SUMMATION OF THE FOURIER SERIES OF ORTHOGONAL FUNCTIONS

CHEN KIEN-KWONG

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直交函數的傅里葉級數之和

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CONTENTS

INTRODUCTION

Let $\varphi(x, \lambda)$ be a function defined for the values of x in some given set of points E, λ being a parameter. The set E may be a linear set, or a p-dimensional set. Let U denote an analytical operation with respect to x; suppose $U(\varphi(x, \lambda))$ and $U(\varphi(x, \lambda)\cdot\varphi(x, \lambda'))$ have definite values, when λ, λ' belong to a given set of values. Then, if

$$U(\varphi(x, \lambda)\, \varphi(x, \lambda')) = 0 \qquad (\lambda \neq \lambda')$$
$$\neq 0 \qquad (\lambda = \lambda'),$$

we call $\varphi(x, \lambda)$ a system of orthogonal functions with respect to the operator U.

A system of orthogonal functions is said to form a system of normalized orthogonal functions when

$$U((\varphi(x, \lambda))^2) = 1$$

for every value of λ in the given set.

In the present work, various types of orthogonal functions are considered.

In Chapter I, an account is given of the systems of orthogonal functions normalized in a finite interval; in this case the operator U means $\int_a^b \cdots dx$. A proof of Menchoff and Rademacher's theorem is given by applying a theorem of Fubini. The logical equivalence between the convergence theorem of Rademacher-Menchoff and the summability theorem of Borgen, Kaczmarz, and Menchoff is established. A theorem on the convergence of a sequence of partial sums of the series in consideration is demonstrated. All these things are given in §1, Chapter I. A discussion of Zygmund's theorem related to the Riesz-summability of the series of orthogonal functions is given in §2. An estimation of Lebesgue's function is contained in §3.

W. Stekloff gave a proof that a normalized orthogonal system is necessarily complete, if Parseval's formula for any polynomial with respect to the system holds good. Another condition for the completeness of the normalized orthogonal system has been given by J. Tamarkin. A general and complete theorem on this direction can be, however, obtained in a simple manner. The theorems of Stekloff as well as of Tamarkin follow as an immediate consequence. These considerations are given in §4. In §5, I have extended Menchoff's inequality.

$$\left[\int_0^1 \left|\sum_{m=1}^{n(x)} c_m \varphi_m(x)\right|^2 dx\right]^{\frac{1}{2}} \leqslant C \log n \sqrt{c_1^2 + \cdots + c_n^2}$$

with $n(x) \leqslant n$ into the type of Riesz and Hausdorff.

Chapter II is concerned with the convergence problem of the classical Fourier series and its allied series. A sufficient condition for the existence of Kronecker's limit is given. New criteria for the convergence of a Fourier series are obtained from the point of view of mean-functions. Indeed, I have proved the following theorem: if $\varphi_2(t) \to s$ as $t \to 0$, and if the function

$$\Phi_k(t) = (\varphi(t) - k\,\varphi_1(t) + \cdots + (-1)^k \varphi_k(t))/t$$

is summable in $(0, \pi)$ for a $k \geqslant 1$, then the Fourier series of $f(x)$ at the point x converges to x. If $\Phi_k(x)$ is summable in $(0, \pi)$, but if the limit $\lim \varphi_2(t)$ does not exist, then the Fourier series of $f(x)$ at x is not summable (C, α), where

$$\varphi(t) = \varphi_0(t) = \frac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\varphi_\nu(t) = \frac{1}{t}\int_0^t \varphi_{\nu-1}(t)\,dt, \qquad \nu \geqslant 1.$$

When k is larger, the criterion is better. However, these criteria do not cover the convergence theorem about the Fourier series of the function $\cos(At^{-a} + B + tl(t))$, which is dealt with in §2, Chapter II. In §3, I give a criterion for the convergence of the allied series of a Fourier series. The theorem is parallel to Gergen's criterion for Fourier series, and is an extension of Misra and Zygmund's theorem. In §4, three theorems concerning the Cesàro summability of Fourier series of functions belonging to Lipschitz class are stated; the proofs are given in Chapter V. In §5, a theorem of Priwaloff is improved.

Chapter III is devoted to theory of absolute convergence of Fourier series. In §1, I give the characteristic property of the functions whose Fourier series are absolutely convergent. The following theorem is established: the necessary and sufficient condition for the absolute convergence of a trigonometrical series is that the series is a Fourier series of a function having the type

$$f(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\xi)\, f_2(x+\xi)\, d\xi,$$

where

$$f_j(x) \in L^2(-\pi, \pi), \quad f_j(x+2\pi) = f_j(x), \quad j = 1, 2.$$

The absolute convergence of the Fourier series of a function $f(t)$ at a given point $t=x$ is not a local property of $f(t)$ at x, but depends on the behaviour of $f(t)$ in $0 \leqslant t \leqslant 2\pi$. I prove in §2 that if both the functions $2\phi(t) = f(x+t) + f(x-t)$ and $t\,\phi'(t)$ are of bounded variation in $0 \leqslant t \leqslant \pi$, then the Fourier series of $f(t)$ converges absolutely at the point $t=x$. Extensions of this theorem are also given in §2. Zygmund proved that if $f(t)$ is of bounded variation in $(-\pi, \pi)$ and belongs to class Lip α, $\alpha > 0$, then the Fourier series of $f(t)$ converges everywhere. This proposition has been extended by other writers. I prove in §3 a theorem for the absolute convergence of Fourier series of bounded variation, which is not a corollary of any of the theorems mentioned above. In §4, I deduce from the absolute convergence of the Fourier series of $f(t)$ at the piint $t=x$ that the function

$$\frac{\rho(t)}{t}\int_0^t (f(x+t)+f(x-t))\,dt$$

is of bounded variation in $0\leqslant t\leqslant A$ whenever $\rho(t)$ is absolutely continuous in $(0, A)$.

Chapter IV contains an exposition of the theory of summabilitv $|C,\alpha|$ of positive order α for a Fourier series at a given point. I prove that if the limit

$$\lim_{\varepsilon\to+0}\int_\varepsilon^t \frac{\phi(u)}{u}\,du = \chi(t)$$

with $\phi(u)=f(x+u)+f(x-u)$ exists and if $t^{-1}\,\chi(t)$ is summable in $(0, \pi)$, then the Fourier series of $f(t)$ at $t=x$ is summable $|C, \alpha|$ for every $\alpha>2$. An extension of this proposition to the summability $|C, \alpha|$ for $\alpha > m+1$ is also given, where m denotes any positive integer.

The summability $|C,\alpha|$, $\alpha<0$, for a Fourier series at a given point is discussed in Chapter V. A series $\Sigma\, u_n$ is summable $|C, \alpha|$ it is summable $|C, \beta|$ if $\beta>\alpha$. This theorem is well known for $\alpha\geqslant 0$. I give a proof for this theorem under the condition $\alpha>0$, and then apply the theorem to the Fourier series. Some parts of the theory are based on the summability $|C,\alpha|$, $\alpha<0$, of power series.

Chapter VI is devoted to the theory of absolute Cesàro summability of the allied series of a Fourier series. Additional conditions must be introduced in the criterion. For example, the boundness of the total variation of the function

$$\frac{d}{dt}\int_0^t \left(\frac{u}{t-u}\right)^a \varphi(u)\,du \qquad (0<a<1)$$

in the interval $(0, \pi)$ involves the absolute convergence of the Fourier series $\Sigma\, A_n(x)$ of $f(t)$ as shown in §2, Chapter III. But for the allied series $\Sigma\, B_n(x)$ of $\Sigma\, A_n(x)$, denoting

$$\chi_0(t) = \frac{d}{dt}\int_0^t \left(\frac{u}{t-u}\right)^a \varphi(u)\,du \qquad (0<a<1),$$

the pair of conditions

$$\int_{-\pi}^{\pi} |d\chi_0(t)| < \infty \quad \text{and} \quad \int_{-\pi}^{\pi}\left|\frac{\chi_0(t)-\chi_0(0)}{t}\right| dt<\infty$$

does not imply the absolute convergence of $\Sigma\, B_n(x)$.

Let p be an integer greater than 3, the coefficients $L_n(\cos\gamma)$ of the expansion

$$(1-2z\cos\gamma+z^2)^{-\frac{p-2}{2}} = \sum_{n=0}^{\infty} L_n(\cos\gamma)\, z^n$$

form an orthogonal system on the hypersphere S:

$$x_1^2 + x_2^2 + \cdots + x_p^2 = 1\,.$$

In this case, the operator U means $\int_S \cdots d\omega$, $d\omega$ being the surface element of S. E. Kogbetliantz has studied the series of hyperspherical functions on the ordinary sphere. In Chapter VII, there will be found a discussion of the Cesaro summability of the series of hyperspherical functions.

CHAPTER I

NORMAL ORTHOGONAL SYSTEM OF FUNCTIONS

§1. Convergence and Summability $(C, 1)$ of the Series of Orthogonal Functions[1]

1. Let $c_1, c_2 \cdots$ be a sequence of real numbers and $\varphi_1(x), \varphi_2(x), \cdots$ be a sequence of normal orthogonal functions in the interval $(0,1)$. Relating to the series

$$c_1 \varphi_1(x) + c_2 \varphi_2(x) \cdots\cdots, \tag{1}$$

we have the following theorems.

(A) The theorems of Rademacher and Menchoff[2]:

If the series $\Sigma (\log \nu)^2 \cdot c_\nu^2$ is convergent, then the series (1) converges almost everywhere in the interval $(0, 1)$.

(B) Theorem of Menchoff-Borgen-Kaczmarz[3]:

If the series $\Sigma (\log \log \nu)^2 \cdot c_\nu^2$ is convergent, then the series (1) is $(C, 1)$ summable almost everywhere in the interval $(0, 1)$.

In the appearance, these two theorems state two facts distinct from each other, yet it can be shown that any one of them is deducible from the other; i.e.

Theorem 1. *The theorems* (A) *and* (B) *are equivalent.*

In fact, both theorems are equivalent to the following theorem:

(C) If the series $\Sigma (\log \log \nu)^2 \cdot c_\nu^2$ is convergent, then the sequence of the partial sums

$$S(x, 2), \quad S(x, 2^2), \cdots, \quad S(x, 2^n), \cdots \tag{2}$$

converges almost everywhere in the interval $(0, 1)$, where

$$S(x, p) = c_1 \varphi_1(x) + c_2 \varphi_2(x) + \cdots + c_p \varphi_p(x).$$

This theorem is due to Borgen and Kaczmarz[4].

2. The contents of §1 are as follows: For the sake of completeness, we

1) Cf. K. K. Chen (1), (2).

2) H. Rademacher (1), D. Menchoff (1).

3) D. Menchoff (4); D. Menchoff (2), Theorems 6; S. Borgen (1); S. Kaczmarz (1).

4) See Borgen and Kaczmarz, (1) c.

give in Section I a proof of the theorem (A). The proof seems simpler than the original works.

Section II_1 establishes the equivalence of the theorems (A) and (C).

Section II_2 completes the proof of the equivalence of the theorems (A) and (B).

Section III is concerned with the convergence of partial sums of the series (1); the following theorem is given there:

Theorem 2. *If the series $\Sigma c_\nu^2 \log \nu$ is convergent, then there exists a sequence of integers, $k_1, k_2, \cdots, k_n, \cdots$ such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = 1$$

for which the sequence of the partial sums

$$S(x, k_1),\ S(x, k_2), \cdots,\ S(x, k_n), \cdots$$

converges almost everywhere in the interval (0, 1).

I. Proof of the Theorem of Rademacher and Menchoff

3. We begin with the introduction of the notations used by Menchoff:

$$\chi(l, s) = 2^m + s\cdot 2^l,$$
$$D(x, l, s) = S(x, \chi(l, s+1)) - S(x, \chi(l, s)),$$

where m, s, l are integers satisfying the conditions

$$0 \leqslant l < m, \qquad 0 \leqslant s < 2^{m-l}.$$

Next, let $\Sigma c_\nu^2 \cdot (\log \nu)^2$ be convergent. We proceed to show the following facts:

1°. If δ be a prescribed positive number, then there is a positive integer $m_0(x)$ such that

$$m \sum_{l,s} [D(x, l, s)]^2 < \delta^2 \quad \text{for} \quad m \geqslant m_0(x),$$

where x is any point of a set T whose measure $|T| = 1$.

To prove this, we put

$$U_m(x) = \int_0^x m \sum_{l,s} (D(x, l, s))^2\, dx,$$

where $0 < x < 1$.

Then

$$U_m(x) \leqslant \int_0^1 \sum_{l,s} (D(x,l,s))^2\,dx =$$

$$= m \sum_{l,s} \sum_{x(l,s)+1}^{x(l,s+1)} c_\nu^2 =$$

$$= m \sum_{l} \sum_{2^m+1}^{2^m+1} c_\nu^2 = m^2 \sum_{2^m+1}^{2^m+1} c_\nu^2 \leqslant$$

$$\leqslant \sum_{2^m+1}^{2^m+1} c_\nu^2 (\log \nu)^2,$$

here and afterwards, logarithm being of the base 2.

Therefore the series

$$U_1(x) + U_2(x) + \cdots$$

converges in (0, 1); and by Fubini's theorem, the series

$$\sum_{m=1}^{\infty} \left\{ m \sum_{l,s} (D(x,l,s))^2 \right\}$$

converges almost everywhere in (0, 1).

Consequently, there exists a set of points T whose measure is 1, such that

$$\lim_{m\to\infty} m \sum_{l,s} (D(x,l,s))^2 = 0 \text{ in } T.$$

This establishes 1°.

2°. If $2^m \leqslant n < 2^{m+1}$, then

$$|S(x,n) - S(x,2^m)| < \delta \quad \text{for} \quad m \geqslant m_0(x),$$

where x is any point of the set T.

Write

$$n = \delta_0 + \delta_1 \cdot 2 + \delta_2 \cdot 2^2 + \cdots + \delta_i \cdot 2^i + \cdots + \delta_{m-1} \cdot 2^{m-1} + \delta_m \cdot 2^m,$$

where $\delta_i = 0$ or 1, $\delta_m = 1$. We have, on writing $S_i = \delta_{i+1} \cdot 2 + \cdots + \delta_{m-1} \cdot 2^{m-i-1}$,

$$S(x,n) - S(x,2^m) = \sum_{i=0}^{m-1} \delta_i D(x,i,S_i).$$

Hence

$$\{S(x,n) - S(x,2^m)\}^2 \leqslant \sum_{\nu=0}^{m-1} \delta_\nu^2 \sum_{i=0}^{m-1} (D(x,i,S_i))^2 \leqslant$$

$$\leqslant m \sum_{l,s} (D(x,l,s))^2.$$

Hence, if x is a point of T, we have

$$\{S(x, n) - S(x, 2^m)\}^2 < \delta^2 \text{ for } m \geqslant m_0(x)$$

by observing 1°. This proves 2°.

3°. If $\delta > 0$, then there is an integer $m_1(x)$ such that

$$|S(x, 2^m) - S(x, 2^{m'})| < \delta \quad \text{for} \quad m, m' \geqslant m_1(x),$$

where x is any point of a set R with $|R| = 1$.

For Σc_ν^2 is convergent, there exists a function $f(x)$, whose square is summable in (0, 1), such that

$$\int_0^1 \{f(x)\}^2 \, dx = \sum_{\nu=1}^{\infty} c_\nu^2, \qquad \int_0^1 f(x)\, \varphi_\nu(x)\, dx = c_\nu .$$

Now consider the series

$$\sum_{m=1}^{\infty} \int_0^x \{f(x) - S(x, 2^m)\}\, dx \qquad (0 < x < 1).$$

We have

$$\int_0^x \{f(x) - S(x, 2^m)^2 \, dx \leqslant \int_0^1 \{f(x) - S(x, 2^m)\}^2 \, dx = \sum_{2^m+1}^{\infty} c_\nu^2$$

and

$$\sum_{m=1}^{\infty} \sum_{2^m+1}^{\infty} c_\nu^2 = \sum_{m=1}^{\infty} m \sum_{2^m+1}^{2^{m+1}} c_\nu^2 \leqslant \sum_{m=1}^{\infty} \sum_{2^m+1}^{2^{m+1}} c_\nu^2 \log \nu = \sum_{\nu=3}^{\infty} c_\nu^2 \log \nu$$

which is convergent.

We thus have proved the convergence of the series

$$\sum_{m=1}^{\infty} \int_0^x \{f(x) - S(x, 2^m)\}^2 \, dx \qquad (0 < x < 1).$$

By a similar reasoning as in 1°, we have

$$\lim_{m \to \infty} \{f(x) - S(x, 2^m)\}^2 = 0$$

in a set of points R whose measure is 1. The remaining proof is easily seen from the inequality

$$|S(x, 2^m) - S(x, 2^{m'})| \leqslant |f(x) - S(x, 2^m)| + |f(x) - S(x, 2^{m'})| .$$

4. Let E be the cross-cut of R and T, i.e., $E = R \cdot T$, then $|E| = 1$. Let $x \in E$, $n_0(x)$ be the greater of $2^{m_0(x)}$ and $2^{m_1(x)}$ then for all pairs of values of

n and n' such that $n, n' \geqslant n_0(x)$ we have

$$2^m \leqslant n < 2^{m+1}, \qquad 2^{m'} \leqslant n' < 2^{m'+1},$$
$$m, m' \geqslant \max\ (m_0(x),\ m_1(x)).$$

It follows from

$$|S(x, n) - S(x, 2^m)| < \delta \qquad (3{,}2^\circ),$$
$$|S(x, 2^m) - S(x, 2^{m'})| < \delta \qquad (3{,}3^\circ),$$
$$|S(x, 2^{m'}) - (x, n')| < \delta \qquad (3{,}2^\circ),$$

that

$$|S(x, n) - S(x, n')| < 3\delta.$$

Hence the series

$$c_1\,\varphi_1(x) + c_2\,\varphi_2(x) + \cdots$$

converges at x. But x is an arbitrary point of E. Thus the theorem is proved.

II_1. Equivalence of (A) and (C)

5.1 DERIVATION OF (C) FROM (A).

Evidently, there is no loss of generality to assume that

$$\gamma_m^2 = \sum_{2^m+1}^{2^{m+1}} c_\nu^2 > 0 \qquad (m=1, 2, 3, \cdots).$$

Accordingly we may put

$$\psi_m(x) = \sum_{2^m+1}^{2^{m+1}} c_\nu\,\varphi_\nu(x)/\gamma_m.$$

We have

$$\int_0^1 \psi_i(x)\,\psi_j(x)\,dx = 0 \quad \text{for} \quad i \neq j,$$
$$\int_0^1 \{\psi_m(x)\}^2\,dx = 1 \qquad (m=1, 2, \cdots),$$

i.e., the sequence $\psi_1(x),\ \psi_2(x),\ \cdots$ forms a normalized orthogonal system of functions in $(0, 1)$.

We have to consider the convergency of the series

$$\gamma_1\,\psi_1(x) + \gamma_2\,\psi_2(x) + \cdots.$$

Now

$$\sum_{m=1}^{\infty} (\log m)^2\,\gamma_m^2 = \sum_{m=1}^{\infty} (\log m)^2 \sum_{2^m+1}^{2^{m+1}} c_\nu^2 \leqslant \sum_{\nu=3}^{\infty} (\log\log \nu)^2\,c_\nu^2$$

which is convergent by the hypothesis in the theorem (C). Then it follows from (A) that the series

$$\sum_{m=1}^{\infty} \gamma_m \psi_m(x) = \sum_{m=1}^{\infty} \left\{ \sum_{2^m+1}^{2^{m+1}} c_\nu \varphi_\nu(x) \right\}$$

converges almost everywhere in (0, 1). This is the conclusion of the theorem (C).

5.2 DERIVATION OF (A) FROM (C).

To do this, we lay down the following definition. If there exists another sequence of orthogonal functions $\psi_1(x)$, $\psi_2(x)$, $\cdots$ such that the addition sequence

$$\{\varphi_n(x)\} + \{\psi_n(x)\} = \varphi_1(x), \psi_1(x), \varphi_2(x), \psi_2(x), \cdots$$

forms a sequence of normal orthogonal functions in (0, 1), then the sequence $\{\varphi_n(x)\}$ is said to be an infinitely incomplete system.

In order to derive the theorem (A) from the theorem (C), we distinguish two cases:

Case 1. $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$ is an infinitely incomplete system.

In this case, there is a sequence of functions

$$\psi_1(x), \psi_2(x), \cdots$$

which forms together with $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$ a normal orthogonal system. We set

$$\Phi_m(x) = \psi_m(x)$$

for such m which is different from a power of 2, and

$$\Phi_{2^n}(x) = \varphi_n(x)\,.$$

The sequence

$$\{\Phi_\nu(x)\}$$

is a normal orthogonal system in (0, 1). Let

$$K_{2^n} = c_n\,, \quad K_m = 0$$

for such m which is different from a power of 2.

Let us consider the series $\sum_1^{\infty} K_\nu \Phi_\nu(x)$. From the convergence of the series

$$\sum_{\nu=1}^{\infty} c_\nu^2 (\log \nu)^2 = \sum_{\nu=1}^{\infty} K_{2^\nu}^2 (\log \nu)^2 = \sum_{\nu=2}^{\infty} K_\nu^2 (\log\log \nu)^2\,,$$

we see, by (C), that the sequence

$$\left\{\sum_{m=1}^{2^n} K_m\, \Phi_m(x)\right\} = \{S(x, n)\}$$

converges almost everywhere in (0, 1). This is nothing but (A) under the restriction that $\varphi_1(x)$, $\varphi_2(x)$, $\cdots$ is infinitely incomplete.

Case 2. $\varphi_1(x)$, $\varphi_2(x)$ is not an infinitely incomplete system.

We see that both the systems

$$\{\varphi_{2n-1}(x)\} \quad \text{and} \quad \{\varphi_{2n}(x)\}$$

are infinitely incomplete, and that the convergence of the series

$$\sum c_\nu^2 (\log \nu)^2$$

implies the convergence of the two series

$$\sum_{n=1}^{\infty} c_{2n-1}^2 (\log n)^2 \text{ and } \sum_{n=1}^{\infty} c_{2n}^2 (\log n)^2 .$$

Then it follows, by case 1, that the two series

$$\sum_{n=1}^{\infty} c_{2n-1}\, \varphi_{2n-1}(x) \quad \text{and} \quad \sum_{n=1}^{\infty} c_{2n}\, \varphi_{2n}(x)$$

converge almost everywhere in (0, 1). So also does the series

$$c_1\, \varphi_1(x) + c_2\, \varphi_2(x) + \cdots .$$

The equivalence of (A) and (C) is established.

II_2. Equivalence of (A) and (B)

6. The equivalence of (A) and (C) being thus established, the equivalence of (A) and (B) will be demonstrated provided that the equivalence of (B) and (C) is proved. The equivalence of (B) and (C) may, however, easily be seen by observing the following two facts:

1°. If $\Sigma\, c_\nu^2$ is convergent, then

$$\lim_{m\to\infty} \{S_{2^m}(x) - S(x, 2^m)\} = 0$$

almost everywhere in (0, 1), where

$$S_n(x) = \frac{S(x, 1) + S(x, 2) + \cdots + S(x, n)}{n} .$$

The proof is contained in Borgen's paper[5].

2°. If Σc_ν^2 is convergent and $\{S(x, 2^n)\}$ is convergent almost everywhere in (0, 1), then the sequence

$$\{S_n(x)\}$$

converges almost everywhere in (0, 1); i.e., the series (1) is summable $(C, 1)$ almost everywhere in (0, 1).

The proof is also contained in Borgen's paper[6].

III. Convergence of the Sequence of Partial Sums of the Series $\sum c_\nu \varphi_\nu(x)$

7. Borgen[7] proved that if $\Sigma c_\nu^2 (\log\log \nu)^2$ is convergent and $k_1, k_2, \cdots$ is a sequence of integers such that

$$\frac{k_\nu}{k_{\nu-1}} > \alpha > 1 \qquad (\nu = 1, 2, \cdots),$$

then the sequence

$$S(x, k_1),\ S(x, k_2), \cdots = \{S(x, k_\nu)\}$$

converges almost everywhere in (0, 1). But if $\Sigma c_\nu^2 \log \nu$ is convergent, then there is a sequence $k_1, k_2, \cdots$ such that

$$\frac{k_\nu}{k_{\nu-1}} \to 1 \quad \text{as} \quad \nu \to \infty$$

for which the sequence

$$S(x, k_1),\ S(x, k_2), \cdots$$

converges almost everywhere in (0, 1).

In fact, if we set

$$k_{\nu-1} = [\nu^{\log \nu}] + 1$$

then, as is easily shown, we have

$$\frac{k_{\nu+1}}{k_\nu} \to 1, \quad k_{\nu+1} - k_\nu \to \infty, \quad \text{as} \quad \nu \to \infty.$$

Without loss of generality, we can assume that

5) The proof of 'Hilfassatz III' contains the above result.

6) The proof of 'Satz I' involves this. Also S. Kaczmarz has already shown the theorem: If Σc_ν is convergent, then the convergence of $\{S(x, 2^m)\}$ is necessary and sufficient for the $(C, 1)$-summability of $\Sigma c_\nu \varphi_\nu(x)$, (almost everywhere in (0, 1)).—S. Kaczmarz (2).

7) L.c. 'Satz II'.