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UNTER MITWIRKUNG DER SCHRIFTLEITUNG DES
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NEUE FOLGE · HEFT 19

POLYNOMIAL EXPANSIONS OF ANALYTIC FUNCTIONS

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WITH 16 FIGURES

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1958

PRINTED IN GERMANY

Druck der Universitätsdruckerei H. Stürtz AG., Würzburg

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REIHE:

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BESORGT

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Preface

This monograph deals with the expansion properties, in the complex domain, of sets of polynomials which are defined by generating relations. It thus represents a synthesis of two branches of analysis which have been developing almost independently. On the one hand there has grown up a body of results dealing with the more or less formal properties of sets of polynomials which possess simple generating relations. Much of this material is summarized in the Bateman compendia (ERDÉLYI [1], vol. III, chap. 19) and in TRUESDELL [1]. On the other hand, a problem of fundamental interest in classical analysis is to study the representability of an analytic function $f(z)$ as a series $\sum c_n p_n(z)$, where $\{p_n\}$ is a prescribed sequence of functions, and the connections between the function f and the coefficients c_n. BIEBERBACH's monograph *Analytische Fortsetzung* (Ergebnisse der Mathematik, new series, no. 3) can be regarded as a study of this problem for the special choice $p_n(z) = z^n$, and illustrates the depth and detail which such a specialization allows. However, the wealth of available information about other sets of polynomials has seldom been put to work in this connection (the application of generating relations to expansion of functions is not even mentioned in the Bateman compendia).

At the other extreme, J. M. WHITTAKER and his students have obtained many results about expansions of analytic functions in the so-called basic series which can be associated with very general sets of polynomials. (See especially WHITTAKER [2].) In that theory the degree of generality is so great that broad rather than refined results are to be expected, and the internal structure of the prescribed sequence of polynomials does not play much of a role. For example, the basic tool of the Whittaker theory is the rearrangement of power series, and so the theory is dominated by the presence of circular regions of convergence.

We have adopted an intermediate position by discussing the expansion of analytic functions in series of polynomials defined by a rather general kind of generating relation. Here we have tried to bring about a certain amount of order and completeness and to formulate results and methods in a fashion which will make them more generally accessible. While we do not obtain as much information about the expansions as is obtainable, for example, about power series, we obtain considerably more than is obtainable for more general sets of polynomials: thus we are not restricted to circular regions, or to the basic series, and we can discuss summability as well as convergence. Results of this kind appear here and there in the literature (mostly for the classical orthogonal polynomials), but as isolated observations and not as part of a coherent theory. (An exception is MARTIN [1], where a theory of the expansion of entire functions is developed by the method which we follow in general.) The chief tool that we use is the method of kernel expansion: this is at least as old as CAUCHY's deduction of TAYLOR's theorem from CAUCHY's integral formula[1].

While little that we have to say is new in principle, some of the general theory that we present is rather more general than anything we have seen elsewhere, and some of the details seem not to have been worked out before. In particular, we have illustrated the general theory by applying it to many of the almost innumerable sets of polynomials which have been introduced into the literature for one reason or another. The material of § 8 on the possible form of multiple expansions of a given function has not been published before[2]. Some open questions are mentioned on pp. 10, 18, 27 and 29.

This study has been developed at intervals during the past twelve years. For financial support during parts of this period, we are indebted to the John Simon Guggenheim Memorial Foundation and Northwestern University (BOAS) and to the Office of Ordnance Research (BUCK).

Evanston (Illinois) June 1957 R. P. BOAS, JR.
Madison (Wisconsin) June 1957 R. C. BUCK

[1] See PRINGSHEIM [1].

[2] (Added in proof). We understand that some of our results, as far as they concern Appell sets, were obtained independently by J. STEINBERG, Application de la théorie des suites d'Appell à une classe d'équations intégrales, Bull. Research Council Israel, Sect. A. 7, no. 2 (to appear in 1958).

Contents

Chapter I

Introduction

§ 1. Generalities

The place of our work in the theory of polynomial expansions will be seen best if we begin with some general remarks. Let $\mathfrak{P}$ be the complex linear space of all polynomials, with the topology of uniform convergence on all compact subsets of a simply-connected region Ω. The completion of $\mathfrak{P}$ is then the space $\mathfrak{A}(\Omega)$ of all functions f which are analytic in Ω. Let $\sigma = \{p_n\}$ be a sequence of polynomials which forms a basis for $\mathfrak{P}$: that is, any $p \in \mathfrak{P}$ has a unique representation as a finite sum $p = \sum c_n p_n$. It is customary to call such a σ a basic set of polynomials. Then every $f \in \mathfrak{A}(\Omega)$ is the limit of a sequence of finite sums of the form $\sum\limits_n a_{k,n} p_n$. Of course this by no means implies that there are numbers c_n such that $f = \Sigma c_n p_n$ with a convergent or even a summable series. One way of attaching a series to a given function is as follows. Since σ is a basis, in particular there is a row-finite infinite matrix, unique among all such matrices, such that

$$z^k = \sum_{n=0}^{\infty} \pi_{k,n} p_n(z), \qquad k = 0, 1, 2, \ldots. \tag{1.1}$$

Suppose that Ω contains the origin, let f be analytic at the origin, and write

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(0)\, z^k/k!\,. \tag{1.2}$$

If we formally substitute (1.1) into (1.2), we obtain

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \sum_{n=0}^{\infty} \pi_{k,n} p_n(z),$$

or

$$f(z) = \sum_{n=0}^{\infty} c_n p_n(z), \tag{1.3}$$

where

$$c_n = \sum_{k=0}^{\infty} \pi_{k,n} f^{(k)}(0)/k!\,. \tag{1.4}$$

The expansion (1.3) with coefficients (1.4) is the so-called basic series, introduced by J. M. WHITTAKER [1], and studied in more detail by him in a recent monograph [2] and by his students in a long series of papers[1]. A typical theorem is the following: with the basic set $\{p_n\}$ we associate two numbers ω ("order") and γ ("type"); then every entire function of growth less than order $1/\omega$, type $1/\gamma$ is represented by its basic series (1.3) with coefficients (1.4); and in general functions of more rapid growth are not so representable.

The present study arose from the observation that this theorem, while it tells nothing but the truth, does not tell the whole truth. The following simple example will serve as an illustration.

Consider the basic set defined by

$$\left.\begin{aligned} p_0 &= -1 \\ p_n(z) &= \frac{z^{n-1}}{(n-1)!} - \frac{z^n}{n!}, \quad n = 1, 2, \ldots \end{aligned}\right\} \tag{1.5}$$

It can easily be calculated that this set is of order 1 and type 1, so that every entire function of exponential type less than 1 is represented by the basic series. As WHITTAKER'S general theory predicts, this result is to be regarded as sharp. For, in this special case formula (1.4) for the coefficients takes the form

$$c_n = -\sum_{k=n}^{\infty} f^{(k)}(0). \tag{1.6}$$

Applying this to $f(z) = e^z$, an entire function of exponential type 1, we obtain $c_n = -(1+1+\cdots)$. Since the formula (1.4) thus fails to define coefficients for the basic series (1.3), it is customary to say that the basic series does not exist, and certainly does not represent e^z. However, it is easy to see that every entire function f, and indeed every function analytic at 0, has a convergent representation

$$f(z) = \sum c_n' p_n(z),$$

where $c_0' = 0$ and

$$c_n' = \sum_{k=0}^{n-1} f^{(k)}(0), \quad n = 1, 2, \ldots. \tag{1.7}$$

Moreover, since $\sum_{n=0}^{\infty} p_n(z)$ converges for every z to the sum 0, we also have $f(z) = \sum (c_n' + a) p_n(z)$ for every a, so that every function analytic at 0 has an infinity of convergent expansions in terms of the $p_n(z)$. In the light of such examples, we have felt that it is advisable to re-examine the subject.

Let us again consider the space $\mathfrak{P}$, its completion $\mathfrak{A}(\Omega)$ and a given basis σ. By $\mathfrak{C}(\sigma)$ we shall mean the subspace of $\mathfrak{A}$ consisting of all f that can be expressed in the form of a convergent series

$$f(z) = \sum_{n=0}^{\infty} c_n p_n(z), \tag{1.8}$$

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[1] See Math. Reviews passim, particularly under DOSS, EWEIDA, MAKAR, MIKHAIL, MURSI, NASSIF, TANTAOUI; also NEWNS [1], FALGAS [1].

irrespective of the mode of formation of the c_n. We call $\mathfrak{E}(\sigma)$ the *expansion class* for σ. If $\mathfrak{E}(\sigma)=\mathfrak{A}$ and if in addition the expansion (1.8) is unique, $\{p_n\}$ is a base for $\mathfrak{A}$ and the c_n may be given the form $c_n=\mathscr{L}_n(f)$, where $\{\mathscr{L}_n\}$ is a sequence of linear functionals orthogonal to $\{p_n\}$. If $\mathfrak{E}(\sigma)=\mathfrak{A}$ and the expansion (1.8) is not necessarily unique, we call $\{p_n\}$ a *semibase*. In this case we have to make precise the notion of an expansion formula.

Let $\mathfrak{S}$ be the space of complex sequences $c=(c_0, c_1, \ldots)$ such that $\sum c_n p_n$ converges to an element of $\mathfrak{A}$. Let U be the linear transformation from $\mathfrak{S}$ onto $\mathfrak{E}(\sigma)$ sending $c\in\mathfrak{S}$ into $U(c)=f=\sum c_n p_n\in\mathfrak{E}(\sigma)$. Let T be any linear transformation whose domain is at least $\mathfrak{P}$ and whose range lies in $\mathfrak{S}$; suppose further that, for every $p\in\mathfrak{P}$, $UT(p)=p$, so that T is a right inverse for U on $\mathfrak{P}$. Then, in the domain of T, we have $UT(f)=f$. Denote the class of such f by $\mathfrak{E}(\sigma,T)$. Then we say that T defines an *expansion formula* applicable to the class $\mathfrak{E}(\sigma,T)$. If we represent T as a sequence of linear functionals $T=(\mathscr{L}_0, \mathscr{L}_1, \ldots)$, so that $T(f)=c$ where $c_n=\mathscr{L}_n(f)$, then for all $f\in\mathfrak{E}(\sigma,T)$ we have $f=UT(f)=U(c)=\sum\mathscr{L}_n(f)\,p_n$. We have thus shown that each right inverse of U on $\mathfrak{P}$ gives rise to an expansion formula and a class to which it is applicable. If U is one-to-one, there is essentially only one right inverse; in this case σ is a base for $\mathfrak{A}$. If U is not one-to-one, U has an infinite number of right inverses; among these, one may be singled out as the principal right inverse, as follows. Since σ is a basis for $\mathfrak{P}$ we may choose T so that $T(p)=(c_0, c_1, \ldots)$, the unique sequence of coefficients for which $p=\sum c_n p_n$ and $c_n=0$ for all large n. The expansion formula corresponding to this choice of T is the basic series.

We may illustrate this with our example (1.5). Here (1.6) defines the sequence $\{\mathscr{L}_n\}$ of functionals given by

$$\mathscr{L}_n(f)=-\sum_{k=n}^{\infty} f^{(k)}(0),$$

and (1.7) defines the sequence $\{\mathscr{L}'_n\}$ given by

$$\mathscr{L}'_0(f)=0,$$

$$\mathscr{L}'_n(f)=\sum_{k=0}^{n-1} f^{(k)}(0), \qquad n=1,2,\ldots.$$

The corresponding linear transformations $T=(\mathscr{L}_0, \mathscr{L}_1, \ldots)$ and $T'=(\mathscr{L}'_0, \mathscr{L}'_1, \ldots)$ are both right inverses for the transformation U associated with the sequence (1.5). The domain of T is a subspace of the domain of T', and so every function f in $\mathfrak{E}(\sigma,T)$ has two essentially different representations as series in the polynomials $\{p_n(z)\}$.

Further illustrations of these ideas will be given for other sets of polynomials as we come to them. Although results can be obtained by our methods for general basic sets of polynomials, the most interesting

1*

results apply only to basic sets that have a sufficient amount of intrinsic structure. We have found that a kind of generalized Appell set is sufficiently specialized to yield interesting results, yet sufficiently general to include many of the better-known polynomial sets, such as those associated with the names of LAGUERRE, LEGENDRE, HERMITE, CHEBYSHEV, GEGENBAUER and JACOBI. Before introducing the class of polynomial sets with which we shall chiefly work, we turn to a discussion of integral representations for analytic functions.

§ 2. Representation formulas with a kernel

We shall use an integral representation for analytic functions which contains both the Cauchy integral formula and the Pólya representation for entire functions of exponential type. It will be developed here in a somewhat more general form than is required for the applications we shall make[1].

Let $K(z, w)$ be analytic for (z, w) in an open set Λ containing the plane $z=0$. For any positive R, the compact set consisting of all points $(0, w)$ with $|w| \leq R$ lies in Λ, so that there is a positive δ such that (z, w) lies in Λ whenever $|z| \leq \delta$, $|w| \leq R$. For fixed R, let $\delta_1(R)$ be the supremum of such δ and set $\delta(R) = \lim\limits_{h \to 0} \delta_1(R+h)$. Then, for any choice of R, all points (z, w) with $|z| < \delta(R)$ and $|w| < R$ lie in Λ.

Let $F(w)$ be regular for $|w| > R$ and let Γ be the circle $|w| = R + \varepsilon$. Then

$$f(z) = (2\pi i)^{-1} \int_{\Gamma} K(z, w)\, F(w)\, dw \tag{2.1}$$

is regular for $|z| < \delta(R + 2\varepsilon)$. If we contract Γ by decreasing ε, we find that $f(z)$ is regular at least in the disk $|z| < \delta(R)$. Thus (2.1) defines a linear transformation $\boldsymbol{T}$ from the class of functions F that are regular at ∞ into the class of functions f that are regular at 0. We can also represent $\boldsymbol{T}$ as a sequence-to-sequence matrix transformation. Let

$$K(z, w) = \sum_{n,k=0}^{\infty} \sum C_{n,k}\, z^n w^k, \qquad |z| < \delta(R),\ |w| < R,$$

and let

$$F(w) = \sum_{k=0}^{\infty} F_k\, w^{-k-1}, \qquad |w| > R.$$

[There is no loss of generality from assuming $F(\infty) = 0$, since $F(w) - F(\infty)$ yields the same $f(z)$ in (2.1) as $F(w)$ does.] Then

$$\boldsymbol{T}(F)(z) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{\infty} C_{n,k} F_k = \sum_{n=0}^{\infty} a_n z^n = f(z). \tag{2.2}$$

[1] For somewhat similar discussions see, in particular, A. J. MACINTYRE [1], [2], EVGRAFOV [1], [2], LOHIN [1], FALGAS [2].

If we write

$$\varphi = (F_0, F_1, \ldots), \qquad \alpha = (a_0, a_1, \ldots),$$

(2.2) is equivalent to the matrix equation $\alpha = \boldsymbol{C}\varphi$, where $\boldsymbol{C} = (C_{n,k})$.

Many familiar integral formulas have the form (2.1) or (2.2). The simplest example is obtained by taking $\boldsymbol{C}$ to be the identity matrix $\boldsymbol{I} = (\delta_{n,k})$. Then

$$K(z, w) = \sum_{n,k=0}^{\infty}\sum \delta_{n,k} z^n w^k = \sum_{n=0}^{\infty} (zw)^n = \frac{1}{1 - zw},$$

so that (2.1) becomes

$$f(z) = (2\pi i)^{-1} \int_{\Gamma} \frac{F(w)\, dw}{1 - wz}. \tag{2.3}$$

In this case f has a simple alternative expression in terms of F, since $\alpha = \boldsymbol{I}\varphi = \varphi$, $f(z) = z^{-1}F(z^{-1})$, and (2.3) can be written

$$F(z^{-1}) = (2\pi i)^{-1} \int_{\Gamma} \frac{F(w)\, dw}{z^{-1} - w}$$

so that (2.1) reduces to CAUCHY's integral formula.

An interesting general class of transforms is obtained by restricting the matrix $\boldsymbol{C}$ to be triangular, with $C_{n,k} = 0$ for $n > k$. In this case $K(z, w)$ can be written

$$K(z, w) = \sum_{k=0}^{\infty} w^k \sum_{n=0}^{k} C_{n,k} z^n = \sum_{k=0}^{\infty} w^k Q_k(z),$$

where $Q_k(z)$ is a polynomial of degree k or less. Alternatively, we can write

$$\begin{aligned} K(z, w) &= \sum_{n=0}^{\infty} z^n \sum_{k=n}^{\infty} C_{n,k} w^k \\ &= \sum_{n=0}^{\infty} (zw)^n \sum_{k=0}^{\infty} C_{n,k+n} w^k \\ &= \Psi(zw, w), \end{aligned}$$

where $\Psi(s, t)$ is analytic in an open set containing the plane $s = 0$.

If $\Psi(s, t)$ is independent of t, then $K(z, w) = \Psi(zw)$, where $\Psi(t) = \sum \Psi_n t^n$ is regular at the origin. In this case, which is the one in which we are chiefly interested, the matrix $\boldsymbol{C}$ is diagonal, with $C_{n,n} = \Psi_n$. We then have

$$f(z) = (2\pi i)^{-1} \int_{\Gamma} \Psi(zw)\, F(w)\, dw, \tag{2.4}$$

or alternatively

$$\left.\begin{aligned} \Psi(t) &= \sum_{n=0}^{\infty} \Psi_n t^n, \\ F(w) &= \sum_{n=0}^{\infty} F_n w^{-n-1}, \\ f(z) &= \sum_{n=0}^{\infty} F_n \Psi_n z^n. \end{aligned}\right\} \tag{2.5}$$

If we assume further that no Ψ_n is zero, we can reconstruct $F(w)$ uniquely from a given $f(z)$, so that we can think of (2.4) as a representation formula for $f(z)$ instead of as defining a transform of $F(w)$. In this case it is convenient to change the notation and replace (2.5) by

$$\left.\begin{aligned} \Psi(t) &= \sum_{n=0}^{\infty} \Psi_n t^n, \quad \Psi_n \neq 0; \\ f(z) &= \sum_{n=0}^{\infty} f_n z^n, \\ F(w) &= \sum_{n=0}^{\infty} \frac{f_n}{\Psi_n w^{n+1}}. \end{aligned}\right\} \tag{2.6}$$

When $\Psi(t) = e^t$, (2.6) describes the correspondence (PÓLYA [1]) between an entire function $f(z)$ of exponential type and its Laplace (or Borel) transform $F(w)$. The general case can be applied, by suitable choice of Ψ, either to entire functions of arbitrary order or to functions that are regular in a prescribed region.

The representation provided by (2.4), (2.6) is most convenient to use when $\Psi(t)$ is restricted by auxiliary conditions on its coefficients Ψ_n. We call $\Psi(t)$ a *comparison function* if $\Psi_n > 0$ and $\Psi_{n+1}/\Psi_n \downarrow 0$. A comparison function is necessarily entire, as the ratio test for convergence shows. When $\Psi(t)$ is a comparison function, we denote by $\mathfrak{K}_\Psi$ the class of entire functions f such that, for some number τ (depending on f),

$$|f(r e^{i\theta})| \leq M \Psi(\tau r), \qquad r \uparrow \infty. \tag{2.7}$$

We call $\mathfrak{K}_\Psi$ the class of functions of finite Ψ-type. The infimum of numbers τ for which (2.7) holds is the (exact) Ψ-type of f; we denote by $\mathfrak{K}_\Psi(\tau)$ the class of functions whose Ψ-type is τ or less. For example, when $\Psi(t) = e^t$, $\mathfrak{K}_\Psi(\tau)$ is the class of functions of exponential type τ, that is, entire functions of order 1 and type not exceeding τ, or of order less than 1.

The Ψ-type of a function can be computed from the coefficients in its power series by applying the following theorem (NACHBIN [1]).

NACHBIN's theorem. *A function* $f(z) = \sum_{n=0}^{\infty} f_n z^n$ *is of* Ψ*-type* τ *if and only if* $\limsup |f_n/\Psi_n|^{1/n} = \tau$.

For the convenience of the reader, we give a proof here. First, let $\limsup |f_n/\Psi_n|^{1/n} = \tau < \infty$. Then, if $\tau_1 > \tau$, we may choose B so that $|f_n/\Psi_n| \leq B\tau_1^n$ for $n = 0, 1, 2, \ldots$. Thus, on the circle $|z| = r$,

$$|f(z)| \leq \sum_{n=0}^{\infty} |f_n| r^n \leq B \sum_{n=0}^{\infty} \tau_1^n \Psi_n r^n = B\Psi(\tau_1 r).$$

Since τ_1 may be arbitrarily close to τ, this shows that f is of Ψ-type at most τ.

In the other direction, we need a simple lemma connecting the rate of growth of Ψ with that of its coefficients.

Lemma. *Let* $\gamma_n = \min_{x>0} \Psi(x)\, x^{-n}$. *Then, for all nonnegative integers* n,

$$1 \leq \gamma_n/\Psi_n \leq (n+1)\, e; \tag{2.8}$$

and consequently $\lim (\gamma_n/\Psi_n)^{1/n} = 1$.

Since $\Psi(x) \geq \Psi_n x^n$, it is evident that $\gamma_n \geq \Psi_n$. To obtain the right-hand side of (2.8), we estimate $\Psi(x)$ for a choice of x near that which minimizes $\Psi(x)\, x^{-n}$. Let $d_n = \Psi_{n-1}/\Psi_n$ and let $0 < \omega < 1$; ω is to be near 1, and will be specified later. Recalling that a restriction on Ψ was that $\{d_n\}$ increases, we observe that $\Psi_k \leq \Psi_n d_n^{n-k}$, both for $k < n$ and $k \geq n$. Setting $x = \omega d_n$, we have

$$\Psi(x) = \sum \Psi_k x^k \leq \Psi_n \sum_0^{\infty} d^{n-k} (\omega\, d_n)^k$$
$$\leq \Psi_n d_n^n/(1-\omega).$$

For this choice of x, we have $\Psi(x)/x^n \leq \Psi_n \omega^{-n}/(1-\omega)$, and so $\gamma_n/\Psi_n \leq \omega^{-n}/(1-\omega)$. Choosing ω as $n/(n+1)$ to minimize the right-hand side, we obtain

$$\gamma_n/\Psi_n \leq (n+1)(1+n^{-1})^n \leq (n+1)\, e.$$

To apply the lemma, suppose that f is of Ψ-type τ. If $\tau_1 > \tau$, then for some M we have $|f(z)| \leq M\Psi(\tau_1 r)$. By Cauchy's inequality,

$$|f_n| \leq M\Psi(\tau_1 r)\, r^{-n} = M\tau_1^n \Psi(\tau_1 r)\ \ \tau_1 r)^{-n}.$$

Choosing r to minimize the right-hand side, we find $|f_n| \leq M\tau_1^n \gamma_n$, and

$$|f_n/\Psi_n|^{1/n} \leq M^{1/n} \tau_1 (\gamma_n/\Psi_n)^{1/n}.$$

Invoking the lemma, we then have $\limsup |f_n/\Psi_n|^{1/n} \leq \tau_1$. The conclusion of Nachbin's theorem follows on letting τ_1 approach τ.

Now let $f \in \mathfrak{K}_\Psi$, let F be defined by (2.6), and let $D(f)$ denote the union of the set of all singular points of F and the set of all points exterior to the domain of F. The contour Γ in (2.4) can then be any contour enclosing $D(f)$. If $f \in \mathfrak{K}_\Psi(\tau)$, then $D(f)$ lies in the disk $|w| \leq \tau$ and Γ may be taken as the circle $|w| = \varrho > \tau$. Additional information about $D(f)$ will lead to more detailed estimates of the growth of f in various directions.

We summarize the relevant parts of the preceding discussion in a formal theorem.

Theorem 2.9. *Let* $\Psi(t)=\sum_0^\infty \Psi_n t^n$ *be a comparison function, i.e.* $\Psi_n>0$ *and* $\Psi_{n+1}/\Psi_n \downarrow 0$. *Let* $f(z)=\sum_{n=0}^\infty f_n z^n$ *belong to the class* $\mathfrak{K}_\Psi$ *[as in (2.7)], and let* $D(f)$ *be the closed set described in the preceding paragraph. Then*

$$f(z)=\frac{1}{2\pi i}\int_\Gamma \Psi(zw)\,F(w)\,dw$$

where Γ *encloses* $D(f)$ *and*

$$F(w)=\sum_{n=0}^\infty \frac{f_n}{\Psi_n\, w^{n+1}}. \tag{2.10}$$

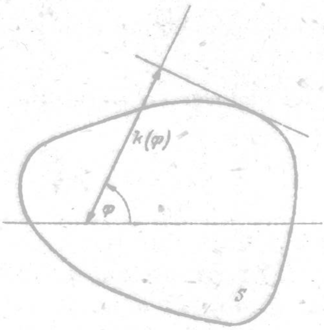

Fig. 1. The supporting function for a convex set

When $\Psi(t)$ is chosen as e^t, this theorem becomes the familiar representation for entire functions of exponential type. If $f(z)=\sum_{n=0}^\infty a_n z^n/n!$ is entire, and of growth at most order 1, finite type, then

$$f(z)=\frac{1}{2\pi i}\int_\Gamma e^{zw}\,F(w)\,dw,$$

where $F(w)=\sum_{n=0}^\infty a_n/w^{n+1}$ and Γ encircles the set $D(f)$. If f is of exponential type τ, then $D(f)$ lies in the closed disk $|w|\le\tau$. Pólya, in his notable memoir [1], showed that the rate of growth of f along radial lines sharply delimits the set $D(f)$. (A detailed treatment is to be found in Boas [3], Bieberbach [2], Cartwright [1] or Levin [1].)

With any closed set S in the plane, one may associate a supporting function

$$k(\varphi)=k(\varphi;S)=\sup_{z\in S}\Re(z e^{-i\varphi}).$$

If S^Δ is the closed convex hull of S, then $k(\varphi;S)=k(\varphi;S^\Delta)$. (See Fig. 1.)

With an entire function f of order 1, one may associate an indicator (growth) function

$$h(\theta)=h(\theta;f)=\limsup r^{-1}\log|f(re^{i\theta})|.$$

If f is of finite type τ, then $h(\theta;f)\le\tau$ for every θ.

The central fact discovered by Pólya was that $h(\theta)$ is also the supporting function for a convex set, namely, the conjugate of the set

$D(f)^{\Delta}$. This is called the conjugate indicator diagram of f. This relation can be stated concisely as follows: if f is any entire function of exponential type, then $k(\theta; D(f)^{\Delta}) = h(-\theta; f)$ for each θ. The proof depends upon the fact that (2.10) can be given an integral form

$$F(w) = w^{-1}\int_0^\infty e^{-t} f(t/w)\,dt = \int_0^\infty e^{-ws} f(s)\,ds$$

from which it easily follows that $F(w)$ is analytic in each half plane $\Re(w\,e^{-i\theta}) > h(-\theta; f)$.

As an illustration of the way in which the relation between $h(\theta)$ and $D(f)$ is used, observe that if f obeys $h(\pm\pi/2) \le c$, then $D(f)$ lies in the strip $|v| \le c$.

For a general comparison function $\Psi(t)$, the relationship between $D(f)$ and growth rates of f is not so precise. Some information can be achieved about the shape of $D(f)$. It is again possible to obtain an integral form for the generalized Borel transform (2.10). Choose a function $\alpha(t)$, of bounded variation on the interval $[0, \infty)$, so that

$$1/\Psi_n = \int_0^\infty t^n\, d\alpha(t). \tag{2.11}$$

Then,

$$F(w) = w^{-1}\int_0^\infty f(t/w)\, d\alpha(t). \tag{2.12}$$

[In § 13, we shall use an analogous approach in dealing with the case in which $\Psi(t)$ is not entire.]

In this direction, the sharpest results have been obtained by A. J. Macintyre [1], using again a specialized choice of $\Psi(t)$. If we wish to discuss entire functions of order ϱ, it would be appropriate to choose $\Psi(t)$ as a function of order ϱ, type 1, whose coefficients Ψ_n have a simple form. Two natural choices are $\sum t^n/(n!)^{1/\varrho}$ and $\sum t^n/\Gamma(1+n/\varrho)$, which reduce to e^t for $\varrho = 1$. Instead of these, Macintyre chose $\Psi(t) = \Sigma t^n/\Gamma((1+n)\varrho^{-1})$. For this, (2.12) takes the form

$$1/\Psi_n = \Gamma(\varrho^{-1} + n\varrho^{-1}) = \int_0^\infty t^n \varrho\, e^{-t^\varrho}\, dt$$

so that

$$F(w) = \frac{\varrho}{w}\int_0^\infty f(t/w)\, e^{-t^\varrho}\, dt. \tag{2.13}$$

Introduce the analogous growth function

$$h_\varrho(\theta) = \limsup r^{-\varrho} \log|f(r e^{i\theta})|.$$