Frank Jones

Lebesgue Integration on Euclidean Space

Revised Edition

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Frank Jones
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Preface

"Though of real knowledge there be little, yet
of books there are plenty"
–Herman Melville, *Moby Dick*, Chapter XXXI.

The treatment of integration developed by the French mathematician Henri Lebesgue (1875–1944) almost a century ago has proved to be indispensable in many areas of mathematics. Lebesgue's theory is of such extreme importance because on the one hand it has rendered previous theories of integration virtually obsolete, and on the other hand it has not been replaced with a significantly different, better theory. Most subsequent important investigations of integration theory have extended or illuminated Lebesgue's work.

In fact, as is so often the case in a new field of mathematics, many of the best consequences were given by the originator. For example, Lebesgue's dominated convergence theorem, Lebesgue's increasing convergence theorem, the theory of the Lebesgue function of the Cantor ternary set, and Lebesgue's theory of differentiation of indefinite integrals.

Naturally, many splendid textbooks have been produced in this area. I shall list some of these below. They are quite varied in their approach to the subject. My aims in the present book are as follows.

1. To present a *slow* introduction to Lebesgue integration. Most books nowadays take the opposite tack. I have no argument with their approach, except that I feel that many students who see only a very rapid approach tend to lack strong intuition about measure and integration. That is why I have made Chapter 2, "Lebesgue measure on $\mathbb{R}^n$," so lengthy and have restricted it to Euclidean space, and why I have (somewhat inconveniently) placed Chapter 3, "Invariance of Lebesgue measure," before Fubini's theorem. In my approach I have omitted much important material, for the sake of concreteness. As the title of the book signifies, I restrict attention almost entirely to Euclidean space.

2. To deal with n-dimensional spaces from the outset. I believe this is preferable to one standard approach to the theory which first thoroughly treats integration on the real line and then generalizes. There are several reasons for this belief. One is quite simply that significant figures are frequently easier to sketch in $\mathbb{R}^2$ than in $\mathbb{R}^1$! Another is that some things in $\mathbb{R}^1$ are so special that the generalization to $\mathbb{R}^n$ is not clear; for example, the structure of the most general open set in $\mathbb{R}^1$ is essentially trivial — it must be a disjoint union of open intervals (see Problem 2.6). A third is that coping with the n-dimensional case from the outset causes the learner to realize that it is not significantly more difficult than the one-dimensional case as far as many aspects of integration are concerned.

3. To provide a thorough treatment of Fourier analysis. One of the triumphs of Lebesgue integration is the fact that it provides definitive answers to many questions of Fourier analysis. I feel that without a thorough study of this topic the student is simply not well educated in integration theory. Chapter 13 is a very long one on the Fourier transform in several variables, and Chapter 14 also a very long one on Fourier series in one variable.

4. To prepare students to become "workers" in real analysis. I do not mean that they should become researchers, but instead that they be able to *apply* to other areas of interest to them the things they have seen in this book. As a certain sort of analyst myself, I have chosen to include those topics which I have found to be of primary importance in my own research. This purpose partially explains the inclusion of the two long Chapters 15 and 16 on differentiation theory. They are also here because of their beauty and depth.

This last aim seems to be ever growing in its importance, as we mathematicians are seeing more and more students from other disciplines taking our advanced courses. It is now commonplace to find engineering graduate students, for example, taking courses in integration theory, differential geometry, etc.

I have written this book under the assumption that the student either is already familiar with certain basic concepts or has a teacher. Thus, the introductory chapter on the basic facts about $\mathbb{R}^n$ is extremely brief, except that I have tried to give a fairly careful account of compactness (in $\mathbb{R}^n$). (I have done so because compactness is a serious stumbling block for many students.)

I confess that I am proud of the problems in this book. There are 600 of them, and I think most of them are interesting and neither trivial nor impossibly difficult. There are a few that are "challenging," and this is another reason for the utility of having a teacher. I have chosen to spread the problems throughout the text, in order to encourage the students and teacher to use them as an integral part of their study. Thus, when a problem appears as a subject is being developed, the indication to the students is that they are now ready for this exercise to check their knowledge and to strengthen their understanding of what is being discussed.

Bibliography

I am placing this in such a prominent position in order to acknowledge some debts and to encourage the reader to engage in further reading.

* Paul R. Halmos, *Measure Theory,* Springer-Verlag, 1988.

* H.L. Royden, *Real Analysis,* third edition, Macmillan, 1988.

The two books just cited are standard texts. They concentrate on theoretical aspects, especially the Carathéodory construction of measurable sets. They contain important topics which are not discussed at all in my book. For example, the Radon-Nikodym theorem, Egorov's theorem, the dual space of L^p.

* Richard L. Wheeden and Antoni Zygmund, *Measure and Integral,* Dekker, 1977.

This excellent book has similar aims to mine (it is quite concrete in its approach), but also concentrates on many technical aspects which I have not included.

* Walter Rudin, *Real and Complex Analysis,* third edition, McGraw-Hill, 1987.

* Herbert Federer, *Geometric Measure Theory,* Springer-Verlag, 1969.

Federer's book is perhaps the ultimate text on this subject. It seems to contain everything but Fourier analysis, including a complete course on measure theory in Chapter Two, "General measure theory." The really serious student should find great benefit in working through this chapter.

* de la Vallée Poussin, *Intégrales de Lebesgue,* Gauthier-Villars, 1950.

When I first learned this subject from Professor Bray in 1958, this was the text he used. A beautiful source for the generalization of the Fourier transform to distributions is

* L. Hörmander, *The Analysis of Linear Partial Differential Operators I,* Springer Study Edition, Springer-Verlag, 1990.

Chapter VII, "The Fourier transformation," contains a wealth of material, and the exercises in the Springer Study Edition are marvelously illuminating.

One deficiency of my book that I am sorely aware of is the dearth of historical information. A very thorough source of such material may be found in the work of

* Thomas Hawkins, *Lebesgue's Theory of Integration,* Univ. of Wisconsin Press, 1970.

I highly recommend the reader of my book to become acquainted with Hawkins' interesting book.

Finally, I have greatly benefited from two papers of Dale E. Varberg concerning differentiation theory. His influence is especially seen in Chapter 15. The references are

* "On absolutely continuous functions," *Amer. Math. Monthly* *72* (1965), 831–841;
* "On differentiable transformations in $\mathbb{R}^n$," *Amer. Math. Monthly* *73* (1966, Number 4, Part II), 111–114.

Acknowledgments

The mathematicians who first introduced me to the subject of this book were Hubert E. Bray and Jim Douglas, Jr. Much of which I have included I originally learned from them. But I must say that it was Professor Bray to a much greater extent, as I took a year-long course from him on integration theory and Fourier series in 1958–59. Professor Bray (1889–1978) earned the first Ph.D. which Rice Institute ever awarded. His 1918 thesis was entitled, *A Green's Theorem in Terms of Lebesgue Integrals.* He was a faculty member of the Rice Department of Mathematics from then until his death. I still have on my shelf the splendid notes I took from his course over thirty years ago.

Many of my students have contributed significantly to this work. I am not going to name them all, but some of them are Jim Fox, Bob Hasse, Randy Mitchell, and Eric Swartz.

I am indebted to Calixto Calderón for discussions about Section 15.I.

I especially am indebted to the following people, who encouraged me to have this book published, and who constantly provided more-or-less friendly goading: John Brickner, Robert Burckel, John Cannon, Beverly Jones, Todd Simpson, and Mary Wheeler. Of these, Professors Burckel and Cannon each provided a detailed reading and very constructive criticism of earlier manuscripts. Just last year, Professor Burckel in fact reread the entire manuscript and provided me with many, many clever and useful observations and suggestions. I gladly acknowledge his splendid assistance.

Special thanks go to Janie McBane, the mathematical typist currently in the Rice mathematics department, and her teamwork with David Mallis; their major tool is a Macintosh computer and software called "TeX." And very special thanks to Anita Poley, who originally typed my manuscript many years ago, on bond paper, using an IBM Model D typewriter (pre-Selectric) and Typ-its!!! Both Janie and Anita are superb craftsmen (craftswomen?) at what they do, and they always do it *cheerfully*! They are absolute delights to work with!

I am most humbly grateful to have been a faculty member here at Rice since 1962. Working in our Department of Mathematics is the most wonderful occupation I can imagine. I have always been blessed with marveleous friendly stimulating colleagues both on the faculty and staff, and have been privileged to teach many truly remarkable and interesting students. Many of the latter have been guinea pigs for earlier drafts of this book.

Our current department coordinator, Sharon McDonough, is the greatest!

Carl Hesler of Jones and Bartlett has been a delight to work with.

Since I originally was taught by Professor Bray, it is especially thrilling to me recently to have had his grandsons Hubert L. Bray and Clark B. Bray as my students at Rice. In fact, young Hubert read my manuscript during a recent summer in order to learn the subject for himself and last year Clark was my student in this very course. I feel especially honored to have been the mediator between Hubert E. Bray and the brothers Bray as the grandfather taught the grandsons this beautiful subject.

To the most important people in my life:

Beverly, my wife
Marianna (and husband Jerry), Elaine, and David, our children
Nathan, Kara, Lauren, and Caroline, our grandchildren
Elizabeth, my mother

and the man who purchased us with his own blood
two thousand years ago.

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Introduction to $\mathbb{R}^n$

1

A. Sets

The real number system will be denoted $\mathbb{R}$. We shall be working on $\mathbb{R}^n$, the set of ordered n-tuples of real numbers, and shall use a notation such as x for points in $\mathbb{R}^n$: $x = (x_1, \ldots, x_n)$. If $n = 1$ we shall simply write x instead of x_1; and if $n = 2$ we shall frequently use the notation (x, y) instead of (x_1, x_2).

In general there is a notion of *Cartesian product*: If A_1, A_2, ..., A_N are sets, then

$$A_1 \times A_2 \times \cdots \times A_N$$

is the set of all ordered N-tuples $(a_1, a_2, \ldots, a_N)$ with $a_k \in A_k$ for $k =$ 1, 2, ..., N. In particular, $\mathbb{R}^n = \mathbb{R} \times \cdots \times \mathbb{R}$ (n factors). We shall not hesitate to write the equality $\mathbb{R}^l \times \mathbb{R}^m = \mathbb{R}^{l+m}$, although, strictly speaking, these two sets are not equal.

Given a set A contained in $\mathbb{R}^n$, the *complement* of A is the set

$$A^c = \{x \in \mathbb{R}^n \mid x \notin A\}.$$

Let $\emptyset$ denote the empty set. Then $(\mathbb{R}^n)^c = \emptyset$ and $\emptyset^c = \mathbb{R}^n$. It is always true that $A^{cc} = A$.

If A and B are sets, and if every member of A is also a member of B, then we say that A is *contained* in B, and we write $A \subset B$. We also write $B \supset A$. Since two sets are equal if and only if they have the same members, a proof that $A = B$ frequently follows the pattern of proving that $A \subset B$ and also that $B \subset A$.

If A and B are sets in $\mathbb{R}^n$, the *union* of A and B is the set

$$A \cup B = \{x \in \mathbb{R}^n \mid x \in A \quad \text{or} \quad x \in B\},$$

and the *intersection* of A and B is the set

$$A \cap B = \{x \in \mathbb{R}^n \mid x \in A \quad \text{and} \quad x \in B\}.$$

The sets A and B are said to be *disjoint* if $A \cap B = \emptyset$.

The *difference* $A \sim B$ is the set

$$A \sim B = A \cap B^c.$$

Now we shall generalize the notions of union and intersection to arbitrary collections of sets. To do this, let the symbol $\mathcal{I}$ stand for an arbitrary set used for indexing. The indices will be denoted by the letter i. Suppose that for each $i \in \mathcal{I}$ there corresponds a set A_i. Then the *union* of these sets is the set

$$\bigcup_{i \in \mathcal{I}} A_i = \{x \mid \text{there exists } i \in \mathcal{I} \text{ such that } x \in A_i\},$$

and the *intersection* is the set

$$\bigcap_{i \in \mathcal{I}} A_i = \{x \mid \text{for every } i \in \mathcal{I},\ x \in A_i\}.$$

We could also denote the union as $\cup\{A_i \mid i \in \mathcal{I}\}$ and the intersection as $\cap\{A_i \mid i \in \mathcal{I}\}$.

We say the sets are *disjoint* if $i \neq i' \Rightarrow A_i \cap A_{i'} = \emptyset$.

The system of *natural numbers* will be denoted $\mathbb{N} = \{1, 2, 3, \ldots\}$. In case the index set is $\mathbb{N}$, we shall usually write

$$\bigcup_{k=1}^{\infty} A_k \quad \text{and} \quad \bigcap_{k=1}^{\infty} A_k$$

for the union and intersection, respectively.

Problem 1 (DE MORGAN'S LAWS).

Prove that

$$\left(\bigcup_{i \in \mathcal{I}} A_i\right)^c = \bigcap_{i \in \mathcal{I}} A_i^c$$

and

$$\left(\bigcap_{i \in \mathcal{I}} A_i\right)^c = \bigcup_{i \in \mathcal{I}} A_i^c.$$