem on continuity up to the circle of convergence."

A-BEL'IAN, *adj.* **Abelian group.** See GROUP.

A-BRIDGED', *adj.* **abridged multiplication.** See MULTIPLICATION.

Plücker's abridged notation. A notation used for studying curves. Consists of the use of a single symbol to designate the expression (function) which, equated to zero, has a given curve for its locus; hence reduces the study of curves to the study of polynomials of the first degree. *E.g.*, if $L_1 = 0$ denotes $2x+3y-5=0$ and $L_2 = 0$ denotes $x+y-2=0$, then $k_1L_1 + k_2L_2 = 0$ denotes the family of lines passing through their common point (1, 1). See PENCIL—pencil of lines through a point.

AB-SCIS'SA, *n.* [*pl.* **abscissas** or **abscissae**]. The horizontal coordinates in a two-dimensional system of rectangular coordinates; usually denoted by x. Also used in a similar sense in systems of oblique coordinates. See CARTESIAN—Cartesian coordinates.

AB'SO-LUTE, *adj.* **absolute constant, continuity, convergence, inequality, maximum (minimum), symmetry.** See CONSTANT, CONTINUOUS, CONVERGENCE, INEQUALITY, MAXIMUM, SYMMETRIC—symmetric function.

absolute moment (*Statistics*). For a random variable X or the associated distribution function, the kth absolute moment about a is the expected value of $|X-a|^k$, whenever this exists. See MOMENT—moment of a distribution.

absolute number. A number represented by figures such as 2, 3, or $\sqrt{2}$, rather than by letters as in algebra.

absolute property of a surface. Same as INTRINSIC PROPERTY OF A SURFACE.

absolute term in an expression. A term which does not contain a variable. *Syn.* constant term. In the expression ax^2+bx+c, c is the only absolute term.

absolute value of a complex number. See MODULUS—modulus of a complex number.

absolute value of a real number. The absolute value of a, written $|a|$, is the nonnegative number which is equal to a if a is nonnegative and equal to $-a$ if a is negative; *e.g.*, $3 = |3|$, $0 = |0|$, and $3 = |-3|$. Useful properties of the absolute value are that $|xy| = |x|\ |y|$ and $|x+y| \le |x|+|y|$ for all real numbers x and y. *Syn.* numerical value.

absolute value of a vector. See VECTOR—absolute value of a vector.

AB'STRACT, *adj.* **abstract mathematics.** See MATHEMATICS—pure mathematics.

abstract number. Any number as such, simply as a number, without reference to any particular objects whatever except in so far as these objects possess the number property. Used to emphasize the distinction between a number, as such, and concrete numbers. See CONCRETE, NUMBER, DENOMINATE.

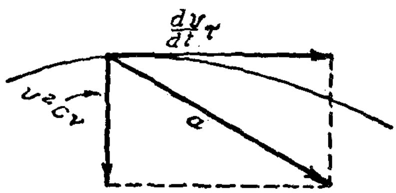
abstract space. A formal mathematical system consisting of undefined objects and axioms of a geometric nature. Examples are Euclidean spaces, metric spaces, topological spaces, and vector spaces.

abstract word or symbol. (1) A word or symbol that is not concrete; a word or symbol denoting a concept built up from consideration of many special cases; a word or symbol denoting a property common to many individuals or individual sets, as yellow, hard, two, three, etc. (2) A word or symbol which has no specific reference in the sense that the concept it represents exists quite independently of any specific cases whatever and may or may not have specific reference.

AC-CEL'ER-A'TION, *n.* The time rate of change of velocity. Since velocity is a directed quantity, the acceleration **a** is a vector equal to $\lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$, where $\Delta\mathbf{v}$ is the increment in the velocity **v** which the moving object acquires in t units of time. Thus, if an airplane moving in a straight line with the speed of 2 miles per minute increases its speed until it is flying at the rate of 5 miles per minute at the end of the next minute, its **average acceleration** during that minute is 3 miles per minute per minute. If the increase in speed during this one minute interval of time is uniform, the average acceleration is equal to the actual acceleration. If the increase in speed in this example is not uniform, the instantaneous acceleration at the time t_1 is determined by evaluating the limit of the quotient $\frac{\Delta \mathbf{v}}{\Delta t}$ as the time interval $\Delta t = t - t_1$ is made to approach zero by making t approach t_1. For a particle moving along a curved path, the velocity **v** is directed along the tangent to the path and the acceleration **a** can be shown to be given by the formula

$$\mathbf{a} = \frac{dv}{dt}\boldsymbol{\tau} + v^2 c\boldsymbol{\nu},$$

where $\frac{dv}{dt}$ is the derivative of speed v along the path, c is the curvature of the path at the point, and τ and ν are vectors of unit lengths directed along the tangent and normal to the path. The first of these terms, $\frac{dv}{dt}$, is called the **tangential component**, and the second, v^2c, the **normal** (or **centripetal**) **component** of acceleration. If the path is a



straight line, the curvature c is zero, and hence the acceleration vector will be directed along the path of motion. If the path is not rectilinear, the direction of the acceleration vector is determined by its tangential and normal components as shown in the figure.

acceleration of Coriolis. *If S' is a reference* frame rotating with the angular velocity $\boldsymbol{\omega}$ about a fixed point in another reference frame S, the acceleration $\mathbf{a}$ of a particle, as measured by the observer fixed in the reference frame S, is given by the sum of three terms: $\mathbf{a} = \mathbf{a}' + \mathbf{a}_t + \mathbf{a}_c$, where $\mathbf{a}'$ is the acceleration of the particle relative to S', $\mathbf{a}_t$ is the acceleration of the moving space, and $\mathbf{a}_c = 2\boldsymbol{\omega} \times \mathbf{v}'$ is the *acceleration of Coriolis.* The symbol $\boldsymbol{\omega} \times \mathbf{v}'$ denotes the vector product of the angular velocity $\boldsymbol{\omega}$, and the velocity $\mathbf{v}'$ relative to S', *so that the acceleration of* Coriolis is normal to the plane determined by the vectors $\boldsymbol{\omega}$ and $\mathbf{v}'$ and has the magnitude $2v' \sin(\boldsymbol{\omega}, \mathbf{v}')$. The acceleration of Coriolis is also called the **complementary acceleration**.

acceleration of a falling body. The acceleration with which a body falls *in vacuo* at a given point on or near a given point on the earth's surface. This acceleration, frequently denoted by g, varies by less than one percent over the entire surface of the earth. Its "average value" has been defined by the International Commission of Weights and Measures as 9.80665 meters (or 32.174 feet) per second per second. Its value at the poles is 9.8321 and at the equator 9.7799. *Syn.* acceleration of gravity.

angular acceleration. The *time rate of* change of angular velocity. If the angular velocity is represented by a vector ω directed along the axis of rotation, then the angular acceleration $\boldsymbol{\alpha}$, in the symbolism of calculus, is given by $\boldsymbol{\alpha} = \frac{d\boldsymbol{\omega}}{dt}$. See VELOCITY—angular velocity.

centripetal, normal, and tangential components of acceleration. See above, ACCELERATION.

uniform acceleration. Acceleration in which there are equal changes in the velocity in equal intervals of time. *Syn.* constant acceleration.

AC'CENT, *n.* A mark above and to the right of a quantity (or letter), as in a' or x'; the mark used in denoting that a letter is primed. See PRIME—prime as a symbol.

AC-CEPT'ANCE, *adj.* **acceptance region.** See HYPOTHESIS—test of a hypothesis.

AC-CU'MU-LAT'ED, *adj.* **accumulated value.** Same as AMOUNT at simple or compound interest. The accumulated value (or **amount**) of an **annuity** at a given date is the sum of the compound amounts of the annuity payments to that date.

AC-CU'MU-LA'TION, *adj., n.* Same as ACCUMULATED VALUE.

accumulation of discount on a bond. Writing up the book value of a bond on each dividend date by an amount equal to the interest on the investment (interest on book value at yield rate) minus the dividend. See VALUE—book value.

accumulation factor. The name sometimes given to the binomial $(1+r)$, or $(1+i)$, where r, or i, is the rate of interest. The formula for compound interest is $A = P(1+r)^n$, where A is the amount accumulated at the end of n periods from an original principal P at a rate r per period. See COMPOUND—compound amount.

accumulation point. An accumulation point *of a* **set of points** *is a point P such that there* is at least one point of the set distinct from P in any neighborhood of the given point; a point which is the limit of a sequence of points of the set (for spaces which satisfy the first axiom of countability). An **accumulation point of a sequence** is a point P such that, for any integer n, each neighborhood of P contains at least one term of the sequence after the nth term; *e.g.*, the sequence $1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, 1, \frac{1}{5}, \cdots$ has two accumulation points, the numbers 0 and 1 (also see SEQUENCE—accumulation point of a sequence). *Syn.* cluster point, limit point. See BOLZANO—Bolzano-Weierstrass theorem, CONDENSATION—condensation point.

accumulation problem. The determination of the amount when the principal, or principals, interest rate, and time for which each principal is invested are given.

accumulation schedule of bond discount. A table showing the accumulation of bond discounts on successive dates. Interest and book values are usually listed also.

AC-CU'MU-LA'TOR, *n.* In a computing machine, an adder or counter that augments its stored number by each successive number is receives.

AC'CU-RA-CY, *n.* Correctness, usually referring to numerical computations. The **accuracy of a table** may mean either: (1) The number of significant digits appearing in the numbers in the table (*e.g.*, in the mantissas of a logarithm table); (2) the number of correct places in computations made with the table. (This number of places varies with the form of computation, since errors may repeatedly combine so as to become of any size whatever.)

AC'CU-RATE, *adj.* Exact, precise, without error. One speaks of an accurate statement in the sense that it is correct or true and of an accurate computation in the sense that it contains no numerical error. **Accurate to a certain decimal place** means that all digits preceding and including the given place are correct and the next place has been made zero if less than 5 and 10 if greater than 5 (if it is equal to 5, the most usual convention is to call it zero or 10 as is necessary to leave the last digit even). *E.g.*, 1.26 is accurate to two places if obtained from 1.264 or 1.256 or 1.255. See ROUNDING—rounding off.

AC'NODE, *n.* See POINT—isolated point.

A-COUS'TI-CAL, *adj.* **acoustical property of conics.** See ELLIPSE—focal property of ellipse, HYPERBOLA—focal property of hyperbola, PARABOLA—focal property of parabola.

A'CRE, *n.* The unit commonly used in the United States in measuring land; contains 43,560 square feet, 4,840 square yards, or 160 square rods.

AC'TION, *n.* A concept in advanced dynamics defined by the line integral $A = \int_{P_1}^{P_2} m\mathbf{v} \cdot d\mathbf{r}$, called the **action integral**, where m is the mass of the particle, $\mathbf{v}$ is its velocity, and $d\mathbf{r}$ is the vector element of the arc of the trajectory joining the points P_1 and P_2. The dot in the integrand denotes the scalar product of the momentum vector $m\mathbf{v}$ and $d\mathbf{r}$. The *action A* plays an important part in the development of dynamics from variational principles. See below, principle of least action.

law of action and reaction. The basic law of mechanics asserting that two particles interact so that the forces exerted by one on another are equal in magnitude, act along the line joining the particles, and are opposite in direction. See NEWTON—Newton's laws of motion.

principle of least action. Of all curves passing through two fixed points in the neighborhood of the natural trajectory, and which are traversed by the particle at a rate such that for each (at every instant of time) the sum of the kinetic and potential energies is a constant, that one for which the action integral has an extremal value is the natural trajectory of the particle. See ACTION.

A-CUTE', *adj.* **acute angle.** An angle numerically smaller than a right angle (usually a positive angle less than a right angle).

acute triangle. See TRIANGLE.

AD'DEND, *n.* One of a set of numbers to be added, as 2 or 3 in the sum $2+3$.

AD'DER, *n.* In a computing machine, any arithmetic component that performs the operation of addition of positive numbers. An arithmetic component that performs the operations of addition and subtraction is said to be an **algebraic adder**. See ACCUMULATOR, COUNTER.

AD-DI'TION, *n.* **addition of angles, directed line segments, integers, fractions, irrational numbers, mixed numbers, matrices, and vectors.** See various headings under SUM.

addition of complex numbers. See COMPLEX—complex numbers.

addition of decimals. The usual procedure for adding decimals is to place digits with like place value under one another, *i.e.*, place decimal points under decimal points, and add as with integers, putting the decimal point of the sum directly below those of the addends. See SUM—sum of real numbers.

addition formulas of trigonometry. See TRIGONOMETRY.

addition of series. See SERIES.

addition of similar terms in algebra. The process of adding the coefficients of terms which are alike as regards their other factors: $2x+3x = 5x$, $3x^2y - 2x^2y = x^2y$ and $ax+bx = (a+b)x$. See DISSIMILAR—dissimilar terms.

addition of tensors. See TENSOR.

algebraic addition. See SUM—algebraic sum, sum of real numbers.

arithmetic addition. See SUM—arithmetic *sum*.

proportion by addition (and **addition and subtraction**). See PROPORTION.

ADD'I-TIVE, *adj.* **additive function.** A function f which has the property that $f(x+y)$ is defined and equals $f(x)+f(y)$ whenever $f(x)$ and $f(y)$ are defined. A *continuous* additive function is necessarily *homogeneous*. A function f is **subadditive** or **superadditive** according as

$$f(x_1+x_2) \leq f(x_1)+f(x_2),$$

or

$$f(x_1+x_2) \geq f(x_1)+f(x_2),$$

for all x_1, x_2, and x_1+x_2 in the domain of f (this domain is usually taken to be an interval of the form $0 \leq x \leq a$).

additive inverse. See INVERSE—inverse of an element.

additive set function. A function which assigns a number $\phi(X)$ to each set X of a family F of sets is **additive** (or **finitely additive**) if the *union* of any two members of F is a member of F and

$$\phi(X \cup Y) = \phi(X)+\phi(Y)$$

for all disjoint members X and Y of F. The function ϕ is **countably additive** (or **completely additive**) if the union of any finite or countable set of members of F is a member of F and

$$\phi(\bigcup X_i) = \sum \phi(X_i)$$

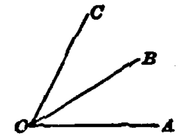
for each finite or countable collection of sets $\{X_i\}$ which are *pairwise disjoint* and belong to F. If $\phi(\bigcup X_i) \leq \sum \phi(X_i)$, then ϕ is said to be **subadditive** (it is then not necessary to assume the sets are pairwise disjoint). See MEASURE—measure of a set.

AD'I-A-BAT'IC, *adj.* **adiabatic curves.** Curves showing the relation between pressure and volume of substances which are assumed to have adiabatic expansion and contraction.

adiabatic expansion (or **contraction**). (*Thermodynamics*) A change in volume without loss or gain of heat.

AD IN'FI-NI'TUM. Continuing without end (according to some law); denoted by three dots, as $\cdots$; used, principally, in writing infinite series, infinite sequences, and infinite products.

AD-JA'CENT, *adj.* **adjacent angles.** Two angles having a common side and common vertex and lying on opposite sides of their common side. In the figure, AOB and BOC are adjacent angles.



AD-JOINED', *adj.* **adjoined number.** See FIELD—number field.

AD'JOINT, *adj.*, *n.* **adjoint of a differential equation.** For a homogeneous differential equation

$$L(y) \equiv p_0 \frac{d^n y}{dx^n}+p_1 \frac{d^{n-1}y}{dx^{n-1}}+\cdots +p_{n-1}\frac{dy}{dx}+p_n y = 0,$$

the adjoint is the differential equation

$$\bar{L}(y) \equiv (-1)^n \frac{d^n(p_0 y)}{dx^n}+(-1)^{n-1}\frac{d^{n-1}(p_1 y)}{dx^{n-1}} +\cdots -\frac{d(p_{n-1}y)}{dx}+p_n y = 0.$$

This relation is symmetric, $L=0$ being the adjoint of $\bar{L} = 0$. A function is a solution of one of these equations if and only if it is an *integrating factor* of the other. There is an expression $P(u, v)$ for which

$$vL(u)-u\bar{L}(v) \equiv \frac{dP(u,v)}{dx}.$$

$P(u, v)$ is linear and homogeneous in $u, u', \cdots, u^{(n-1)}$, and in $v, v', \cdots, v^{(n-1)}$. It is known as the **bilinear concomitant**. An equation is **self-adjoint** if $L(y) = \bar{L}(y)$. *E.g.*, Sturm-Liouville differential equations and Legendre differential equations are self-adjoint.

adjoint of a matrix. The *transpose* of the matrix obtained by replacing each element by its *cofactor*; the matrix obtained by replacing each element a_{rs} (in row r and column s) by the cofactor of the element a_{sr} (in row s and column r). The adjoint is defined only for square matrices. Sometimes (rarely) the adjoint is called the **adjugate**, although adjugate has also been used for the square matrix of order $\binom{n}{r}$ formed from a square matrix of order n by arranging all rth-order minors in some specified order. The *Hermitian conjugate* matrix is frequently called the adjoint matrix by writers on quantum mechanics.

adjoint of a transformation. For a bounded linear transformation T which maps a **Hilbert space** H into H (with domain of T equal to H), there is a unique bounded linear transformation T^*, the *adjoint* of T, such that the inner products (Tx, y) and (x, T^*y) are equal for all x and y of H. It follows that $\|T\| = \|T^*\|$. Two linear transformations T_1 and T_2 are said to be *adjoint* if $(T_1x, y) = (x, T_2y)$ for each x in the domain of T_1 and y in the domain of T_2. If T is a linear transformation whose domain is dense in H, there is a unique transformation T^* (called the *adjoint* of T) such that T and T^* are adjoint and, if S is any other transformation adjoint to T, then the domain of S is contained in the domain of T^* and S and T^* coincide on the domain of S. For a **finite dimensional space** and a transformation T which maps vectors $x = (x_1, x_2, \cdots, x_n)$ into $Tx = (y_1, y_2, \cdots, y_n)$ with $y_i = \sum_j a_{ij}x_j$ (for each i), the adjoint of T is the transformation for which $T^*x = (y_1, y_2, \cdots, y_n)$ with $y_i = \sum_j \bar{a}_{ji}x_i$ and the matrices of the coefficients of T and of T^* are *Hermitian conjugates* of each other. If T is a bounded linear transformation which maps a **Banach space** X into a Banach space Y, and X^* and Y^* are the *first conjugate spaces* of X and Y, then the adjoint of T is the linear transformation T^* which maps Y^* into X^* and is such that $T^*(g) = f$ (for f and g members of X^* and Y^*, respectively) if f is the continuous linear functional defined by $f(x) = g[T(x)]$. For two bounded linear transformations T_1 and T_2, the adjoints of $T_1 + T_2$ and $T_1 \cdot T_2$ are $T_1^* + T_2^*$ and $T_2^* \cdot T_1^*$, respectively. If T has an inverse whose domain is all of H (or Y), then $(T^*)^{-1} = (T^{-1})^*$. For Banach spaces, the adjoint T^{**} of T^* is a mapping of X^{**} into Y^{**} which is a norm-preserving extension of T (T maps a subset of X^{**}, which is isometric with X, into Y^{**}). For Hilbert space, T^{**} is identical with T if T is bounded with domain H; T^{**} is a linear extension of T otherwise. **See** SELF—self-adjoint transformation.

adjoint space. See CONJUGATE—conjugate space.

AD'JU-GATE, *n.* See ADJOINT—adjoint of a matrix.

AD-MIS'SI-BLE, *adj.* **admissible hypothesis.** See HYPOTHESIS.

AF-FINE', *adj.* **affine transformation.** (1) A transformation of the form

$$x' = a_1x + b_1y + c_1, \quad y' = a_2x + b_2y + c_2,$$

where $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0.$

(2) A transformation of the form given in (1) except that the determinant of the coefficients may or may not be zero (it is **singular** or **nonsingular** according as this determinant is zero or nonzero). The determinant of the coefficients is denoted by Δ. The following are important special cases of the affine transformation, $\Delta \neq 0$: (a) **translations** ($x' = x + a$, $y' = y + b$); (b) **rotations** ($x' = x \cos\theta + y \sin\theta$, $y' = -x \sin\theta + y \cos\theta$); (c) **stretchings and shrinkings** ($x' = kx$, $y' = ky$), called *transformations of similitude* or *homothetic transformations*; (d) **reflections** in the x-axis and y-axis, respectively ($x' = x, y' = -y$ or $x' = -x, y' = y$); (e) **simple elongations and compressions** ($x' = x, y' = ky$ or $x' = kx, y' = y$); (f) **simple shear transformations** ($x' = x + ky$, $y' = y$ or $x' = x$, $y' = kx + y$). The affine transformation carries parallel lines into parallel lines, finite points into finite points, and leaves the line at infinity fixed. An affine transformation can always be factored into the product of transformations belonging to the above special cases. An **homogeneous affine transformation** is an affine transformation in which the constant terms are zero; an affine transformation which does not contain a translation as a factor. Its form is

$$x' = a_1x + b_1y, \quad y' = a_2x + b_2y,$$
$$\Delta = a_1b_2 - a_2b_1 \neq 0.$$

An **isogonal affine transformation** is an affine transformation which does not change the size of angles. It has the form

$$x' = a_1x + b_1y + c_1,$$
$$y' = a_2x + b_2y + c_2,$$

where either $a_1 = b_2$ and $a_2 = -b_1$ or $-a_1 = b_2$ and $a_2 = b_1$.

AGE, *n.* (*Life Insurance*) The **age at issue** is the age of the insured at his birthday nearest the policy date. The **age year** is a year in the lives of a group of people of a certain age. The age year l_x refers to the year from x to $x+1$, the year during which the group is x years old.

AG'GRE-GA'TION, *n.* **signs of aggregation: Parenthesis,** ( ); **bracket,** [ ]; **brace,** { }; and **vinculum** or **bar,** ——. Each means that the terms enclosed are to be treated as a single term. *E.g.*, $3(2-1+4)$ means 3 times 5, or 15. See various headings under DISTRIBUTIVE.

AGNESI, Maria Gaetana (1718–1799). Distinguished Italian mathematician. See WITCH.

AHLFORS, Lars Valerian (1907–). Finnish-American mathematician and Fields Medal recipient (1936). Research in complex-variable theory and theory of quasiconformal mappings, with important contributions to Riemann surfaces and meromorphic functions.

AHMES (RHYND or RHIND) PAPYRUS. Probably the oldest mathematical book known, written 2000 to 1800 B.C. and copied by the Egyptian scribe Ahmes about 1650 B.C. See RHIND.

ALBERT, Abraham Adrian (1905–1972). American algebraist. Made fundamental contributions to the theory of Riemannian matrices and to the structure theory of associative and nonassociative algebras, Jordan algebras, quasigroups, and division rings.

ALBERTI, Leone Battista (1404–1472). Italian mathematician and architect. Wrote on art, discussing perspective and raising questions that pointed toward the development of projective geometry.

A'LEPH, *n.* The first letter of the Hebrew alphabet, written $\aleph$.

aleph-null or **aleph-zero.** The *cardinal number* of countably infinite sets, written $\aleph_0$. See CARDINAL—cardinal number.

ALEXANDER, James Waddell (1888–1971). American algebraic topologist who did research in complex-variable theory, homology and ring theory, fixed points, and the theory of knots.

Alexander's subbase theorem. A topological space is *compact* if and only if there is a subbase S for its topology which has the property that, whenever the union of a collection of members of S contains X, then X is contained in the union of a finite number of members of this collection.

AL'GE-BRA, *n.* (1) A generalization of arithmetic. *E.g.*, the arithmetic facts that $2+2+2 = 3\times 2$, $4+4+4 = 3\times 4$, etc., are all special cases of the (general) algebraic statement that $x+x+x = 3x$, where x is any number. Letters denoting any number, or any one of a certain set of numbers, such as all real numbers, are related by laws that hold for any numbers in the set; *e.g.*, $x+x = 2x$ for all x (all numbers). On the other hand, conditions may be imposed upon a letter, representing any one of a set, so that it can take on but one value, as in the study of equations; *e.g.*, if $2x+1 = 9$, then x is restricted to 4. Equations are met in arithmetic, although not so named. For instance, in percentage one has to find one of the unknowns in the equation, interest = principal × rate, or $I = p\times r$, when the other two are given. (2) A system of logic expressed in algebraic symbols, or a Boolean algebra (see BOOLEAN). (3) See below, algebra over a field.

algebra over a field. An **algebra** (or **linear algebra**) over a field F is a ring R that is also a vector space with members of F as scalars and satisfies $(ax)(by) = (ab)(xy)$ for all scalars a and b and all members x and y of R. The dimension of the vector space is the **order** of R. The algebra is a **commutative algebra,** or an **algebra with unit element,** according as the ring is a commutative ring, or a ring with unit element. A **division algebra** is an algebra that is also a division ring. A **simple algebra** is an algebra that is a simple ring. The set of real numbers is a commutative division algebra over the field of rational numbers; for any positive integer n, the set of all square matrices of order n with complex numbers (or real numbers) as elements is an algebra (noncommutative) over the field of real numbers. Any algebra consisting of all n by n matrices with elements in a given field is a simple algebra. An algebra of order n with a unit element is isomorphic to an algebra of n by n square matrices.

algebra of propositions. See BOOLE—Boolean algebra.

algebra of subsets. An algebra of subsets of a set X is a class of subsets of X which contains the *complement* of each of its members and the *union* of any two of its members (or the *intersection* of any two of its members). It is called a **σ-algebra** if it also contains the union of any sequence of its members. An algebra of subsets is a *Boolean algebra* relative to the operations of union and intersection. A *ring* of subsets of a set X is an algebra of subsets of X if and only if it contains X as a member. For any class C of subsets of a set X, the intersection of all algebras (or σ-algebras) which contain C is the smallest algebra (σ-algebra) which contains C and is said to be the algebra (σ-algebra) generated by C. For the real line (or n-dimensional space) examples of σ-algebras are the system of all measurable sets, the system of all Borel sets, and the system of *all* sets having the property of Baire. See RING—ring of sets.

Banach algebra. An algebra over the field of real numbers (or complex numbers) which is also a real (or complex) Banach space for which $\|xy\| \le \|x\|\cdot\|y\|$ for all x and y. It is called a **real** or a **complex Banach algebra** according as the field is the real or the complex number field. The set of all functions which are continuous on the closed interval [0, 1] is a Banach algebra over the field of real numbers if $\|f\|$ is defined to be the largest value of $|f(x)|$ for $0 \leq x \leq 1$. *Syn.* normed vector ring.

Boolean algebra. See BOOLE.

fundamental theorem of algebra. See FUNDAMENTAL—fundamental theorem of algebra.

measure algebra. See MEASURE—measure ring and measure algebra.

AL'GE-BRAIC, *adj.* **algebraic adder.** See ADDER.

algebraic addition. See SUM—algebraic sum, sum of real numbers.

algebraic curve. See CURVE.

algebraic deviation. See DEVIATION.

algebraic expression, equation, function, operation, etc. An expression, etc., containing or using only algebraic symbols and operations, such as $2x+3$, x^2+2x+4, or $\sqrt{2}-x+y=3$. **Algebraic operations** are the operations of addition, subtraction, multiplication, division, extraction of roots, and raising to integral or fractional powers. A **rational algebraic expression** is an expression that can be written as a quotient of polynomials. An **irrational algebraic expression** is one that is not rational, as $\sqrt{x+4}$. See FUNCTION—algebraic function and various headings under RATIONAL.

algebraic extension of a field. See EXTENSION.

algebraic hypersurface. See HYPERSURFACE.

algebraic multiplication. See MULTIPLICATION.

algebraic number. (1) Any ordinary positive or negative number; any real directed number. (2) Any number which is a root of a polynomial equation with rational coefficients; the degree of the polynomial is said to be the **degree** of the algebraic number α, and the equation is the **minimal equation** of α, if α is not a root of such an equation of lower degree. An **algebraic integer** is an algebraic number which satisfies some *monic* equation,

$$x^n + a_1x^{n-1} + \cdots + a^n = 0,$$

with *integers* as coefficients. The minimal equation of an algebraic integer is also monic. A rational number is an algebraic integer if and only if it is an ordinary integer. The set of all algebraic numbers is an integral domain (see DOMAIN—integral domain). (3) Let F^* be a field and F a subfield of F^*. A member c of F^* is **algebraic** with respect to F if c is a zero of a polynomial with coefficients in F; otherwise, c is **transcendental** with respect to F.

algebraic operations. Addition, subtraction, multiplication, division, evolution and involution (extracting roots and raising to powers).

algebraic proofs and solutions. Proofs and solutions which use algebraic symbols and no operations other than those which are algebraic. See above, algebraic operations.

algebraic subtraction. See SUBTRACTION.

algebraic symbols. Letters representing numbers, and the various operational symbols indicating *algebraic operations.* See MATHEMATICAL SYMBOLS in the appendix.

algebraic variety. Let V be an n-dimensional vector space with F the field of scalars. An **algebraic variety** is a subset of V that is the set of all points $(x_1, \cdots, x_n)$ which satisfy a finite set of polynomial equations $f_k(x_1, \cdots, x_n) = 0$, $k = 1, 2, \cdots, m$.

irrational algebraic surface. See IRRATIONAL.

AL-GE-BRA'IC-AL-LY, *a.* **algebraically complete field.** A field F which has the property that every polynomial equation with coefficients in F has a root in F. The field of algebraic numbers and the field of complex numbers are algebraically complete. Every field has an extension that is algebraically complete. *Syn.* algebraically closed field.

AL'GO-RITHM, *n.* Some special process of solving a certain type of problem, particularly a method that continually repeats some basic process.

division algorithm. See DIVISION.

Euclid's algorithm. A method of finding the greatest common divisor (G.C.D.) of two numbers—one number is divided by the other, then the second by the remainder, the first remainder by the second remainder, the second by the third, etc. When exact division is finally reached, the last divisor is the greatest common divisor of the given numbers (integers). In algebra, the same process can be applied to polynomials. *E.g.*, to find the greatest common divisor of 12 and 20, we have $20 \div 12$ is 1 with remainder 8; $12 \div 8$ is 1 with remainder 4; and $8 \div 4 = 2$; hence 4 is the G.C.D.

AL'IEN-A'TION, *n.* **coefficient of alienation.** See CORRELATION—normal correlation.

A-LIGN'MENT, *adj.* **alignment chart.** Same as NOMOGRAM.

AL'I-QUOT PART. Any exact divisor of a quantity; any factor of a quantity; used almost entirely when dealing with integers. *E.g.*, 2 and 3 are *aliquot parts* of 6.

AL'MOST, *adj.* **almost all** and **almost everywhere.** See MEASURE—measure zero.

almost periodic. See PERIODIC.

AL'PHA, *n.* The first letter in the Greek alphabet: lower case, α; capital, A.

AL-TER'NANT, *n.* *A determinant for which* there are n functions $f_1, f_2, \cdots, f_n$ (if the determinant is of order n) and n quantities $r_1, r_2, \cdots, r_n$ for which the element in the ith column and jth row is $f_i(r_j)$ for each i and j (this determinant with rows and columns interchanged is also called an alternant). The Vandermonde determinant is an alternant (see DETERMINANT).

AL'TER-NATE, *adj.* Two **alternate angles** are angles on opposite sides of a transversal cutting two lines, each having one of the lines for one of its sides. They are **alternate exterior angles** if neither lies between the two lines cut by the transversal. They are **alternate interior angles** if both lie between the two lines. See ANGLE—angles made by a transversal.

AL-TER-NAT'ING, *adj.* **alternating function.** A function f of more than one variable for which $f(x_1, x_2, \cdots, x_n)$ changes sign if two of the variables are interchanged. If f is an alternating multilinear function of n vectors in an n-dimensional vector space and $\{e_1, e_2, \cdots, e_n\}$ is a basis for V, then

$$f(v_1, \cdots, v_n) = \det(v_{ij}) f(e_1, \cdots, e_n),$$

where $v_k = \sum_{i=1}^{n} v_{ik} e_i$ and $\det(v_{ij})$ is the determinant with v_{ij} in the ith row and jth column.

alternating group. See GROUP—alternating group.

alternating series. A series whose terms are alternately positive and negative, as

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots + (-1)^{n-1}/n + \cdots,$$

An alternating series converges if each term is numerically equal to or less than the preceding and if the nth term approaches zero as n increases without limit. This is a sufficient, but not a necessary set of conditions—the term-by-term sum of any two convergent series converges and, if one series has all negative terms and the other all positive terms, their indicated sum may be a convergent alternating series and not have its terms monotonically decreasing. The series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{8} + \tfrac{1}{27} - \tfrac{1}{16} + \cdots$$

is such a series. See NECESSARY—necessary condition for convergence.

AL'TER-NA'TION, *n.* (1) In logic, the same as DISJUNCTION. (2) See PROPORTION.

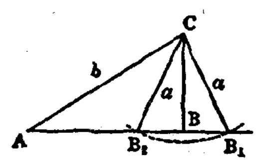
AL-TER'NA-TIVE, *adj.* **alternative hypothesis.** See HYPOTHESIS—test of a hypothesis.

AL'TI-TUDE, *n.* A line segment indicating *the height of a figure in some sense (or the* length of such a line segment). See CONE, CYLINDER, PARABOLIC—parabolic segment, PARALLELOGRAM, PARALLELEPIPED, PRISM, PYRAMID, RECTANGLE, SEGMENT—spherical segment, TRAPEZOID, TRIANGLE, ZONE.

altitude of a celestial point. Its angular distance above, or below, the observer's horizon, measured along a great celestial circle (vertical circle) passing through the point, the zenith, and the nadir. The altitude is taken positive when the object is above the horizon and negative when below. See figure under HOUR—hour-angle and hour-circle.

AM-BIG'U-OUS, *adj.* *Not uniquely determinable.*

ambiguous case in the solution of triangles. For a **plane triangle,** the case in which two sides and the angle opposite one of them is given. One of the other angles is then found by use of the law of sines; but there are always two angles less than 180° corresponding to any given value of the sine (unless the sine be unity, in which case the angle is 90° and the triangle is a right triangle). When the sine gives two distinct values of the angle, two triangles result if the side opposite is less than the side adjacent to the given angle (assuming the data are not such that there is no triangle possible, a situation that may arise in any case, ambiguous or nonambiguous). In the figure, angle A and sides a and b are given ($a < b$); triangles AB_1C and AB_2C are both solutions. If $a = b \sin A$, the right triangle ABC is the unique solution.



For a **spherical triangle,** the ambiguous case is the case in which a side and the opposite angle are given (the given parts may then be either two sides and an angle opposite one side, or two angles and the side opposite one angle).

A-MER'I-CAN, *adj.* **American experience table of mortality.** See MORTALITY.

AM'I-CA-BLE, *adj.* **amicable numbers.** Two numbers, each of which is equal to the sum of all the exact divisors of the other except the number itself. *E.g.*, 220 and 284 are amicable numbers, for 220 has the exact divisors 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110, whose sum is 284; and 284 has the exact divisors 1, 2, 4, 71, and 142, whose sum is 220.

A-MOR-TI-ZA'TION, *n.* **amortization of a debt.** The discharge of the debt, including interest, by periodic payments, usually equal, which continue until the debt is paid without any renewal of the contract. The mathematical principles are the same as those used for annuities.

amortization equation. An equation relating the amount of an obligation to be amortized, the interest rate, and the amount of the period payments. See AMORTIZE.

amortization of a premium on a bond. Writing down (decreasing) the book value of the bond on each dividend date by an amount equal to the difference between the dividend and the interest on the investment (interest on the book value at the yield rate). See VALUE—book value.

amortization schedule. A table giving the annual payment, the amount applied to principal, the amount applied to interest, and the balance of principal due. See AMORTIZE.

A-MOUNT', *n.* **amount of a sum of money at a given date.** The sum of the principal and interest (simple or compound) to the date; designated as *amount at simple interest* or *amount at compound interest* (or *compound amount*), according as interest is simple or compound. In practice, the word *amount* without any qualification usually refers to amount at compound interest.

amount of an annuity. See ACCUMULATED—accumulated value of an annuity at a given date.

compound amount. See COMPOUND.

AM'PERE, *n.* A unit of measure of electric current. The legal standard of current since 1950 is the **absolute ampere**, which is the current in each of two long parallel wires which carry equal currents and for which there is a force of $2 \cdot 10^{-7}$ newton per meter acting on each wire. The legal standard of current before 1950 was the **international ampere,** which is the current which when passed through a standard solution of silver nitrate deposits silver at the rate of .001118 gram per sec. One international ampere equals 0.999835 absolute ampere. See COULOMB, OHM.

AM'PLI-TUDE, *n.* **amplitude of a complex number.** The angle that the vector representing the complex number makes with the positive horizontal axis. *E.g.*, the *amplitude* of $2+2i$ is 45°. See POLAR—polar form of a complex number.

amplitude of a curve. Half the difference between the greatest and the least values of the ordinates of a periodic curve. The *amplitude* of $y = \sin x$ is 1; of $y = 2 \sin x$ is 2.

amplitude of a point. See POLAR—polar coordinates in the plane.

amplitude of simple harmonic motion. See HARMONIC—simple harmonic motion.

AN'A-LOG, *adj.* **analog computer.** See COMPUTER.

A-NAL'O-GY, *n.* A form of inference sometimes used in mathematics to set up new theorems. It is reasoned that, if two or more things agree in some respects, they will probably agree in others. Exact proofs must, of course, be made to determine the validity of any theorems set up by this method.

A-NAL'Y-SIS, *n.* [*pl.* **analyses**]. That part of mathematics which uses, for the most part, algebraic and calculus methods—as distinguished from such subjects as synthetic geometry, number theory, and group theory.

analysis of a problem. The exposition of the principles involved; a listing, in mathematical language, of the data given in the statement of the problem, other related data, the end sought, and the steps to be taken.

analysis of variance. See VARIANCE.

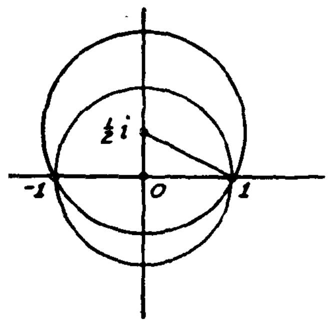
analysis situs. The field of mathematics now called *topology*.

diophantine analysis. See DIOPHANTUS.

proof by analysis. Proceeding from the thing to be proved to some known truth, as opposed to synthesis which proceeds from the true to that which is to be proved. The most common method of *proof by analysis* is, in fact, by *analysis* and *synthesis*, in that the steps in the analysis are required to be reversible.

unitary analysis. A system of analysis that proceeds from a given number of units to a unit, then to the required number of units. Consider the problem of finding the cost of 7 tons of hay if 2½ tons cost $25.00. Analysis: If 2½ tons cost $25.00, 1 ton costs $10.000. Hence 7 tons cost $70.00.

AN-A-LYT'IC, *adj.* **analytic continuation.** If f is given to be a single-valued analytic function in a domain D, then possibly there is a function F analytic in a domain of which D is a proper subdomain, and such that $F(z) = f(z)$ in D. If so, the function F is necessarily unique.



The process of obtaining F from f is called **analytic continuation.** *E.g.*, the function f defined by $f(z) = 1+z+z^2+z^3+\cdots$ is thereby defined only for $|z| < 1$, the radius of convergence of the series being 1 and the circle of convergence having center at 0. The series represents the function $1/(1-z)$, but if this function is given a new representation, say by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{1}{2}i)}{n!}\left(z-\frac{1}{2}i\right)^n,$$

where the coefficients are determined from the original series, the new circle of convergence extends outside the old one (see the figure). The given function f, usually given as a power series (but not necessarily so), is called a **function-element** of F. The analytic continuation might well lead to a many-sheeted Riemann surface of definition of F. See MONOGENIC—monogenic analytic function.

analytic curve. A curve in n-dimensional Euclidean space which, in the neighborhood of each of its points, admits a representation of the form $x_j = x_j(t)$, $j = 1, 2, \cdots, n$, where the $x_j(t)$ are real analytic functions of the real variable t. If in addition we have $\sum_{j=1}^{n} (x_j')^2 \neq 0$, the curve is said to be a **regular analytic curve** and the parameter t is a **regular parameter** for the curve. For three-dimensional space, an analytic curve is a curve which has a parametric representation $x = x(t)$, $y = y(t)$, $z = z(t)$, for which each of these functions is an analytic function of the real variable t; it is a regular analytic curve if dx/dt, dy/dt and dz/dt do not vanish simultaneously. See POINT—ordinary point of a curve.

analytic function of a complex variable. A *single-valued function*, f, or a *multiple-valued* function considered as a single-valued function on its Riemann surface, which is differentiable at each point of its *domain* (a nonnull connected open set) of definition D is **analytic in D**. It can be shown that an analytic function f of a complex variable has continuous derivatives of all orders and can be expanded as a Taylor series in a neighborhood of each point z_0 of D:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n.$$

A function is sometimes said to be analytic in D if it is continuous in D and has a derivative at all except at most a finite number of points of D. If f is differentiable at all points of D, it is a **regular function**, or a **regular analytic function**, or a **holomorphic function**, in D. See CAUCHY—Cauchy-Riemann partial differential equations, MONOGENIC.

analytic function of a real variable. A function f is analytic at h if it can be represented by a Taylor's series in powers of $(x-h)$ whose sum is equal to the value of the function at each x in some neighborhood of h. The function is said to be analytic in the interval (a, b) if the above is true for every h in the interval (a, b). See TAYLOR—Taylor's theorem.

analytic geometry. See GEOMETRY—analytic geometry.

analytic at a point. A single-valued function f of the complex variable z is **analytic** at the point z_0 if there is a neighborhood N of z_0 such that f is differentiable at every point of N. *I.e.*, f is analytic at z_0 if it is analytic in a neighborhood of z_0. *Syn.* holomorphic, regular, or monogenic at a point. See above, analytic function of a complex variable.

analytic proof or solution. A proof or solution which depends upon that sort of procedure in mathematics called analysis; a proof which consists, essentially, of algebraic (rather than geometric) methods and/or of methods based on limiting processes (such as the methods of differential and integral calculus).

analytic structure for a space. A covering of a *locally Euclidean space* of dimension n by a set of open sets each of which is homeomorphic to an open set in n-dimensional Euclidean space E_n and which are such that whenever any two of these open sets overlap, the coordinate transformation (in both directions) is given by analytic functions (*i.e.*, *functions which can be expanded in power* series in some neighborhood of any point). If neighborhoods U and V overlap and P is in their intersection, then the homeomorphisms *of U and V with open sets of n-dimensional* Euclidean space define coordinates $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ for P, and the functions $x_i = x_i(y_1, \cdots, y_n)$ and $y_i = y_i(x_1 \cdots, x_n)$ are the functions required to be analytic. The analytic structure is **real** or **complex** according as the coordinates of points in E_n are taken as real or complex numbers. See EUCLIDEAN—locally Euclidean space, MANIFOLD.

a-point of an analytic function. An a-point of the analytic function $f(z)$ of the complex variable z is a *zero point* of the analytic function $f(z)-a$. The order of an a-point is the order of the zero of $f(z)-a$ at the point. See ZERO—zero point of an analytic function of a complex variable.

normal family of analytic functions. See NORMAL.

quasi-analytic function. For a sequence of positive numbers $\{M_1, M_2, \cdots\}$ and a closed interval $[a, b] = I$, the *class of quasi-analytic functions* is the set of all functions which possess derivatives of all orders on I and which are such that for each function f there is a constant k such that

$$|f^{(n)}(x)| < k^n M_n \quad \text{for} \quad n \geq 1 \quad \text{and} \quad x \in I,$$

provided this set of functions has the property that if f is a member of the set and $f^{(n)}(x_0) = 0$ for $n \geq 0$ and $x_0 \in I$, then $f(x) \equiv 0$ on I. If $M_n = n!$, or $M_n = n^n$, then the corresponding class of functions is precisely the class of all analytic functions on I. Every function which possesses derivatives of all orders on I (*e.g.*, e^{-1/x^2} on $[0, 1]$) is the sum of two functions each of which belongs to a quasi-analytic class. Even if the class defined by $M_1, \cdots$ and I is not quasi-analytic, certain subclasses are sometimes said to be quasi-analytic if they do not contain a nonzero function f for which $f^{(n)}(x_0) = 0$ for $n \geq 0$ and $x_0 \in I$. Quasi-analyticity is one of the most important properties of analytic functions, but there exist classes of quasi-analytic functions which contain nonanalytic functions.

singular point of an analytic function. See SINGULAR.

AN'A-LYT'I-CAL-LY, *adj.* Performed by analysis, by analytic methods, as opposed to synthetic methods.

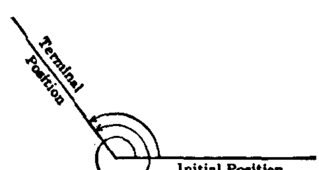
AN'A-LY-TIC'I-TY, *n.* **point of analyticity.** A point at which a function of the complex variable z is analytic.

ANCHOR RING or TORUS. A surface in the shape of a doughnut with a hole in it; a surface generated by the rotation, in space, of a circle about an axis in its plane but not cutting the circle. If r is the radius of the circle, k the distance from the center to the axis of revolution, in this case the z-axis, and the equation of the generating circle is $(y-k)^2+z^2 = r^2$, then the equation of the anchor ring is

$$(\sqrt{x^2+y^2}-k)^2+z^2 = r^2.$$

Its volume is $2\pi^2kr^2$ and the area of its surface is $4\pi^2kr$.

AN'GLE, *n.* A **geometric angle** (or simply **angle**) is a set of points consisting of a point P and two rays extending from P (sometimes it is required that the rays do not lie along the same straight line). The point P is the **vertex** and the rays are the **sides** (or **rays**) of the angle. Two geometric angles are **equal** if and only if they are congruent. When the two rays of an angle do not extend along the same line in opposite directions from the vertex, the set of points between the rays is the **interior** of the angle. The **exterior** of an angle is the set of all points in the plane that are not in the union of the angle and its interior. A directed angle, then a **radian measure** of the angle for which one ray is designated as the initial side and the other as the terminal side. There are two commonly used signed measures of directed angles. If a circle is drawn with unit radius and center at the vertex of a directed angle, then a **radian measure** of the angle is the length of an arc that extends counterclockwise along the circle from the initial side to the terminal side of the angle, or the negative of the length of an arc that extends clockwise along the circle from the initial side to the terminal side. The arc may wrap around the circle any number of times. For example, if an angle has radian measure $\frac{1}{2}\pi$, it also has radian measure $\frac{1}{2}\pi+2\pi$, $\frac{1}{2}\pi+4\pi$, etc., or $\frac{1}{2}\pi-2\pi$, $\frac{1}{2}\pi-4\pi$, etc. **Degree measure** of an angle is defined so that 360° corresponds to radian measure of 2π [see SEXAGESIMAL—



sexagesimal measure of an angle]. A **rotation angle** consists of a directed angle *and* a signed measure of the angle. The angle is a **positive angle** or a **negative angle** according as the measure is positive or negative. **Equal** rotation angles are rotation angles that have the same measure. Usually, **angle** means rotation angle (*e.g.*, see below, angle of depression, angle of inclination, obtuse angle). A rota-