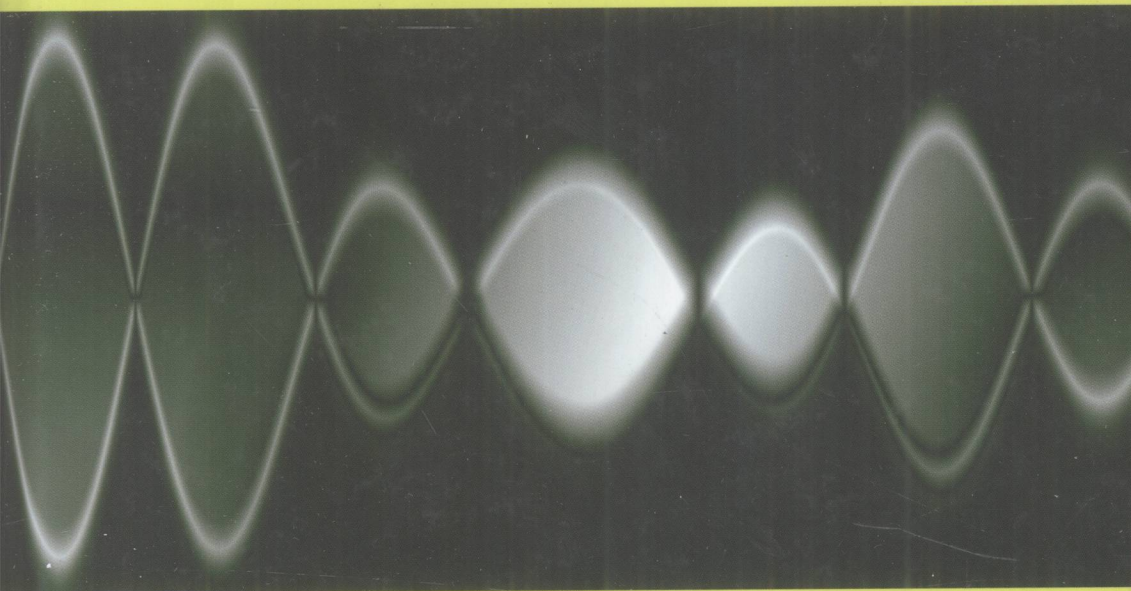


*APPLIED STOCHASTIC METHODS SERIES*

# **Switching Processes in Queueing Models**

**Vladimir V. Anisimov**



**ISTE**

 **WILEY**

## Switching Processes in Queueing Models



IN 916.4  
A599

# Switching Processes in Queueing Models

Vladimir V. Anisimov

*Series Editor*  
*Nikolaos Limnios*



ISTE



 WILEY

First published in Great Britain and the United States in 2008 by ISTE Ltd and John Wiley & Sons, Inc.

Apart from any fair dealing for the purposes of research or private study, or criticism or review, as permitted under the Copyright, Designs and Patents Act 1988, this publication may only be reproduced, stored or transmitted, in any form or by any means, with the prior permission in writing of the publishers, or in the case of reprographic reproduction in accordance with the terms and licenses issued by the CLA. Enquiries concerning reproduction outside these terms should be sent to the publishers at the undermentioned address:

ISTE Ltd  
6 Fitzroy Square  
London W1T 5DX  
UK  
www.iste.co.uk

John Wiley & Sons, Inc.  
111 River Street  
Hoboken, NJ 07030  
USA  
www.wiley.com

© ISTE Ltd, 2008

The rights of Vladimir V. Anisimov to be identified as the author of this work have been asserted by him in accordance with the Copyright, Designs and Patents Act 1988.

---

Library of Congress Cataloging-in-Publication Data

Anisimov, V. V. (Vladimir Vladislavovich)

Switching processes in queueing models / Vladimir V. Anisimov.  
p. cm.

Includes bibliographical references and index.

ISBN 978-1-84821-045-5

1. Telecommunication--Switching systems--Mathematical models. 2. Telecommunication--Traffic--Mathematical models. 3. Queuing theory. I. Title.

TK5102.985.A55 2008

519.8'2--dc22

2008008995

---

British Library Cataloguing-in-Publication Data

A CIP record for this book is available from the British Library

ISBN: 978-1-84821-045-5

---

Printed and bound in Great Britain by CPI Antony Rowe, Chippenham, Wiltshire.



# Contents

<b>Preface</b> . . . . .	13
<b>Definitions</b> . . . . .	17
<b>Chapter 1. Switching Stochastic Models</b> . . . . .	19
1.1. Random processes with discrete component . . . . .	19
1.1.1. Markov and semi-Markov processes . . . . .	21
1.1.2. Processes with independent increments and Markov switching . . . . .	21
1.1.3. Processes with independent increments and semi-Markov switching . . . . .	23
1.2. Switching processes . . . . .	24
1.2.1. Definition of switching processes . . . . .	24
1.2.2. Recurrent processes of semi-Markov type (simple case) . . . . .	26
1.2.3. RPSM with Markov switching . . . . .	26
1.2.4. General case of RPSM . . . . .	27
1.2.5. Processes with Markov or semi-Markov switching . . . . .	27
1.3. Switching stochastic models . . . . .	28
1.3.1. Sums of random variables . . . . .	29
1.3.2. Random movements . . . . .	29
1.3.3. Dynamic systems in a random environment . . . . .	30
1.3.4. Stochastic differential equations in a random environment . . . . .	30
1.3.5. Branching processes . . . . .	31
1.3.6. State-dependent flows . . . . .	32
1.3.7. Two-level Markov systems with feedback . . . . .	32
1.4. Bibliography . . . . .	33
<b>Chapter 2. Switching Queueing Models</b> . . . . .	37
2.1. Introduction . . . . .	37
2.2. Queueing systems . . . . .	38
2.2.1. Markov queueing models . . . . .	38

2.2.1.1. A state-dependent system $M_Q/M_Q/1/\infty$ . . . . .	39
2.2.1.2. Queueing system $M_{M,Q}/M_{M,Q}/1/m$ . . . . .	40
2.2.1.3. System $\overline{M}_{\overline{Q},B}/\overline{M}_{\overline{Q},B}/1/\infty$ . . . . .	41
2.2.2. Non-Markov systems . . . . .	42
2.2.2.1. Semi-Markov system $SM/M_{SM,Q}/1$ . . . . .	42
2.2.2.2. System $M_{SM,Q}/M_{SM,Q}/1/\infty$ . . . . .	43
2.2.2.3. System $M_{SM,Q}/M_{SM,Q}/1/V$ . . . . .	44
2.2.3. Models with dependent arrival flows . . . . .	45
2.2.4. Polling systems . . . . .	46
2.2.5. Retrial queueing systems . . . . .	47
2.3. Queueing networks . . . . .	48
2.3.1. Markov state-dependent networks . . . . .	49
2.3.1.1. Markov network $(M_Q/M_Q/\overline{m}/\infty)^r$ . . . . .	49
2.3.1.2. Markov networks $(M_{Q,B}/M_{Q,B}/\overline{m}/\infty)^r$ with batches . . . . .	50
2.3.2. Non-Markov networks . . . . .	50
2.3.2.1. State-dependent semi-Markov networks . . . . .	50
2.3.2.2. Semi-Markov networks with random batches . . . . .	52
2.3.2.3. Networks with state-dependent input . . . . .	53
2.4. Bibliography . . . . .	54
<b>Chapter 3. Processes of Sums of Weakly-dependent Variables</b> . . . . .	<b>57</b>
3.1. Limit theorems for processes of sums of conditionally independent random variables . . . . .	57
3.2. Limit theorems for sums with Markov switching . . . . .	65
3.2.1. Flows of rare events . . . . .	67
3.2.1.1. Discrete time . . . . .	67
3.2.1.2. Continuous time . . . . .	69
3.3. Quasi-ergodic Markov processes . . . . .	70
3.4. Limit theorems for non-homogenous Markov processes . . . . .	73
3.4.1. Convergence to Gaussian processes . . . . .	74
3.4.2. Convergence to processes with independent increments . . . . .	78
3.5. Bibliography . . . . .	81
<b>Chapter 4. Averaging Principle and Diffusion Approximation for Switching Processes</b> . . . . .	<b>83</b>
4.1. Introduction . . . . .	83
4.2. Averaging principle for switching recurrent sequences . . . . .	84
4.3. Averaging principle and diffusion approximation for RPSMs . . . . .	88
4.4. Averaging principle and diffusion approximation for recurrent processes of semi-Markov type (Markov case) . . . . .	95
4.4.1. Averaging principle and diffusion approximation for SMP . . . . .	105
4.5. Averaging principle for RPSM with feedback . . . . .	106
4.6. Averaging principle and diffusion approximation for switching processes . . . . .	108

4.6.1. Averaging principle and diffusion approximation for processes with semi-Markov switching . . . . .	112
4.7. Bibliography . . . . .	113

**Chapter 5. Averaging and Diffusion Approximation in Overloaded Switch-  
ing Queueing Systems and Networks . . . . . 117**

5.1. Introduction . . . . .	117
5.2. Markov queueing models . . . . .	120
5.2.1. System $\overline{M}_{\overline{Q},B}/\overline{M}_{\overline{Q},B}/1/\infty$ . . . . .	121
5.2.2. System $M_Q/M_Q/1/\infty$ . . . . .	124
5.2.3. Analysis of the waiting time . . . . .	129
5.2.4. An output process . . . . .	131
5.2.5. Time-dependent system $M_{Q,t}/M_{Q,t}/1/\infty$ . . . . .	132
5.2.6. A system with impatient calls . . . . .	134
5.3. Non-Markov queueing models . . . . .	135
5.3.1. System $GI/M_Q/1/\infty$ . . . . .	135
5.3.2. Semi-Markov system $SM/M_{SM,Q}/1/\infty$ . . . . .	136
5.3.3. System $M_{SM,Q}/M_{SM,Q}/1/\infty$ . . . . .	138
5.3.4. System $SM_Q/M_{SM,Q}/1/\infty$ . . . . .	139
5.3.5. System $G_Q/M_Q/1/\infty$ . . . . .	142
5.3.6. A system with unreliable servers . . . . .	143
5.3.7. Polling systems . . . . .	145
5.4. Retrial queueing systems . . . . .	146
5.4.1. Retrial system $M_Q/G/1/w.r$ . . . . .	147
5.4.2. System $\overline{M}/\overline{G}/\overline{1}/w.r$ . . . . .	150
5.4.3. Retrial system $M/M/m/w.r$ . . . . .	154
5.5. Queueing networks . . . . .	159
5.5.1. State-dependent Markov network $(M_Q/M_Q/1/\infty)^r$ . . . . .	159
5.5.2. Markov state-dependent networks with batches . . . . .	161
5.6. Non-Markov queueing networks . . . . .	164
5.6.1. A network $(M_{SM,Q}/M_{SM,Q}/1/\infty)^r$ with semi-Markov switching . . . . .	164
5.6.2. State-dependent network with recurrent input . . . . .	169
5.7. Bibliography . . . . .	172

**Chapter 6. Systems in Low Traffic Conditions . . . . . 175**

6.1. Introduction . . . . .	175
6.2. Analysis of the first exit time from the subset of states . . . . .	176
6.2.1. Definition of $S$ -set . . . . .	176
6.2.2. An asymptotic behavior of the first exit time . . . . .	177
6.2.3. State space forming a monotone structure . . . . .	180
6.2.4. Exit time as the time of first jump of the process of sums with Markov switching . . . . .	182
6.3. Markov queueing systems with fast service . . . . .	183



6.3.1. $M/M/s/m$ systems . . . . .	183
6.3.1.1. System $M_M/M/\bar{l}/m$ in a Markov environment . . . . .	185
6.3.2. Semi-Markov queueing systems with fast service . . . . .	188
6.4. Single-server retrial queueing model . . . . .	190
6.4.1. Case 1: fast service . . . . .	191
6.4.1.1. State-dependent case . . . . .	194
6.4.2. Case 2: fast service and large retrial rate . . . . .	195
6.4.3. State-dependent model in a Markov environment . . . . .	197
6.5. Multiserver retrial queueing models . . . . .	201
6.6. Bibliography . . . . .	204
<b>Chapter 7. Flows of Rare Events in Low and Heavy Traffic Conditions . .</b>	<b>207</b>
7.1. Introduction . . . . .	207
7.2. Flows of rare events in systems with mixing . . . . .	208
7.3. Asymptotically connected sets ( $V_n$ - $S$ -sets) . . . . .	211
7.3.1. Homogenous case . . . . .	211
7.3.2. Non-homogenous case . . . . .	214
7.4. Heavy traffic conditions . . . . .	215
7.5. Flows of rare events in queueing models . . . . .	216
7.5.1. Light traffic analysis in models with finite capacity . . . . .	216
7.5.2. Heavy traffic analysis . . . . .	218
7.6. Bibliography . . . . .	219
<b>Chapter 8. Asymptotic Aggregation of State Space . . . . .</b>	<b>221</b>
8.1. Introduction . . . . .	221
8.2. Aggregation of finite Markov processes (stationary behavior) . . . . .	223
8.2.1. Discrete time . . . . .	223
8.2.2. Hierarchic asymptotic aggregation . . . . .	225
8.2.3. Continuous time . . . . .	227
8.3. Convergence of switching processes . . . . .	228
8.4. Aggregation of states in Markov models . . . . .	231
8.4.1. Convergence of the aggregated process to a Markov process (finite state space) . . . . .	232
8.4.2. Convergence of the aggregated process with a general state space . . . . .	236
8.4.3. Accumulating processes in aggregation scheme . . . . .	237
8.4.4. MP aggregation in continuous time . . . . .	238
8.5. Asymptotic behavior of the first exit time from the subset of states (non-homogenous in time case) . . . . .	240
8.6. Aggregation of states of non-homogenous Markov processes . . . . .	243
8.7. Averaging principle for RPSM in the asymptotically aggregated Markov environment . . . . .	246
8.7.1. Switching MP with a finite state space . . . . .	247
8.7.2. Switching MP with a general state space . . . . .	250

8.7.3. Averaging principle for accumulating processes in the asymptotically aggregated semi-Markov environment . . . . .	251
8.8. Diffusion approximation for RPSM in the asymptotically aggregated Markov environment . . . . .	252
8.9. Aggregation of states in Markov queueing models . . . . .	255
8.9.1. System $M_Q/M_Q/r/\infty$ with unreliable servers in heavy traffic . .	255
8.9.2. System $M_{M,Q}/M_{M,Q}/1/\infty$ in heavy traffic . . . . .	256
8.10. Aggregation of states in semi-Markov queueing models . . . . .	258
8.10.1. System $SM/M_{SM,Q}/1/\infty$ . . . . .	258
8.10.2. System $M_{SM,Q}/M_{SM,Q}/1/\infty$ . . . . .	259
8.11. Analysis of flows of lost calls . . . . .	260
8.12. Bibliography . . . . .	263

## Chapter 9. Aggregation in Markov Models with Fast Markov Switching . . . . . 267

9.1. Introduction . . . . .	267
9.2. Markov models with fast Markov switching . . . . .	269
9.2.1. Markov processes with Markov switching . . . . .	269
9.2.2. Markov queueing systems with Markov type switching . . . . .	271
9.2.3. Averaging in the fast Markov type environment . . . . .	272
9.2.4. Approximation of a stationary distribution . . . . .	274
9.3. Proofs of theorems . . . . .	275
9.3.1. Proof of Theorem 9.1 . . . . .	275
9.3.2. Proof of Theorem 9.2 . . . . .	277
9.3.3. Proof of Theorem 9.3 . . . . .	279
9.4. Queueing systems with fast Markov type switching . . . . .	279
9.4.1. System $M_{M,Q}/M_{M,Q}/1/N$ . . . . .	279
9.4.1.1. Averaging of states of the environment . . . . .	279
9.4.1.2. The approximation of a stationary distribution . . . . .	280
9.4.2. Batch system $BM_{M,Q}/BM_{M,Q}/1/N$ . . . . .	281
9.4.3. System $M/M/s/m$ with unreliable servers . . . . .	282
9.4.4. Priority model $M_Q/M_Q/m/s, N$ . . . . .	283
9.5. Non-homogenous in time queueing models . . . . .	285
9.5.1. System $M_{M,Q,t}/M_{M,Q,t}/s/m$ with fast switching – averaging of states . . . . .	286
9.5.2. System $M_{M,Q}/M_{M,Q}/s/m$ with fast switching – aggregation of states . . . . .	287
9.6. Numerical examples . . . . .	288
9.7. Bibliography . . . . .	289

## Chapter 10. Aggregation in Markov Models with Fast Semi-Markov Switching . . . . . 291

10.1. Markov processes with fast semi-Markov switches . . . . .	292
10.1.1. Averaging of a semi-Markov environment . . . . .	292

10.1.2. Asymptotic aggregation of a semi-Markov environment . . . . .	300
10.1.3. Approximation of a stationary distribution . . . . .	305
10.2. Averaging and aggregation in Markov queueing systems with semi-Markov switching . . . . .	309
10.2.1. Averaging of states of the environment . . . . .	309
10.2.2. Asymptotic aggregation of states of the environment . . . . .	310
10.2.3. The approximation of a stationary distribution . . . . .	311
10.3. Bibliography . . . . .	313
<b>Chapter 11. Other Applications of Switching Processes . . . . .</b>	<b>315</b>
11.1. Self-organization in multicomponent interacting Markov systems . . .	315
11.2. Averaging principle and diffusion approximation for dynamic systems with stochastic perturbations . . . . .	319
11.2.1. Recurrent perturbations . . . . .	319
11.2.2. Semi-Markov perturbations . . . . .	321
11.3. Random movements . . . . .	324
11.3.1. Ergodic case . . . . .	324
11.3.2. Case of the asymptotic aggregation of state space . . . . .	325
11.4. Bibliography . . . . .	326
<b>Chapter 12. Simulation Examples . . . . .</b>	<b>329</b>
12.1. Simulation of recurrent sequences . . . . .	329
12.2. Simulation of recurrent point processes . . . . .	331
12.3. Simulation of RPSM . . . . .	332
12.4. Simulation of state-dependent queueing models . . . . .	334
12.5. Simulation of the exit time from a subset of states of a Markov chain .	337
12.6. Aggregation of states in Markov models . . . . .	340
<b>Index . . . . .</b>	<b>343</b>

To my wife, Zoya



## Preface

Contemporary communication systems and computer networks usually have a rather complex structure and therefore require creating more complicated mathematical models of queues and developing new approaches for modeling and asymptotic investigation. The main features of these systems are the stochasticity of the processes describing the behavior in time, influence of various internal and external events which may change (switch) the behavior of the system, the presence of different time scales for different subsystems (very fast internal computer time and user interaction time, etc), and the hierarchic structure. Wide classes of such systems can be adequately described with the help of so-called “switching” stochastic processes.

Switching processes (SP) have been developed by the author for describing the operation of stochastic systems with the property that their development in time varies spontaneously (switches) at some random points of time which may depend on the previous system trajectory. According to Kolmogorov, these processes can be called random processes with discrete interference of chance or with discrete components. Processes of this type often appear in the theory of queueing and communication systems and networks, branching, population and migration processes, in the analysis of stochastic dynamical systems with random perturbations, random movements and various other applications.

SP can be represented as a two-component process  $(x(t), \zeta(t))$ ,  $t \geq 0$ , with the property that there exists a sequence of Markov points of time  $t_1 < t_2 < \dots$  such that in each interval  $[t_k, t_{k+1})$ ,  $x(t) = x(t_k)$ , and the behavior of the process  $\zeta(t)$  in this interval depends only on the value  $(x(t_k), \zeta(t_k))$ .  $x(t)$  is a discrete switching component and the points of time  $\{t_k\}$  are called switching times. SP can be described in terms of constructive characteristics and is very suitable in analyzing and asymptotic investigating of stochastic systems with “rare” and “fast” switching.

The class of SPs is the natural generalization of well-known classes of random processes such as Markov processes that are homogenous in the 2nd component, processes with independent increments and Markov or semi-Markov switches, piecewise

Markov aggregates, and Markov processes with Markov and semi-Markov switching (random evolutions). Wide classes of queueing models can be described in terms of SPs. The class of switching queueing models includes, as examples, various types of state-dependent queueing systems and networks in a Markov or semi-Markov environment, queueing models under the influence of flows of external events or internal perturbations, unreliable systems, retrial queues, hierarchic queueing systems, etc. Therefore, the asymptotic theory of SPs can be effectively applied to the investigation of wide classes of queueing systems and networks.

In the book several large directions of asymptotic results for SP are investigated and successfully applied to various classes of switching queueing models.

The first direction is devoted to the limit theorems of averaging principle (AP) and diffusion approximation (DA) type in the case of fast switching. Theorems on the convergence of the trajectory of an SP to a solution of a differential equation (AP) and the convergence of the normalized difference to a diffusion process (DA) are proved for different subclasses of SP: recurrent processes of a semi-Markov type (RPSMs), processes with semi-Markov switching and general SP with feedback between both components. The results are based on the investigation of the asymptotic properties of a special subclass of SP – RPSMs theorems on the convergence of recurrent sequences with Markov switching to the solutions of stochastic differential equations and the convergence of superpositions of random functions.

This class of theorems is the basis of a new approach to the investigation of transient phenomena for service processes in overloading queueing systems and Markov and semi-Markov type networks, retrial queues, etc. Numerous examples for the illustration AP and DA for queueing models are considered.

The second direction is devoted to the limit theorems for SP with slow switching. Models of this type appear at the investigation of hierarchic systems in different scales of time (slow and fast). The conditions, when an SP of a rather complicated structure can be approximated by an SP of a simpler structure, in particular, by a Markov or semi-Markov process, are established and various applications to processes with Markov and semi-Markov switching are considered. The method of investigation uses the results on the convergence of the accumulating type processes constructed on the trajectory of Markov or semi-Markov process satisfying some form of the asymptotic mixing condition in triangular scheme to processes with independent increments (homogenous or non-homogenous in time). A special class of non-homogenous in time Markov processes with transition probabilities slowly varying in the expanding time scale is introduced. These processes have quasi-ergodic properties and are called quasi-ergodic Markov processes. Under rather general conditions it is proved a Poisson approximation of the flows of rare events governed by a Markov process satisfying an asymptotic mixing condition, in particular with the state space forming a so-called  $S$ -set (asymptotically connected set), and the exponential approximation of the exit

time from  $S$ -set. Special attention is paid to the analysis of the flow of rare events defined on stochastic systems satisfying asymptotic mixing conditions, in particular, with state space forming an  $S$ -set. These models naturally appear at study queueing models with asymptotically “fast” service (or low traffic). Applications of a method of  $S$ -sets are considered for different classes of queueing systems.

Using these results and the results on the convergence of SP with slow switching, the models of the asymptotic aggregation of the state space of Markov and semi-Markov processes (homogenous and non-homogenous in time) are investigated. These results create the basis for a theory of the asymptotic decreasing dimension and aggregation (consolidation) of the state space of stochastic systems. Special attention is paid to the hierarchic Markov and semi-Markov systems operating in different time scales. These systems under rather general conditions can be approximated by a simpler Markov system with averaged transition characteristics. The applications to the asymptotic aggregation of a state space and approximation by Markov models with averaged characteristics are considered for different classes of Markov and non-Markov queueing models in a random environment.

The asymptotic aggregation of SP in different time scales is the next natural level of development. The conditions of the convergence of SP to solutions of differential and stochastic differential equations with coefficients depending on a limiting aggregated Markov or semi-Markov process are obtained. Various applications to the asymptotic aggregation of overloaded queueing systems and networks under the influence of hierarchic random environment in different time scales are considered.

The results of the book were obtained while the author was working at Kiev University as Head and Professor of Applied Statistics Department at the Faculty of Cybernetics (1978–2002) and also as Visiting Professor at Bilkent University, Ankara (1997–2002). Some results were reflected in different courses on stochastic processes and queueing models that the author taught at Kiev University and Bilkent University for graduate and post-graduate students.

The book contains many practical examples of asymptotic results for queueing models and is directed to applied mathematicians and researchers, post-graduate students and engineers working in the field of stochastic systems, queueing models and applications to computer sciences, biology, ecology, physical and social sciences. Some theoretical results are illustrated by examples of simulation in R.

The author is sincerely grateful to professors Vladimir Korolyuk, Anatoli Skorokhod, Igor Kovalenko and Nikolaos Limnios for a fruitful long-term collaboration and useful discussions.

Vladimir V. Anisimov  
March 2008



