MATHEMATICAL MODELING AND SIMULATION IN HYDRODYNAMIC STABILITY

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Preface

In the last several decades, there has been great progress on the subject of hydrodynamic stability mainly due to mathematical modeling and numerical simulation efforts which were based on various analytical and computational methods. The subject of hydrodynamic stability is of fundamental importance in the mechanics of fluids and its applications in various fields of engineering, geophysical, astrophysical and environmental sciences. An important problem that hydrodynamic stability is concerned is that of instabilities that can take place and their transition sequences towards turbulence regime.

The initial motivation for preparing the present volume was the interest expressed by several invited speakers at the 14th World Congress of the International Association for Mathematics and Computers in Simulation (IMACS) (Atlanta, GA, July 1994) who participated in special sessions on hydrodynamic stability one of which was organized by the present editor. As a latest research volume on the subject of hydrodynamic stability, this book has developed from manuscripts submitted by internationally recognized applied mathematicians and scientists who have made important contributions to the mathematical modeling and simulation aspects of hydrodynamic stability. This book brings together current developments in the theory and applications of hydrodynamic stability, which were due to the mathematical modeling and simulation efforts in the subject, and it is hoped that they provide sufficient stimulation and directions for future research investigations on the subject.

The chapters in this book are placed in a sequence based on the alphabetical order of the main authors' last name. The first chapter, by A. Bottaro and P. Luchini, presents development of a mathematical model for the problem of the stability of the disturbances for the Görtler vortices. Chapter two, by F.H. Busse and R.M. Clever, present a general outline of the bifurcation type approach towards an understanding of complex flows. The evolution from simple to complex flows is studied by a sequence of mathematical models and simulations through the Galerkin procedure. Chapter three, by A.L. Frenkel and K. Indireshkumar, discusses the evolution equations and the associated theories for wavy flows of viscous films on solid surfaces. Both perturbation analysis and numerical simulations are employed and some unresolved questions are posed and discussed. Chapter four, by L. Hadji, considers the influence of thermal Soret diffusion on the instabilities in a dilute binary mixture during solidification process. The issue of constitutional super cooling and the importance of the coupling between solidification and Soret-driven flow are analyzed and discussed. Chapter five, by W.R.C. Phillips, reviewes rotational waves and their nonlinear interaction with shear flows. The interaction process is described through a Lagrangian formulation and a theory. Chapter six, by D.N. Riahi, reviews nonlinear stability analysis and modeling for convective flows, and the latest research results for several convective flow problems are presented. Chapter seven, by D.N. Riahi, reviews primary hydrodynamic instabilities and their mathematical modeling and simulation for several shear flow cases, and the latest research results for these shear flows are presented. Chapter eight, by A. Zebib, A. Bottaro and B.G.B. Klingmann, determines spatial development of longitudinal vortices through modeling and simulation, and various results are presented for a range of rotation numbers.

I would like to thank the authors for their effective contributions which demonstrate the power of mathematical modeling and simulation for the advancement of the research in the subject of hydrodynamic stability. I also would like to thank Ms. Kim Tan, editor at World Scientific Publishing Company, Inc., for her help in publishing this monograph.

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CONTENTS

THE LINEAR STABILITY OF GÖRTLER VORTICES REVISITED

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ABSTRACT

The linear stability theory for the Görtler vortex problem is reexamined. Because of the non-parallelism of the base flow, a set of parabolic, partial differential equations rules the stability of the disturbances. A unique solution is not available for this problem since each different initial condition provides a different, equally valid, answer. However, by exploiting the fact that two streamwise scales appear far enough downstream of the leading edge of the curved plate, an asymptotic formulation is constructed which allows to consistently take into account non-parallel effects in the evaluation of the total amplification factor of the instability.

1. Introduction

Görtler vortices (or Görtler-like vortices) occur in a vast variety of flow configurations, in technical applications and in nature. They are streamwise-oriented vortices, with the fluid slowly spiralling around an axis, of dimensions comparable to those of the local boundary layer thickness. Although Görtler vortices can occur in both laminar and turbulent environments, the discussion is here limited to the primary Görtler instability, where a laminar boundary layer developing along a curved surface becomes destabilized by the action of centrifugal forces, and near-wall streamwise vortices are amplified. The basic flow situation is sketched in figure 1. The flow developing along a curved passage follows a curvilinear trajectory in the plane of the figure and is subject to the action of centrifugal forces which tend to deviate the fluid particles from their trajectories. The instability mechanism is of inviscid nature and arises from a local disequilibrium between the centrifugal force term $\rho U^2/r$ and the restoring normal pressure gradient $\partial p/\partial r$. The new stable flow that ensues is sketched in figure 2.

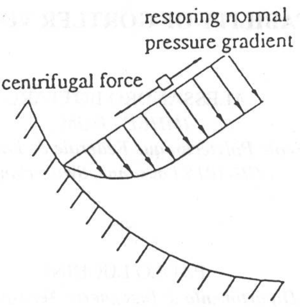

Figure 1. Curved boundary layer. This flow is susceptible to an instability when the centrifugal force differs from the restoring normal pressure gradient. The instability results in the formation of streamwise vortices.

Although the vortices thus created can themselves be unstable to other kinds of instabilities (secondary instabilities), the scope of the present paper is to outline the linear theory of the primary instability alone.

Since Görtler's original theory[1], which provided a theoretical framework for the instability developing along a curved boundary layer, much work has been devoted toward the understanding of what became known as "Görtler vortices". A comprehensive review of the topic is provided by Saric[2]. Here it suffices to say that until Hall[3] only local linear stability studies were carried out, in which the streamwise growth of the basic boundary layer and the convective nature of the instability were ignored. Later, it became clear that an appropriate description of the vortices required consideration of non-parallel effects.

2. Mathematical description of the primary instability

We consider the flow over a concave surface of constant radius of curvature *R*. The dimensional Navier-Stokes and continuity equations in cylindrical coordinates (r, θ, z) and corresponding velocity components (v, u, w) are the starting point of the analysis:

$$(rv)_r + u_\theta + rw_z = 0, \tag{1.1}$$

$$u_t + vu_r + \frac{uu_\theta}{r} + wu_z + \frac{uv}{r} + \frac{p_\theta}{r\rho} = \nu\,[\nabla^2 u + \frac{2v_\theta}{r^2} - \frac{u}{r^2}], \tag{1.2}$$

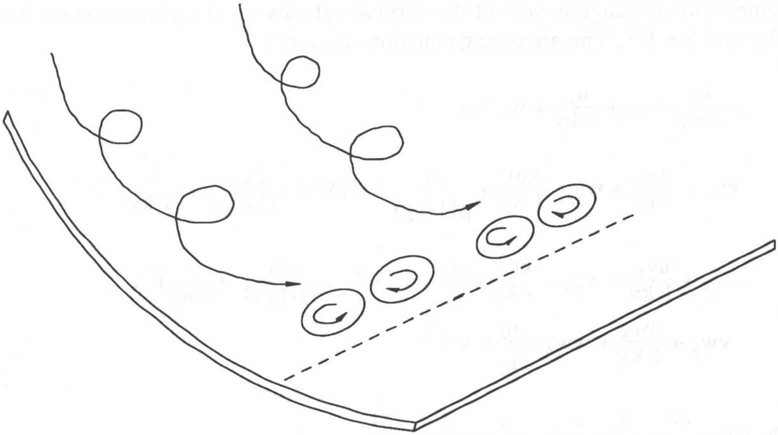

Figure 2. **Sketch of particle paths and of secondary flow in a cross-section perpendicular to a curved surface.**

$$v_t + v v_r + \frac{u v_\theta}{r} + w v_z - \frac{u^2}{r} + \frac{p_r}{\rho} = \nu \, [\nabla^2 v - \frac{2u_\theta}{r^2} - \frac{v}{r^2}], \tag{1.3}$$

$$w_t + v w_r + \frac{u w_\theta}{r} + w w_z + \frac{p_z}{\rho} = \nu \, \nabla^2 w, \tag{1.4}$$

with $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r \, \partial r} + \frac{\partial^2}{r^2 \, \partial \theta^2} + \frac{\partial^2}{\partial z^2}$, ν kinematic viscosity and ρ density.

There is experimental and theoretical evidence (Herron[4], Swearingen and Blackwelder[5]) that the primary instability is stationary. Hence we can safely drop the time-dependent terms. Furthermore, we can express the equations in cartesian-like coordinates by the introduction of

$$x = R\,\theta\,, \tag{2.1}$$

$$y = R - r\,, \tag{2.2}$$

together with a change in sign of the vertical velocity v. We also introduce the curvature of the wall $k = R^{-1}$. The governing equations become:

$$-\frac{kv}{1-ky} + v_y + \frac{u_x}{1-ky} + w_z = 0, \tag{3.1}$$

$$vu_y + \frac{uu_x}{1-ky} + wu_z - \frac{kuv}{1-ky} + \frac{p_x}{\rho(1-ky)} = \nu\left[\nabla^2 u - \frac{2kv_x}{(1-ky)^2} - \left(\frac{k}{1-ky}\right)^2 u\right], \tag{3.2}$$

$$vv_y + \frac{uv_x}{1-ky} + wv_z + \frac{ku^2}{1-ky} + \frac{p_y}{\rho} = \nu\left[\nabla^2 v + \frac{2ku_x}{(1-ky)^2} - \left(\frac{k}{1-ky}\right)^2 v\right], \tag{3.3}$$

$$vw_y + \frac{uw_x}{1-ky} + ww_z + \frac{p_z}{\rho} = \nu\, \nabla^2 w, \tag{3.4}$$

with $\nabla^2 = \frac{\partial^2}{\partial y^2} - \frac{k}{1-ky}\frac{\partial}{\partial y} + \frac{1}{(1-ky)^2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$.

If l is a typical length along the wall and U_l is a typical value of the free stream velocity, the dimensionless boundary layer coordinates X, Y, and Z can be introduced via the equations:

$$X = \frac{x}{l}, \qquad Y = \frac{y}{\delta}, \qquad Z = \frac{z}{\delta}, \tag{4}$$

where $Re = \frac{U_l \delta}{\nu} \gg 1$ is the Reynolds number, with $\delta = \left(\frac{\nu l}{U_l}\right)^{1/2}$. We also define $K = k\,\delta$, $K \ll 1$. The velocity components (u, v, w) and the pressure p are made nondimensional and expanded in terms of a small parameter ε as follows:

$$\frac{u}{U_l} = U(X,Y) + \varepsilon\, u'(X,Y,Z) + O(\varepsilon^2), \tag{5.1}$$

$$\frac{v}{U_l} Re = V(X,Y) + \varepsilon\, v'(X,Y,Z) + O(\varepsilon^2), \tag{5.2}$$

$$\frac{w}{U_l} Re = \varepsilon\, w'(X,Y,Z) + O(\varepsilon^2), \tag{5.3}$$

$$\frac{p}{\rho U_l^2} Re^2 = P(X)\, Re^2 + p_0(X,Y) + \varepsilon\, p'(X,Y,Z) + O(\varepsilon^2). \tag{5.4}$$

Grouping together all terms proportional to the same power of ε and neglecting terms of order Re^{-2} and smaller, we find:

$O(1)$:

$$-\frac{KV}{1-KY}+V_Y+\frac{U_X}{1-KY}=0, \tag{6.1}$$

$$VU_Y+\frac{UU_X}{1-KY}-\frac{KUV}{1-KY}+\frac{P_X}{1-KY}=U_{YY}-\frac{KU_Y}{1-KY}-\left(\frac{K}{1-KY}\right)^2 U\,, \tag{6.2}$$

$$VV_Y+\frac{UV_X}{1-KY}+\frac{(GU)^2}{1-KY}+p_{0Y}=V_{YY}-\frac{KV_Y}{1-KY}+\frac{2KU_X}{(1-KY)^2}-\left(\frac{K}{1-KY}\right)^2 V\,, \tag{6.3}$$

$O(\varepsilon)$: (note that the primes have been dropped)

$$-\frac{Kv}{1-KY}+v_Y+\frac{u_X}{1-KY}+w_Z=0, \tag{6.4}$$

$$Vu_Y+vU_Y+\frac{Uu_X+uU_X}{1-KY}-\frac{K(Uv+uV)}{1-KY}=u_{YY}+u_{ZZ}-\frac{Ku_Y}{1-KY}-\left(\frac{K}{1-KY}\right)^2 u\,, \tag{6.5}$$

$$Vv_Y+vV_Y+\frac{Uv_X+uV_X}{1-KY}+\frac{2G^2Uu}{1-KY}+p_Y=v_{YY}+v_{ZZ}-\frac{Kv_Y}{1-KY}+ + \frac{2Ku_X}{(1-KY)^2}-\left(\frac{K}{1-KY}\right)^2 v\,, \tag{6.6}$$

$$Vw_Y+\frac{Uw_X}{1-KY}+p_Z=w_{YY}+w_{ZZ}-\frac{Kw_Y}{1-KY}, \tag{6.7}$$

with the (finite) Görtler number G defined by

$$G^2=\frac{l}{R}\,Re=K\,Re^2. \tag{7}$$

Although K is of order Re^{-2}, the derivation of the final perturbation equations is carried out keeping all the K terms. Some of these terms (particularly the product KY) might not be negligible when small spanwise wavenumbers are considered. The leading order equations (6.1-2) allow the determination of the basic flow and pressure fields (U, V, P). Equation (6.3) is not used; it shows that a normal gradient of p_0 is set up to balance the centrifugal term $(G\,U)^2$. For the limiting case of $K \rightarrow 0$, a self-similar boundary layer solution can be obtained for the general case of non-zero streamwise pressure gradients by assuming the outer flow to vary as X^m, and by enforcing no-slip conditions at the wall. This procedure yields the well known Falkner-Skan similarity solution, so that a nondimensional stream-function $f(\eta)$ satisfying the ordinary differential equation

$$f''' + \frac{1}{2}(m+1)\, f f'' + m\,(1 - f'^2) = 0 \qquad (8)$$

with boundary conditions

$$f = f' = 0 \qquad \text{at } \eta = 0, \qquad (9.1)$$
$$f' = 1 \qquad \text{at } \eta \rightarrow \infty, \qquad (9.2)$$

is sought. The similarity variable η is defined by

$$\eta = Y\, X^{(m-1)/2}\,, \qquad (10)$$

and the velocity components U and V are given by

$$U = X^m f', \qquad (11.1)$$

$$V = X^{(m-1)/2}\left(\frac{1-m}{2}\eta f' - \frac{1+m}{2} f\right). \qquad (11.2)$$

For m=0 the Blasius solution is recovered, and the stability of the Blasius flow constitutes the classical Görtler problem. For positive values of m the outer flow is accelerated (favourable or negative pressure gradient) and a unique Falkner-Skan solution is available for each value of m; the case m = -0.0904 corresponds to the limiting situation of decelerating flow without separation at the wall. For decelerating flows with -0.0904 < m < 0 there are an infinite number of solutions available, the laminar wall jet in an external stream being an example[6]. Although such flows are of interest, we focus here only on the classical solutions where U is a monotonically increasing function of η. The combined Görtler-Coriolis instability of wall jets with spanwise system rotation has been recently studied by Matsson[7].

Eqs. (6.4-7) define the linear stability problem for the leading order perturbations (u,v,w) and for p. When $K \rightarrow 0$ the equations already given by Floryan and Saric[8] and Hall[3,9] are recovered. The boundary conditions appropriate to Eqs. (6.4-7) are

$$u = v = w = 0 \qquad \text{at } Y = 0 \text{ and } Y \rightarrow \infty. \qquad (12)$$

A form for the perturbation must be chosen and introduced into Eqs. (6.4-7). Because of the lack of translational invariance along X we have:

$$(u,v,w,p) = (\tilde{u}(X,Y)\cos\alpha Z, \tilde{v}(X,Y)\cos\alpha Z, \tilde{w}(X,Y)\sin\alpha Z, \tilde{p}(X,Y)\cos\alpha Z), \tag{13}$$

with α real spanwise wavenumber. The final set of disturbance equations can be obtained through elimination of the pressure term and w (note that tildes have been dropped):

$$u_{YY} - \left(V + \frac{K}{1-KY}\right) u_Y - \frac{U}{1-KY} u_X - \left[\alpha^2 + \frac{U_X}{1-KY} - \frac{KV}{1-KY} + \left(\frac{K}{1-KY}\right)^2\right] u + \\ + \left(- U_Y + \frac{KU}{1-KY}\right) v = 0, \tag{14.1}$$

$$v_{YYYY} - \left(V + \frac{2K}{1-KY}\right) v_{YYY} - \frac{U}{1-KY} v_{XYY} - \left[2\alpha^2 + V_Y - \frac{KV}{1-KY} + 3\left(\frac{K}{1-KY}\right)^2\right] v_{YY} + \\ - \frac{KU}{(1-KY)^2} v_{XY} + \left[\alpha^2 V - 3\left(\frac{K}{1-KY}\right)^3 + \frac{3K^2V-2KU_X}{(1-KY)^2} + \frac{2K\alpha^2+U_{XY}}{1-KY}\right] v_Y + \\ + \left[\frac{-2K^2U}{(1-KY)^3} + \frac{3KU_Y}{(1-KY)^2} + \frac{\alpha^2 U+U_{YY}}{1-KY}\right] v_X + \\ + \left[\alpha^4 + \alpha^2 V_Y - 3\left(\frac{K}{1-KY}\right)^4 + \frac{-5K^2U_X+3K^3V}{(1-KY)^3} + \frac{2\alpha^2K^2+2KU_{XY}}{(1-KY)^2} + \frac{U_{XYY}}{1-KY}\right] v + \\ + \frac{V_X}{1-KY} u_{YY} + \left[\frac{8K^2}{(1-KY)^3} + \frac{2U_X}{(1-KY)^2}\right] u_{XY} + \frac{2KU}{(1-KY)^3} u_{XX} + \frac{3KV_X}{(1-KY)^2} u_Y + \\ + \left[\frac{8K^3}{(1-KY)^4} + \frac{10KU_X-8K^2V}{(1-KY)^3} + \frac{2U_{XY}}{(1-KY)^2}\right] u_X + \\ + \left[\frac{5KU_{XX}-5K^2V_X}{(1-KY)^3} + \frac{U_{XXY}}{(1-KY)^2} + \frac{2\alpha^2G^2U+\alpha^2V_X}{1-KY}\right] u = 0, \tag{14.2}$$

with boundary conditions

$$u = v = v_Y = 0 \qquad \text{at } Y = 0 \text{ and } Y \to \infty, \tag{15}$$

and initial conditions

$$u = u_0 \text{ and } v = v_0 \qquad \text{at } X = X_0. \tag{16}$$

These equations, together with Eqs. (6.1-3), are necessary to handle the stability at small values of the wavenumber α, where a very strong sensitivity to small variations of the basic state exists[10-12]. The stability equations reduce to those given by Hall[3] when K is set equal to 0. The parameters ruling this problem are the Görtler number G, the curvature parameter K, and the initial position X_0 at which the perturbations are provided. Furthermore, the initial perturbation distributions u_0 and v_0 plays a crucial role in the instability development, particularly in proximity of X_0; as a consequence it was shown[3] that the concept of a unique neutral curve is not tenable. To monitor the development of the instability one can define a perturbation energy, for example

$$E(X) = \int u^2(X,Y)\, dY \tag{17}$$

and introduce a growth rate σ as

$$\sigma = \frac{1}{2E}\frac{dE}{dX} \, . \tag{18}$$

The position of "neutral" stability is defined by the value of X for which σ vanishes: since a Görtler number $G_X = G\, X^{(3+m)/4}$ is associated with each X, a "neutral stability curve" can then be constructed. Goulpié *et al.*[13] have, however, demonstrated that different definitions of energy can produce very different and discording predictions of neutral points; this reinforces the argument that a unique marginal curve can not be defined. Figure 3 attests to this. Analytical progress towards an "amplification threshold" is, however, possible if one exploits the fact that two longitudinal scales exist: one for the development of the base flow and one for the development of the perturbation. In the following section, a local procedure that consistently allows consideration of non-parallel effects will be outlined.

3. Multiple-scale analysis

In the presence of a spanwise-varying perturbation with wavenumber α, the centrifugal term in Eq. (3.3) provides an amplification rate of disturbances of the order of $\alpha(\delta/R)^{1/2}$, which is contrasted by a viscous damping of the order of $\alpha^2\nu/U_l$. Therefore, whereas the balance between amplification and damping becomes positive at very small wavenumbers even for Görtler numbers less than unity (i.e. provided that $\alpha\delta < (\delta/R)^{1/2}$ $\delta U_l/\nu = G$), the scale of length over which the perturbation evolves, namely $[\alpha(\delta/R)^{1/2} - \alpha^2\nu/U_l]^{-1}$, is comparable to or greater than the longitudinal boundary layer scale l unless $\alpha = O(\delta^{-1})$ and $(\delta R)^{1/2} << l$. When, on the other hand, this condition is satisfied, a multiple-

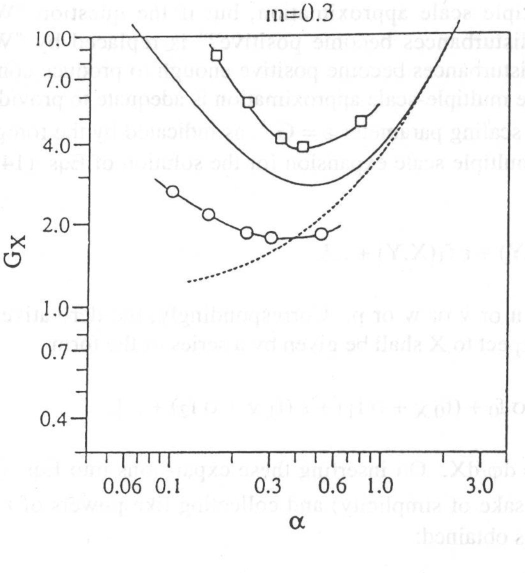

Figure 3. Some of the different neutral curves that can be obtained from local and marching analyses for a boundary layer with negative streamwise pressure gradient (m = 0.3), K → 0. The continuous line represents the prediction using Floryan and Saric's local theory[8], the dashed line is the prediction from Görtler's original equations[1], and the symbols correspond to predictions from the nonlocal theory[3] with two different growth criteria.

scale approximation can be employed in which $(\delta R)^{1/2}$ plays the role of the "fast" scale and l plays the role of the "slow" one. Since the small ratio $(\delta R)^{1/2}/l$ (which is the same as G^{-1} with the definition (7) of the Görtler number G) is not likely to be exceedingly small in practice, a multiple scale approximation of higher than leading order is well suited for this problem. On the other hand, when the Görtler number is of order unity or less and α is small compared to δ^{-1}, the multiple scale approximation fails; this range of parameters, however, appears to be relatively uninteresting, even if a positive rate of amplification (whose definition in this range is not obvious) can be found, because the driving centrifugal force is too small to produce an important total amplification before the properties of the boundary layer change completely. This is in contrast to what happens in curved channel flow (the Dean problem), where even a very small amplification rate can produce considerable integrated effects in a channel of suitable length; the "available length" of the boundary layer (i.e., the length over which its properties do not change too much) is limited, and the product of the order of magnitude of the amplification rate times this length is exactly what the Görtler number measures. It may look disturbing that this approach entails renouncing to the concept of a "neutral curve", or at least renouncing to finding one by a multiple scale approximation, but if the question "When does the amplification rate of disturbances become positive?" is replaced by "When does the amplification rate of disturbances become positive enough to produce considerable total amplification?", then the multiple-scale approximation is adequate to provide an answer.

Using the small scaling parameter $\varepsilon = G^{-1}$, as indicated by the foregoing order-of-magnitude analysis, a multiple scale expansion for the solution of Eqs. (14) can be set up in the form:

$$f = e^{\varphi(X)/\varepsilon}\,[f_0(X,Y) + \varepsilon\, f_1(X,Y) + \ldots], \tag{19}$$

where f denotes either u or v or w or p. Correspondingly, the derivative of any one of these quantities with respect to X shall be given by a series of the form

$$f_X = e^{\varphi(X)/\varepsilon}\,[\varepsilon^{-1}\,\sigma\, f_0 + (f_{0,X} + \sigma\, f_1) + \varepsilon\,(f_{1,X} + \sigma\, f_2) + \ldots], \tag{20}$$

with the definition $\sigma = d\varphi/dX$. On inserting these expansions into Eqs. (14) (where we have set K = 0 for the sake of simplicity) and collecting like powers of ε, the following hierarchy of equations is obtained:

$O(\varepsilon^{-1})$:

$$-\sigma\, U\, u_0 = 0, \tag{21.1}$$

$$-\sigma\, U\, v_{0,YY} + (\alpha^2 U + U_{YY})\,\sigma\, v_0 + 2\,\alpha^2\, U\, u_1 = 0, \tag{21.2}$$

$O(1)$:

$$-\sigma\, U\, u_1 - U_Y\, v_0 = 0, \tag{21.3}$$

$$\begin{aligned} -\sigma\, U\, v_{1,YY} + (\alpha^2 U + U_{YY})\,\sigma\, v_1 + 2\,\alpha^2\, U\, u_2 = {} & -v_{0,YYYY} + V\, v_{0,YYY} + \\ & + U\, v_{0,XYY} + (2\alpha^2 + V_Y)\, v_{0,YY} - (\alpha^2 V + U_{XY})\, v_{0,Y} - (\alpha^2 U + U_{YY})\, v_{0,X} + \\ & - (\alpha^4 + \alpha^2 V_Y + U_{XYY})\, v_0 - 2\,\sigma\, U_X\, u_{1,Y} - 2\,\sigma\, U_{XY}\, u_1\,, \end{aligned} \tag{21.4}$$

$O(\varepsilon)$:

$$-\sigma\, U\, u_2 - U_Y\, v_1 = -u_{1,YY} + V\, u_{1,Y} + U\, u_{1,X} + (\alpha^2 + U_X)\, u_1, \tag{21.5}$$

etc.

However, a difficulty, common to many other boundary layer perturbation expansions, is encountered: the expansion given above is not uniformly valid in Y, because the highest Y-derivatives are lost at leading order and with them disappears the possibility of enforcing a certain number of boundary conditions. Exactly the same difficulty turns up, for instance, in the derivation of the Orr-Sommerfeld equation as applied to non-parallel problems. The possible solutions to this difficulty are twofold. One approach is to develop a multiple-deck description of the Y-structure of the problem, as, for instance, in Hall's[9] analysis of the large wavenumber limit in which the amplification turns back into damping far enough downstream. Alternatively, one can promote the largest Y-derivative terms to an earlier order in the hierarchy than their formal dependence on ε would suggest, thus obtaining the modified equations

$$- \sigma U v_{0,YY} + (\alpha^2 U + U_{YY}) \sigma v_0 + 2 \alpha^2 U u_1 + \varepsilon [v_{0,YYYY} - V v_{0,YYY} + - (2\alpha^2 + V_Y) v_{0,YY} + (\alpha^2 V + U_{XY}) v_{0,Y} + (\alpha^4 + \alpha^2 V_Y + U_{XYY}) v_0 + + 2 \sigma U_X u_{1,Y} + 2 \sigma U_{XY} u_1] = 0, \quad (22.1)$$

$$- \sigma U u_1 - U_Y v_0 + \varepsilon [u_{1,YY} - V u_{1,Y} - (\alpha^2 + U_X) u_1] = 0, \quad (22.2)$$

$$- \sigma U v_{1,YY} + (\alpha^2 U + U_{YY}) \sigma v_1 + 2 \alpha^2 U u_2 + \varepsilon [v_{1,YYYY} - V v_{1,YYY} + -(2\alpha^2 + V_Y) v_{1,YY} + (\alpha^2 V + U_{XY}) v_{1,Y} + (\alpha^4 + \alpha^2 V_Y + U_{XYY}) v_1 + + 2 \sigma U_X u_{2,Y} + 2 \sigma U_{XY} u_2] = U v_{0,XYY} - (\alpha^2 U + U_{YY}) v_{0,X}, \quad (22.3)$$

$$- \sigma U u_2 - U_Y v_1 + \varepsilon [u_{2,YY} - V u_{2,Y} - (\alpha^2 + U_X) u_2] = U u_{1,X}, \quad (22.4)$$

The price to be paid for this modification is, just like in the Orr-Sommerfeld problem, that the single terms f_0, f_1, etc. are now themselves functions of ε, so that the final result of truncating the series will not be a polynomial in ε; the reward is that a definite most amplified mode shape and amplification factor will be obtained independently of the initial transient from which the mode is generated. Of course, if the initial transient itself is to be studied other techniques must, and can, be employed; but this is a different subject (the so-called receptivity problem).

In the just derived hierarchy, Eqs. (22.1-2) form a coupled system of homogeneous ordinary differential equations for the unknowns v_0 and u_1 which must be solved with σ in the role of eigenvalue. The coefficients of these equations depend parametrically on X and so will the solution (i.e., the numerical solution will be different at successive X-stations). The X-derivatives of this leading-order solution then appear as known right-hand-sides in Eqs. (22.3-4), which must be solved as ordinary differential equations in the