Partial Differential Equations of First Order and Their Applications to Physics

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PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER AND THEIR APPLICATIONS TO PHYSICS

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PREFACE

This book is a collection of notes of the course of partial differential equations given by *Dr* Gustavo López during different periods of time at different Universities. The main body of this book was put together during a seminar, given by Dr. G. López in collaboration with M. Murguía and M. Romero at the Physics Institute of the University of Guanajuato, from August to November of 1988. This collection was made with the help of M. A. Murguia, M. Romero, E. Benitez, and C. Melo. The final revision was made by Dr. López at the University of Guadalajara and at Los Alamos National Laboratory (Department of Non Linear Dynamics) in 1995–1996. I want to point out that without the help and enthusiasm of M. Murguía, M. Romero and C. Melo., the elaboration of these notes would not have been possible. I want to thank also to A. Taylor for her collaboration during the revision of the text.

In this book I try to point out the mathematical importance of Partial Differential Equations of First Order in Physics and Applied Sciences. The intention is to give to mathematicians a wide view of the application of this branch in physics, and to give to physicists and applied scientists a powerful tool for solving some problems appearing in Classical Mechanics, Quantum Mechanics, Optics, and General Relativity. This books is intended for senior or first year graduate students in mathematics, physics or engineering curricula.

Gustavo López

Mathematics is a gift ...
for man to understand the laws
which make up the whole Universe .

CONTENTS

CHAPTER I

Geometric Concepts and Generalities

In this chapter we shall study some geometric concepts that are basic to understand the geometric meaning of the partial differential equation in R^3.

1. Surfaces and Curves in Three Dimensions

By a surface S in R^3 we mean any relation between the rectangular cartesian coordinates (x, y, z) of a point in this space given by following expressions

$$(\text{ explicit}) \quad z = f(x, y), \tag{1.1}$$

$$(\text{ implicit}) \quad F(x, y, z) = 0, \tag{1.2}$$

$$(\text{parametric }) \quad x = f_1(u, v), y = f_2(u, v), z = f_3(u, v) \tag{1.3}$$

where to each pair of values of u, v there corresponds a set of numbers (x, y, z) and hence a point in space. While the expression for the surface (1.1) and (1.2) are unique, the parametric expression (1.3) is not unique. For example, the spherical surface ($x^2+y^2+z^2 = a^2$, see Fig. 1) can be parameterized by

$$x = a \sin u \cos v, \quad y = a \sin u \sin v, \quad z = a \cos u$$

or

$$x = a\frac{1-v^2}{1+v^2}\cos u, \quad y = a\frac{1-v^2}{1+v^2}\sin u, \quad z = \frac{2av}{1+v^2}$$

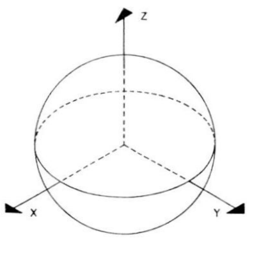

Fig. 1

or more general

$$x = a\frac{1-g(v)}{1+g(v)}\cos u, \quad y = a\frac{1-g(v)}{1+g(v)}\sin u, \quad z = \frac{2a\sqrt{g(v)}}{1+g(v)} \ ,$$

where $g(v)$ is such that $1 + g(v) > 0$ for all $v \in \Re$. By a curve Γ in R^3 we understand any relation between a point (x, y, z) in this space of the form

$$(\text{ non} - \text{parametric }) \qquad f(x, y, z) = 0; \quad g(x, y, z) = 0 \tag{1.4}$$

or

$$(\text{parametric }) \qquad x = f_1(t), \quad y = f_2(t), \quad z = f_3(t) \tag{1.5}$$

where t is a continuous variable called the parameter of the curve (a usual parameter is the length of a curve measured from some fixed point). The relation (1.4) expresses in fact the intersection of two surfaces (see Fig. 2).

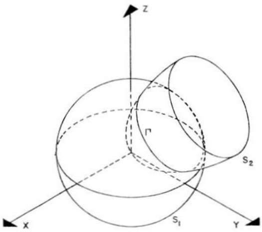

Fig. 2

A surface can be thought of as being generated by a set of curves in the following way, the surface $f(x, y, z) = 0$ is generated by the set of curves Γ_k defined by

$$z = k, \qquad f(x, y, k) = 0 \tag{1.6}$$

where k takes a certain interval of values, for example, the sphere (see Fig. 3) can be seen as generated by the curves Γ_k given by

$$z = k, \quad x^2 + y^2 = a^2 - k^2 \ .$$

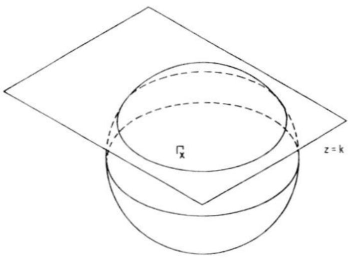

Fig. 3

Let a curve Γ be parameterized by the length of the curve s, and let

$$P_0 = (x(0), y(0), z(0)) \ ,$$

$$P = (x(s), y(s), z(s)) \ ,$$

and

$$Q = (x(s + \delta s), y(s + \delta s), z(s + \delta s))$$

be three points on Γ (see Fig. 4).

If δc is the Euclidean distance between the points P and Q, we restrict ourselves to those kind of curves which satisfy

$$\lim_{\delta s \to 0} \frac{\delta c}{\delta s} = 1 \; . \tag{1.7}$$

This means, for example, that we will not be interested in such a curves which turn around and cross themselves in some point. The direction cosines of the chord PQ are

$$\left(\frac{x(s+\delta s) - x(s)}{\delta c}, \frac{y(s+\delta s) - y(s)}{\delta c}, \frac{z(s+\delta s) - z(s)}{\delta c} \right) ,$$

and due to the Taylor's theorem,

$$x(s+\delta s) - x(s) = \delta s (dx/ds) + O(\delta s^2) \; ,$$

these direction cosines are reduced to

$$\frac{\delta s}{\delta c} \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right) + O(\delta s^2)$$

as δs tends to zero. The chord PQ takes up the direction of the tangent to the curve at P, and according to Eq. (1.7), the direction cosines of this tangent are

$$\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right) . \tag{1.8}$$

By a curve Γ given in parametric form, with s as the parameter, and passing upon a surface S given by expression (1.2) (cf. Fig. 5), we understand that the following identity is satisfied

$$F(x(s), y(s), z(s)) = 0 \tag{1.9}$$

for all the values s in the curve which lies on the surface.

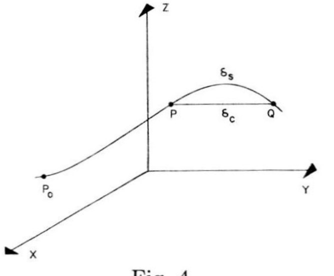

Fig. 4

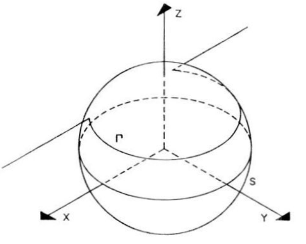

Fig. 5

If Eq. (1.9) is satisfied for all values of s, then the curve lies completely on the surface. Of course, if the curve is caused by the intersection of two surfaces, this curve lies completely on both surfaces. Differentiating Eq. (1.9) with respect to s, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds} + \frac{\partial F}{\partial z}\frac{dz}{ds} = 0 \; . \tag{1.10}$$

From the relations (1.8) and (1.10) we see that the tangent T to the curve Γ at any point P on the surface S is perpendicular to the gradient of F

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) , \tag{1.11}$$

and this is true for any curve Γ lying on S passing through P, then the vector ∇F is normal to the surface S at the point P (see Fig. 6).

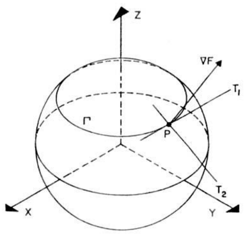

Fig. 6

If the equation of the surface S is given in the form $z = f(x, y)$, defining p and q as

$$p = \frac{\partial z}{\partial x} \text{ , and } \quad q = \frac{\partial z}{\partial y} , \tag{1.12}$$

and making $F = f(x, y) - z$, it follows that $F_x = p$, $F_y = q$, $F_z = -1$ and the unitary vector $\hat{n}$ normal to the surface at any point is

$$\hat{n} = \frac{1}{\left[p^2 + q^2 + 1\right]^{1/2}} (p, q, -1) . \tag{1.13}$$

Let $P = (x, y, z)$ be a point on the surface S defined by $F(x, y, z) = 0$ and let π_1 be the tangent plane at this point, if (X, Y, Z) is any other point on π_1 then, from the above discussion, the vector $(X - x, Y - y, Z - z)$ lying on the plane π_1, must be perpendicular to the normal direction ∇F at P, so the equation of the tangent plane π , (see Fig. 7) is

$$(X - x)\frac{\partial F}{\partial x} + (Y - y)\frac{\partial F}{\partial y} + (Z - z)\frac{\partial F}{\partial z} = 0 . \tag{1.14}$$

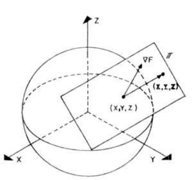

Fig. 7

Similarly, let S_2 be other surface defined by $G(x,y,z)=0$ which intersects the surface S_1 generating a curve Γ that passes through the point P. The equation for the tangent plane π_2 of this surface at the point P is

$$(X'-x)\frac{\partial G}{\partial x}+(Y'-y)\frac{\partial G}{\partial y}+(Z'-z)\frac{\partial G}{\partial z}=0\ , \tag{1.15}$$

where (X',Y',Z') is now any other point on this tangent plane π_2 (see Fig. 8).

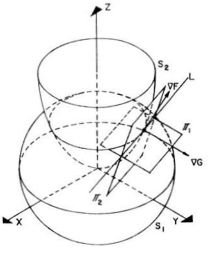

Fig. 8

The equation of the line L generated by the intersection o both planes π_1 and π_2 must be such that its direction cosines vector $(X''-x,\ Y''-y,\ Z''-z)$, where (X'',Y'',Z'') is now any other point on the line L, is perpendicular to ∇F and ∇G, that is, it must be parallel to the cross product of ∇F with ∇G,

$$\nabla F\times\nabla G=\left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial z}-\frac{\partial F}{\partial z}\frac{\partial G}{\partial y},\ \frac{\partial F}{\partial z}\frac{\partial G}{\partial x}-\frac{\partial F}{\partial x}\frac{\partial G}{\partial z},\ \frac{\partial F}{\partial x}\frac{\partial G}{\partial y}-\frac{\partial F}{\partial y}\frac{\partial G}{\partial x}\right)\ , \tag{1.16}$$

and therefore is proportional to this vector, establishing the following equations

$$\frac{X''-x}{\dfrac{\partial(F,G)}{\partial(y,z)}}=\frac{Y''-y}{\dfrac{\partial(F,G)}{\partial(z,x)}}=\frac{Z''-z}{\dfrac{\partial(F,G)}{\partial(x,y)}}\ , \tag{1.17}$$

where $\partial(F,G)/\partial(y,z)$ is given by

$$\frac{\partial(F,G)}{\partial(y,z)}=\det\begin{pmatrix}F_y & F_z\\ G_y & G_z\end{pmatrix}=F_yG_z-F_zG_y\ , \tag{1.18}$$

and so on. Choosing the point on L close enough to (x,y,z), i.e. $X''=x+dx, Y''=y+dy, Z''=z+dz$, and given F and G, then (1.17) has the following form

$$\frac{dx}{P(x,y,z)}=\frac{dy}{Q(x,y,z)}=\frac{dz}{R(x,y,z)}\ , \tag{1.19}$$

where P,Q, and R are known functions. The solution of (1.19) gives us the lines with tangents parallel to the vector field $(P(x,y,z),Q(x,y,z),R(x,y,z))$. These integral curves form a two-parameter family of curves in three dimensional space.

EXAMPLE 1. Give the tangent planes at the point

$P = \left(0, \sqrt{\sqrt{17}-1}/\sqrt{2}, (\sqrt{17}-1)/2\right)$ of the surface $x^2+y^2+z^2 = 4$ and $z = x^2+y^2$. Give their normal vectors and the equation of the tangent line generated by the intersection of the planes at this point. Give the curve Γ generated by the intersection of both surfaces.

In this case

$$F = x^2 + y^2 + z^2 - 4$$

and

$$G = x^2 + y^2 - z \ .$$

$\nabla F = (2x, 2y, 2z)$, the normal vector of the surface F at P is

$$\hat{n}_1 = \frac{1}{4}\left[0, [2\sqrt{17}-2]^{1/2}, \sqrt{17}-1\right] \ ,$$

$\nabla G = (2x, 2y, -1)$, the normal vector of the surface G at P is

$$\hat{n}_2 = \frac{1}{[\sqrt{17}-1]^{1/2}}\left[0, [2\sqrt{17}-2]^{1/2}, -1\right] .$$

$\dfrac{\partial(F,G)}{\partial(y,z)}, \dfrac{\partial(F,G)}{\partial(z,x)}$, and $\dfrac{\partial(F,G)}{\partial(x,y)}$ are given by $-2y(1+2z), 2x(1+4z)$, and 0. Therefore, at the point P they have the values $-[34(\sqrt{17}-1)]^{1/2}$, 0, and 0 respectively. The equations for the planes at the point P are according to Eq. (1.14) and Eq. (1.15)

$$[2\sqrt{17}-2]^{1/2}\, Y + (\sqrt{17}-1)Z = 8 \ ,$$

where Y, Z are coordinates of the plane π_1 and

$$[2\sqrt{17}-2]^{1/2}\, Y' - Z' = \frac{\sqrt{17}-1}{2} \ ,$$

where Y', Z' are coordinates of the plane π_2. The equations for the line lying on both tangent planes which is tangent to the surface at the point P is given, according to (1.17), by the equations

$$\frac{X''}{-[34(\sqrt{17}-1]^{1/2}} = \frac{Y'' - [(\sqrt{17}-1)/2]^{1/2}}{0} = \frac{Z'' - [(\sqrt{17}-1)/2]^{1/2}}{0} \ ,$$

or writing these equations in parametric way, it follows

$$X'' = -[34(\sqrt{17}-1)]^{1/2}\, s \ \ , \ Y'' = [(\sqrt{17}-1)/2]^{1/2} \ \ , \ Z'' = [(\sqrt{17}-1)/2]^{1/2}$$

(see Fig. 9).

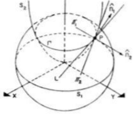

Fig. 9

The equation of the curve Γ, which is generated by the intersection of both surfaces and passes for the point P, is given by

$$x^2 + y^2 = \frac{\sqrt{17} - 1}{2}$$

EXERCISE 1. Find the tangent planes at the point $P = (0, \sqrt{3}, 1)$ of the surfaces $x^2+y^2+z^2 = 4$ and $z = 1$. Find the normal vectors to these tangent planes at that point, the line generated by the intersection of these planes, and the curve generated by the intersection of both surfaces.

2. Method of Solution of $dx/P = dy/Q = dz/R$

We pointed out in the last section that the integral curves of the set of differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{1.20}$$

form a two - parameter family of curves in three dimensional space. Suppose we are able to derive from Eq. (1.20) two relations of the form

$$u_1(x, y, z) = c_1 \quad \text{and} \quad u_2(x, y, z) = c_2, \tag{1.21}$$

where c_1 and c_2 are the constants of integration, then by varying these constants, we obtain a two - parameter family of curves satisfying the differential equations (1.20).

METHOD (I). Since any tangential direction (dx, dy, dz) at the point (x, y, z) on the surface $u_1(x, y, z) = c_1$ satisfies the relation

$$\frac{\partial u_1}{\partial x}\, dx + \frac{\partial u_1}{\partial y}\, dy + \frac{\partial u_1}{\partial z}\, dz = 0\ , \tag{1.22}$$

and according with the relations (1.19) we also have

$$\frac{\partial u_1}{\partial x}\, P + \frac{\partial u_1}{\partial y}\, Q + \frac{\partial u_1}{\partial z}\, R = 0\ . \tag{1.23}$$

To find u_1, we look for functions P', Q', and R' such that

$$P'P + Q'Q + R'R = 0\ , \tag{1.24}$$

i.e. a vector field $\mathbf{E}' = (P', Q', R')$ which is perpendicular to $\mathbf{E} = (P, Q, R)$ at every point (x, y, z). Because of Eq. (1.23), this vector field satisfies

$$P' = \frac{\partial u_1}{\partial x}, \quad Q' = \frac{\partial u_1}{\partial y}, \quad R' = \frac{\partial u_1}{\partial z}\ . \tag{1.25}$$

Then, with (1.22) we would have that

$$P'dx + Q'dy + R'dz$$

is an exact differential, du_1. The same procedure can be followed to obtain the other family of curves u_2.

EXAMPLE 2. Find the integral curves of the equations

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

the vector field **E** is given as

$$\mathbf{E} = (x(y-z), y(z-x), z(x-y)) \ .$$

If we take the vectors fields $\mathbf{E}' = (1,1,1)$ and $\mathbf{E}'' = (zy,\ zx,\ xy)$ the condition (1.24) is satisfied and the functions u_1, u_2 of equation (1.25) are

$$u_1 = x + y + z, \quad u_2 = xyz$$

hence, the integral curves of the given differential equations are the members of the two - parameter family

$$x + y + z = c_1, \quad xyz = c_2.$$

We must note that this method depends very much on the intuition and skill in determining the form of the vectors fields $\mathbf{E}'$, $\mathbf{E}''$.

EXERCISE 2. Find the integral curves of the equations

$$\frac{dx}{xz-y} = \frac{dy}{yz-x} = \frac{dz}{1-z^2}$$

METHOD (II). Suppose we find two vectors fields $\mathbf{E}' = (P', Q', R')$ and $\mathbf{E}'' = (P'', Q'', R'')$ such that the differentials

$$dw' = \frac{P'dx + Q'dy + R'dz}{PP' + QQ' + RR'} \tag{1.26}$$

and

$$dw'' = \frac{P''dx + Q''dy + R''dz}{PP' + RR' + QQ'} \tag{1.27}$$

are exact and are equal to each other. Then, it follows

$$w' = w'' + c_1 \tag{1.28}$$

where c_1 is the integration constant.

EXERCISE 3. Let us find the integral curves of the equations

$$\frac{dx}{y+az} = \frac{dy}{z+bx} = \frac{dz}{x+cy}.$$

Let us choose the vector $\mathbf{E}'$ of the form $\mathbf{E}' = (\lambda, \mu, \nu)$, where $\lambda, \mu,$ and ν are constants and find the conditions that they must satisfy in order for the differential form (1.26), given by

$$\frac{1}{\rho}\frac{\lambda x + \mu y + \nu z}{\lambda x + \mu y + \nu z}, \tag{1.29}$$

to be an exact differential, where ρ is another constant. From Eq. (1.26), this is possible only if the determinant of the matrix in the expression

$$\begin{pmatrix} -\rho & b & 1 \\ 1 & -\rho & c \\ a & 1 & -\rho \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \\ \nu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{1.30}$$

is zero, that is if ρ is a root of the equation

$$-\rho^3 + (a+b+c)\rho + 1 + \text{ abc } = 0 \; .$$

this equation has three complex roots $\rho_i, i = 1,2,3$ and for each of them there exists a vector

$$\begin{pmatrix} \lambda_i \\ \mu_i \\ \nu_i \end{pmatrix} \quad i = 1,2,3$$

which satisfies Eq. (1.30), and thus we have with Eq. (1.29) three possible exact differentials

$$dW' = d \; \log(\lambda_1 x + \mu_1 y + \nu_1 z)^{1/\rho_1},$$

$$dW'' = d \; \log(\lambda_2 x + \mu_2 + \nu_2 z)^{1/\rho_2} \; ,$$

and

$$dW''' = d \; \log(\lambda_3 x + \mu_3 + \nu_3 z)^{1/\rho_3} \; .$$

According to Eq. (1.28), we have the integral curves

$$(\lambda_1 x + \mu_1 y + \nu_1 z)^{1/\rho_1} = c_1(\lambda_2 x + \mu_2 y + \nu_2 z)^{1/\rho_2}$$

and

$$(\lambda_1 x + \mu_1 y + \nu_1 z)^{1/\rho_1} = c_2(\lambda_3 x + \mu_3 y + \nu_3 z)^{1/\rho_3} \; .$$

This method depends also on the intuition in determining the form of the vector fields $\mathbf{E}', \mathbf{E}''$.

EXERCISE 4. Find the integral curves of the equations

$$\frac{adx}{(b-c)yz} = \frac{bdy}{(c-a)zx} = \frac{cdz}{(a-b)xy}$$

METHOD (III). When one of the variables is absent from one of the equations of the set (1), it is possible to make a partial separation of variables, and we can derive the integral curves in a simple way. Suppose that the equation

$$\frac{dy}{Q} = \frac{dz}{R}$$

can be written in the form

$$\frac{dy}{dz} = f(y, z)$$

this equation has a solution of the form

$$\phi_1(y, z, c_1) = 0,$$

where c_1 is the integration constant. Solving this equation for z ($z = \psi(y, c_1)$) and substituting this value in the equation

$$\frac{dx}{P} = \frac{dy}{Q} ,$$

we obtain an ordinary differential equation of the type

$$\frac{dy}{dx} = g(x, y, c_1)$$

whose solution

$$\phi_2(x, y, c_1, c_2) = 0$$

may be readily obtained.

EXEMPLE 4. Find the integral curves of the equations

$$\frac{dx}{ye^z} = \frac{dy}{xe^y} = \frac{dz}{y/x} \tag{1.31}$$

Using the first and third terms, we obtained the ordinary differential equation

$$\frac{dx}{dz} = xe^{-z}$$

which has the solution

$$x = c_1 e^{-e^{-z}} .$$

Substituting this in the second and third term of (1.31), we obtain the ordinary differential equation

$$\frac{dy}{dz} = c_1^2 \frac{e^y}{y} e^{-2e^{-z}} .$$

This equation has the two parametric solution

$$(y+1)e^{-y} + c_1^2 \int e^{-2e^{-z}} \, dz = c_2 .$$