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INTRODUCTION TO THE THEORY OF NONLINEAR ELLIPTIC EQUATIONS

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Introduction to the Theory of Nonlinear Elliptic Equations

by

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Preface

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These lecture notes are a very free continuation of my book "Les méthodes directes en théorie des équations elliptiques", Prague–Paris, 1967, in the nonlinear case. The realization of the existence-regularity scheme for nonlinear systems had to be preceded by the effort of mathematicians from the whole world. Finally, in last years, some new ideas have appeared that enabled us to present a theory complete in some sense. I underline that I omit completely the spectral, bifurcation, multiplicity, genericity, and other problems that belong rather to functional analysis then to the theory of elliptic differential equations. Nevertheless, in Chapters 3 and 4, which are concerned with the existence of solution and with approximate methods, many subjects have the same character. The main part of the lecture notes is Chapters 5 and 6 on the regularity questions. I added some applications to elasticity, which are far from being immediate and which show in a large extent some fundamental questions remaining still open.

A lot of the topics of these lecture notes were discussed in the seminar on partial differential equations in the Mathematical Institute of the Czechoslovak Academy of Sciences and in the lectures that I held at the Faculty of Mathematics and Physics of the Charles University, at Scuola Normale Superiore di Pisa and at the University of Pierre et Marie Curie in Paris.

It is the author's pleasant duty to thank all his colleagues and friends for variable discussions: O. A. Oleĭnik, S. Campanato, M. Giaquinta, P. Ciarlet, G. Tronel, J. Frehse, A. Kufner, J. Stará, O. John, R. Švarc, M. Krbec, M. Šilhavý and P. Drábek. I also wish to thank K. Segeth for his help with the English translation, R. Pachtová for her excellent typing, and the TEUBNER Publishing House, Leipzig, for their collaboration and patience.

Prague, May 1982 J. Nečas

Chapter 1

The topic of the lecture notes and something on modelling by partial differential equations

1.1. Introduction

The significance of nonlinear elliptic partial differential equations grows up both from the theoretical point of view and as a consequence of their numerous applications. Taking into account the main results, the classical as well as the most recent ones, we can affirm that the theory of these equations is just creating a harmonic entirety where the basic questions are answered. These lecture notes have been written as an introduction though they are in some directions complete enough. I will not use the introduction to describe the history of the development of nonlinear partial differential equations and I recommend the reader the books by O. A. LADYŽENSKAJA, N. N. URALCEVA [1], CH. B. MORREY [2], J. L. LIONS [3], D. GILBARG, N. S. TRUDINGER [4], S. FUČÍK, A. KUFNER [5] and E. GIUSTI [6]. We shall touch some historical features in the sequel.

In these lecture notes, the study of second order systems in the divergence form

$$(1.1.1)\qquad -\frac{\partial}{\partial x_i}[a_i^r(x, u, \nabla u)] + a^r(x, u, \nabla u) = -\frac{\partial f_i^r}{\partial x_i} + f^r$$

will be in the centre of our interest. Here the summation over the repeated subscript is understood and

$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \qquad u = (u_1, u_2, \ldots, u_m),$$

$$\nabla u = (\nabla u_1, \nabla u_2, \ldots, \nabla u_m), \qquad \nabla u_r = \left(\frac{\partial u_r}{\partial x_1}, \frac{\partial u_r}{\partial x_2}, \ldots, \frac{\partial u_r}{\partial x_n}\right).$$

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One comes to such systems, for example, in the study of the critical points of the functionals

$$\int_\Omega F(x, u, \nabla u)\,\mathrm{d}x. \tag{1.1.2}$$

Elliptic partial differential equations, together with boundary conditions, are models for many physical, mechanical, and technical phenomena and we shall touch such models in these lecture notes in order to clarify that the construction of modern models as well as the study of these modern models is considerably neglected as compared with the study of the classical ones where very subtle results have often nearly nothing in common with the reality. Such a trivial case is, for example, the functional of the total potential energy of a membrane

$$\Phi(u) \overset{\text{def}}{=} \int_\Omega \left[T(x) \left(\sqrt{1 + \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i} \right)^2} - 1 \right) - uf \right] \mathrm{d}x, \tag{1.1.3}$$

which is correct only in the case of $T(x) = \text{const}$ because otherwise one does not obtain the conditions of equilibrium for the momentum, as we shall see in 1.2. In this case one must study a parametrical "minimal surface" problem. Also the usual trick

$$\sqrt{1 + \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i} \right)^2} - 1 \doteq \frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial u}{\partial x_i} \right)^2 \tag{1.1.4}$$

leads to the Poisson equation $-\dfrac{\partial}{\partial x_i}\left(T\dfrac{\partial u}{\partial x_i}\right) = f$ the theory of which completely differs from the theory of the equation for the surface with prescribed mean curvature $-\dfrac{1}{T} f$ (if T is constant).

The main questions concerning the systems (1.1.1), if we add some boundary conditions (for example if we prescribe $u = h$ on the boundary $\partial\Omega$ of the considered domain $\Omega \subset \mathbb{R}^n$), are

(i) the existence and uniqueness of the solution; if the uniqueness does not occur, then the structure of the solutions, generic properties, and the bifurcation,

(ii) approximate methods for finding the solution,

(iii) the regularity of solution,

(iv) what such systems can model.

In these lecture notes we shall not be concerned with systems of higher order; roughly speaking, those differ from (1.1.1) by more indices. We shall make only brief remarks and some references to this topic. In these lecture notes, we shall be interested both in variational methods and in methods of monotone operators. Nowadays, these are the classical results and the reader can consult also other lecture notes of this series, see S. Fučík, J. Nečas, J. Souček [7] and E. Zeidler [8].

The main point of these lecture notes is the solvability of elliptic systems and the qualitative behaviour of their solutions. Hence all the functional-analytic methods are presented from this point of view.

Surveying the historical development of the theory of nonlinear elliptic equations, we find that one of the fundamental steps was the introduction of the Sobolev spaces $W^{k,p}(\Omega)$, see S. L. SOBOLEV [9], the development of the abstract variational methods, see for example M. M. VAJNBERG [10], and the methods of monotone operators, see J. L. LIONS [3]. Let us mention that the fundamentals of those direct methods were given much earlier; if we will not go back to works of RIEMANN and DIRICHLET we can find them in the book by R. COURANT and D. HILBERT [11]. The works of F. RELLICH, E. TREFFTZ, S. L. SOBOLEV, S. G. MICHLIN, R. CACCIOPPOLI, C. MIRANDA, L. SCHWARTZ, L. GÅRDING, K. O. FRIEDRICHS, and others have established the fundamentals of the theory of weak solutions to elliptic equations and this theory was accomplished, especially for nonlinear equations, by M. M. VAJNBERG, G. MINTY, J. LERAY, M. I. VIŠIK, J. L. LIONS, F. E. BROWDER, H. BRÉZIS and others. The Sobolev spaces $[W^{1,p}(\Omega)]^m$ are the spaces of vector functions $u = (u_1, u_2, \ldots, u_m)$ that are L^p-integrable in the considered domain along with their first derivatives.

The solvability of the systems (1.1.1) in $[W^{1,p}(\Omega)]^m$ requires some growth conditions for coefficients, for example

$$(1.1.5) \qquad |a_i^r(x, u, \nabla u)| + |a^r(x, u, \nabla u)| \leqq c(1 + |u| + |\nabla u|)^{p-1}.$$

On the other hand, when such systems are models (physical, technical, ...), the response functions a_i^r, a^r are defined for u and ∇u only in some bounded regions, and the conditions (1.1.5) are fictitious extrapolations. This incongruity can be eliminated if one considers solutions from $[W^{1,\infty}(\Omega)]^m$. In general, it is not known whether such solutions do exist under reasonable conditions. One can go further and look for solutions in $[C^1(\bar{\Omega})]^m$ or $[C^{1,\alpha}(\bar{\Omega})]^m$ and, as we shall see later, this is just the question (iii), which is also the focal point of these lecture notes. There are books or lecture notes on the questions (i), (ii), and (iv) that are more complete than these lecture notes and the author added this topic rather because of the integrity of the material presented.

As far as the point (ii) is concerned, it does not surpass a standard treatment too much. For more details, the reader is recommended to consult the book by J. CÉA [12]. We look for the steepest descent methods and, also from this point of view, for the Newton method. In general, this method requires the C^1 regularity of solutions, which is not generally known. Surprisingly, the imbedding method requires practically only weak solutions. (The imbedding method is, roughly speaking, a continuous analogue of the Newton method.)

The central point (iii) of these lecture notes is the $[C^{1,\alpha}(\Omega)]^m$ or $[C^{1,\alpha}(\bar{\Omega})]^m$ regularity of weak solutions. In general under standard assumptions on the systems (1.1.1) and for a weak solution from $[W^{1,p}(\Omega)]^m$ (a solution in the sense of distributions), there exists a set M, closed in Ω, of zero measure (more precisely, see later), and such that the solution is from $[C^{1,\alpha}(\Omega \setminus M)]^m$. This is the so-called partial regularity. Under some further assumptions that are sufficient and "necessary", in principle the same method "of partial regularity" gives that $M = \emptyset$. This condition is for the interior regularity, i.e. for proving that a Lipschitz continuous solution to (1.1.1) lies in $[C^{1,\alpha}(\Omega)]^m$ (which means that it is Hölder continuous on every compact $K \subset \Omega$) denoted as $L(\mathbb{R}^n)$—the Liouville-type condition.

For the system (1.1.1) it means:

$\forall x^0 \in \Omega$, $\forall \xi \in \mathbb{R}^m$ the solutions v to the system

$$-\frac{\partial}{\partial x_i}[a_i^r(x^0, \xi, \nabla v)] = 0 \tag{1.1.6}$$

in R^n with a bounded gradient $|\nabla v(x)| \leqq c < \infty$ are polynomials of at most first degree. It is known that $C^{1,\alpha}(\Omega)$ regularity holds for the dimension $n = 2$ and for $m = 1$, $n \geqq 2$, and we reprove these results in Chapters 5 and 6. The regularity of weak solutions to (1.1.1) was proved for $n = 2$ by CH. B. MORREY [13], for $m = 1$, $n \geqq 2$ by E. DE GIORGI [14], and J. NASH [14¹] and the condition (1.1.6) was discovered by M. GIAQUINTA, J. NEČAS [15]. There are many important steps in the regularity problem and we removed all the details to Chapters 5 and 6. Let us mention that we shall suppose the system (1.1.1) to be *very strongly* elliptic throughout these lecture notes, i.e. such that

$$\frac{\partial a_i^r}{\partial \eta_j^s}(x, \xi, \eta)\, \zeta_i^r \zeta_j^s > 0 \quad \text{for} \quad \zeta \neq 0. \tag{1.1.7}$$

Also, under this condition, there exists system (1.1.1) with real-analytic coefficients for which the condition (1.1.6) is not satisfied, namely systems with the solution $u^{rs} = \dfrac{x_r x_s}{|x|} - \dfrac{1}{n}\delta_{rs}|x|$, $r, s = 1, 2, \ldots, n$, for $n \geqq 3$, see M. GIAQUINTA, J. NEČAS [16].

Nevertheless, we shall close this introduction by an important historical remark. The regularity problem in words of the analyticity of the extremum of the functional (1.1.2) with $F(x, \xi, \eta)$ analytic was formulated by D. HILBERT in 1900 as his 19th problem and the way of its solution connected with the names of S. N. BERNSTEIN, J. LERAY, J. SCHAUDER, H. WEYL and other mathematicians may serve as a beautiful description of the development of mathematics.

1.2. Partial differential equations in modelling

In modelling, one supposes very often that response functions and quantities to be considered are differentiable enough. On the other hand, the mathematical formulation is often given in terms of generalized solutions. The connection between these two points of view is not, up to this time, clarified enough.

Less often the formulation corresponds to the mathematical model. It will be very useful to consider also this way in modelling.

1.2.1. EXAMPLE. Everybody knows the heat conduction equation. Let us derive it once again.

Let Ω be the considered body. We suppose that Ω is a domain in $\mathbb{R}^3$. But it is more reasonable to consider $\bar{\Omega}$. Is it true that $\partial\Omega$ is always insignificant? Let us suppose the time running through the interval $[0, \infty)$. Let $u(t, x)$ be the temperature of Ω at the point (t, x); let us accept this point of view. Let $D \subset \Omega$ be a subdomain and let $S \subset \partial D$. We suppose that the Newton law for the heat flux is valid: in the time interval $[t_1, t_2]$ the heat flux through S from $\Omega \setminus D$ to D is

$$q_{[t_1,t_2]} = + \int_{t_1}^{t_2} \mathrm{d}t \int_S k(x, u, \nabla u) \frac{\partial u}{\partial n} \mathrm{d}S, \tag{1.2.2}$$

where $k(x, u, \nabla u)$ is the heat conductivity. One writes (1.2.2) also in the form

$$\mathrm{d}q = -k \frac{\partial u}{\partial n} \mathrm{d}t\, \mathrm{d}S. \tag{1.2.3}$$

How many assumptions are hidden in (1.2.3)! Clearly first (1.2.3) must make sense; then $\mathrm{d}q$ is a sign measure on $[t_1, t_2] \times \partial D$. Of course, a surface measure on ∂D must be defined. Further (1.2.3) implies that the measure $\mathrm{d}q$ has density $-k\dfrac{\partial u}{\partial n}$ that depends linearly on $\dfrac{\partial u}{\partial n}$. So ∂D must be an oriented surface. On the other hand, $k = k(x, u(t, x), \nabla u(t, x))$: the principle of locality ...

The reader sees from these simplest remarks that (1.2.3) is not obvious and, under some conditions, it can be false. One way continue with the classical balance of heat flow. The increase of the temperature of the body D during the time $[t_1, t_2]$ needs the heat

$$\int_{t_1}^{t_2} \int_D \varrho(x)\, c(x, u) \frac{\partial u}{\partial t}(t, x)\, \mathrm{d}t\, \mathrm{d}x, \tag{1.2.4}$$

where ϱ is the density and c is the specific heat. (One can make as many remarks as before.) If there is a heat source in Ω, it yields the heat

$$\int_{t_1}^{t_2} \int_D f(t, x)\, \mathrm{d}t\, \mathrm{d}x \tag{1.2.5}$$

during the time $[t_1, t_2]$ in D.

The total heat flux from $\Omega \setminus D$ to D is

$$\int_{t_1}^{t_2} \int_{\partial D} k \frac{\partial u}{\partial n} \mathrm{d}t\, \mathrm{d}S. \tag{1.2.6}$$

Hence,

$$\int_{t_1}^{t_2} \int_{\partial D} k \frac{\partial u}{\partial n} \mathrm{d}t\, \mathrm{d}S + \int_{t_1}^{t_2} \int_D f\, \mathrm{d}t\, \mathrm{d}x = \int_{t_1}^{t_2} \int_D \varrho c \frac{\partial u}{\partial t} \mathrm{d}t\, \mathrm{d}x. \tag{1.2.7}$$

From Green's formula we have

$$(1.2.8)\qquad \int_{t_1}^{t_2}\int_D \left(\frac{\partial}{\partial x_i}\left[k\frac{\partial u}{\partial x_i}\right] + f\right) dt\, dx = \int_{t_1}^{t_2}\int_D \varrho c \frac{\partial u}{\partial t}\, dt\, dx;$$

we let the reader formulate the assumptions for the validity of (1.2.8). If D is regularly shrinking to a point x and $t_1 = t$, $t_2 \to t$, then, if the integrand is continuous, for example, the classical heat-conduction equation follows from (1.2.8).

If the temperature approaches a steadyvalue for $t \to \infty$, i.e. if $u(t, x) \to U(x)$ in some sense, it may be excepted that

$$(1.2.9)\qquad -\frac{\partial}{\partial x_i}\left[k(x, U, \nabla U)\frac{\partial U}{\partial x_i}\right] = F(x) \quad \text{in } \Omega$$

and, for example,

$$(1.2.10)\qquad U(x) = H(x) \quad \text{on } \partial\Omega,$$

where H is prescribed and $F(x) = \lim_{t\to\infty} f(t, x)$. The classical formula for the Cauchy problem

$$(1.2.11)\qquad \frac{\partial u}{\partial t} = \Delta u \quad \text{in} \quad \mathbb{R}^{n+1}_+ = \{(t, x) \mid t > 0\},$$

$$(1.2.12)\qquad u(0, x) = \phi(x):$$

$$(1.2.13)\qquad u(t, x) = \frac{1}{2^n \pi^{n/2} t^{n/2}} \int_{R^n} \exp\left(-\frac{|x - y|^2}{4t}\right) \phi(y)\, dy$$

gives that the heat flow has an infinite speed, which is impossible. Where is the mistake?

1.2.14. EXAMPLE. Let us consider a membrane M and let us suppose first that it is a surface given in $\mathbb{R}^3$, as usually, by

$$(1.2.15)\qquad y_1 = x_1, \qquad y_2 = x_2, \qquad y_3 = u(x_1, x_2),$$

where $(x_1, x_2) \in \Omega \subset \mathbb{R}^2$. Let us consider first the steady state; let us suppose that a surface force $f_3(x_1, x_2)$ (in the direction of the axis x_3) related to the unit volume of Ω is acting on the membrane. The total work of the surface forces, if we start from the configuration $u_0(x_1, x_2)$, therefore is

$$(1.2.16)\qquad \int_\Omega f_3(u - u_0)\, dx_1\, dx_2.$$

Let us consider a part S of the membrane M the projection of which is $O \subset \Omega$ with ∂O smooth enough. Let us suppose that we can define a tension vector σ at the points $(x_1, x_2) \in \partial O$, related to the unit surface of ∂O, such that $\sigma = \sigma((x_1, x_2), \nu)$, where ν is the outer normal to ∂O at the point (x_1, x_2). We suppose that the vector σ lies in the plane tangent to the surface. If we put $p = \sqrt{1 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2}$ then the unit normal to the surface is

$$n = \left(-\frac{u_{x_1}}{p}, -\frac{u_{x_2}}{p}, \frac{1}{p}\right). \tag{1.2.17}$$

Hence

$$(n, \sigma) = 0. \tag{1.2.18}$$

If t is the tangent vector to ∂S at the point y, then we suppose

$$(t, \sigma) = 0. \tag{1.2.19}$$

So we can suppose that the tension vector $g(x_1, x_2)$ acts on some part of "∂S" the projection of which is $\Gamma \subset \partial\Omega$. The total work of this tension vector is

$$\int_\Gamma g_3(u - u_0)\,\mathrm{d}s. \tag{1.2.20}$$

Let us suppose that the total potential energy of tension vectors is

$$\int_\Omega T(x_1, x_2)\,[p - p_0]\,\mathrm{d}x_1\,\mathrm{d}x_2, \tag{1.2.21}$$

which means that the increment $\mathrm{d}W$ of the stored energy of the membrane is proportional to the change of the surface of the membrane. So the functional of the total potential energy of the membrane (apart from a constant) is

$$\Phi(u) = \int_\Omega (Tp - f_3 u)\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_\Gamma g_3 u\,\mathrm{d}s. \tag{1.2.22}$$

Let us consider (formally) the minimum of $\Phi(u)$ over the set of u such that $u = u^0$ on $\partial\Omega \setminus \Gamma$. Let u be such a minimum. If $h = 0$ on $\partial\Omega \setminus \Gamma$, it follows that

$$\begin{aligned} D\Phi(u, h) &= \frac{\mathrm{d}}{\mathrm{d}t}[\Phi(u + th)]_{t=0} \\ &= \int_\Omega \left[\frac{T}{p}(\nabla u \cdot \nabla h) - f_3 h\right]\mathrm{d}x_1\,\mathrm{d}x_2 \\ &\quad - \int_\Omega g_3 h\,\mathrm{d}s = 0. \end{aligned} \tag{1.2.23}$$

If $O \subset \bar{O} \subset \Omega$, then it follows in the same manner that

$$\int_O \left[\frac{T}{p}\nabla u\,\nabla h - f_3 h\right]\mathrm{d}x_1\,\mathrm{d}x_2 - \int_{\partial O} \sigma_3 h\,\mathrm{d}s = 0. \tag{1.2.24}$$

So we get *Euler's equation*

$$\frac{\partial}{\partial x_i}\left(\frac{T}{p}\frac{\partial u}{\partial x_i}\right) + f_3 = 0 \quad \text{in } \Omega \tag{1.2.25}$$

and

$$\frac{T}{p}\frac{\partial u}{\partial n} = g_3 \quad \text{on } \Gamma, \tag{1.2.26}$$

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