

**C.A.J. Fletcher**

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**Computational  
Techniques for  
Fluid Dynamics  
Volume II**

Specific Techniques  
for Different Flow Categories



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C.A.J. Fletcher



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# Computational Techniques for Fluid Dynamics 2

Specific Techniques  
for Different Flow Categories

With 183 Figures



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Dr. Clive A. J. Fletcher

Department of Mechanical Engineering, The University of Sydney  
New South Wales 2006, Australia

*Editors*

J.-L. Armand

Department of Mechanical Engineering  
University of California  
Santa Barbara, CA 93106, USA

M. Holt

College of Engineering and  
Mechanical Engineering  
University of California  
Berkeley, CA 94720, USA

P. Hut

The Institute for Advanced Study  
School of Natural Sciences  
Princeton, NJ 08540, USA

H. B. Keller

Applied Mathematics 101-50  
Firestone Laboratory  
California Institute of Technology  
Pasadena, CA 91125, USA

J. Killeen

Lawrence Livermore Laboratory  
P.O. Box 808  
Livermore, CA 94551, USA

S. A. Orszag

Applied and Computational Mathematics,  
218 Fine Hall, Princeton University,  
Princeton, NJ 08544, USA

V. V. Rusanov

Keldysh Institute of Applied Mathematics  
4 Miusskaya Pl.  
SU-125047 Moscow, USSR

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# Preface

As indicated in Vol. 1, the purpose of this two-volume textbook is to provide students of engineering, science and applied mathematics with the specific techniques, and the framework to develop skill in using them, that have proven effective in the various branches of computational fluid dynamics

Volume 1 describes both fundamental and general techniques that are relevant to all branches of fluid flow. This volume contains specific techniques applicable to the different categories of engineering flow behaviour, many of which are also appropriate to convective heat transfer.

The contents of Vol. 2 are suitable for specialised graduate courses in the engineering computational fluid dynamics (CFD) area and are also aimed at the established research worker or practitioner who has already gained some fundamental CFD background. It is assumed that the reader is familiar with the contents of Vol. 1.

The contents of Vol. 2 are arranged in the following way: Chapter 11 develops and discusses the equations governing fluid flow and introduces the simpler flow categories for which specific computational techniques are considered in Chaps. 14–18.

Most practical problems involve computational domain boundaries that do not conveniently coincide with coordinate lines. Consequently, in Chap. 12 the governing equations are expressed in generalised curvilinear coordinates for use in arbitrary computational domains. The corresponding problem of generating an interior grid is considered in Chap. 13.

Computational techniques for inviscid flows are presented in Chap. 14 for incompressible, supersonic and transonic conditions. In Chapt. 15 methods are described for predicting the flow behaviour in boundary layers.

For many steady flows with a dominant flow direction it is possible to obtain accurate flow predictions, based on reduced forms of the Navier-Stokes equations, in a very efficient manner. Such techniques are developed in Chap. 16. In Chaps. 17 and 18 specific computational methods are discussed for separated flows, governed by the incompressible and compressible Navier-Stokes equations respectively.

In preparing this textbook I have been assisted by many people, some of whom are acknowledged in the Preface of Vol. 1. However, the responsi-

bility for any errors or omissions remaining rests with me. Any comments, criticism and suggestions that will improve this textbook are most welcome and will be gratefully received.

Sydney, October 1987

*C. A. J. Fletcher*

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## 11. Fluid Dynamics: The Governing Equations

In this chapter, equations will be developed that govern the more common categories of fluid motion. Subsequently, various simplifications of these equations will be presented and the physical significance of these simpler equations discussed. The simplifications often coincide with limiting values of particular nondimensional numbers (Sect. 11.2.5), e.g. incompressible flow is often associated with very small values of the Mach number.

A fluid is categorised as a substance which cannot withstand any attempt to change its shape when at rest. Consequently, a fluid cannot sustain a shear force when at rest, as can a solid. However, a fluid can sustain and transmit a shear force when in motion. The proportionality between the shear force per unit area (or stress) and an appropriate velocity gradient defines the viscosity of the fluid (Sect. 11.1). Fluids include both liquids and gases. The two fluids that occur most often, naturally or in flow machinery, are water (often in the liquid phase) and air.

Fluids, whether liquids or gases, consist of molecules which are individually in a state of random motion. The large-scale motion of a fluid adds a uniform or slowly varying velocity vector to the motion of each molecule. If a large enough sample of molecules is considered, (one cubic millimetre of air at normal temperature (15°C) and pressure (101 kPa) contains approximately  $3 \times 10^{16}$  molecules), the individual molecular motion is not detectable and only the large-scale (macroscopic) motion is perceived. By assuming that the various properties of the fluid in motion, pressure, velocity, etc., vary continuously with position and time (continuum hypothesis) it is possible to derive the equations that govern fluid motion without regard to the behaviour of the individual molecules.

However, for flows at very low density, e.g. re-entry vehicles travelling through the outer parts of the atmosphere, the continuum hypothesis is not appropriate and the molecular nature of the flow must be taken into account. This also dictates the choice of appropriate computational techniques (Bird 1976).

### 11.1 Physical Properties of Fluids

The thermodynamic state of a small volume of fluid in equilibrium (i.e. uniform in space and time) is defined uniquely by specifying two independent thermodynamic properties e.g. for air, pressure and temperature would be appropriate. Other

thermodynamic properties, such as density or internal energy, are then functions of the two primary thermodynamic properties.

For air at normal temperature and pressure the various thermodynamic properties are related by the ideal gas equation

$$p = \rho RT , \quad (11.1)$$

where  $p$  is the pressure, measured in kPa,  $\rho$  is the density, measured in  $\text{kg/m}^3$ ,  $T$  is the (absolute) temperature, measured in K, and  $R$  is the gas constant. For air,  $R = 0.287 \text{ kJ/kgK}$ . It is not possible to write down a simple algebraic equation of state connecting the thermodynamic properties for water, but the relationship is contained implicitly in steam tables (e.g. van Wylen and Sonntag 1976, pp. 645–669).

The pressure is defined as the force per unit area and has the same dimensions as a stress. The pressure on a surface acts normal to the surface. Pressure is an important property since an integration of the pressure distribution over the surface of an immersed body will determine major forces (e.g. form drag, lift) and moments acting on the body. For fluids at rest the forces acting on a small volume of fluid due to the local pressure gradient are typically balanced by the force due to gravity, which gives rise to the following equation for the hydrostatic pressure:

$$\Delta p = \rho gh , \quad (11.2)$$

where  $h$  is the height over which the change in pressure is measured and  $g$  is the acceleration due to gravity. Equation (11.2) may also hold for fluids in motion, under certain circumstances. For many geophysical flows the pressure variation in the vertical direction is given approximately by (11.2).

Variation in temperature of a fluid may be due to the processes of heat transfer if the fluid is in contact with a substance at a different temperature or if latent heat release occurs. The temperature variation may also be influenced by the compression of the fluid, which might be due to the motion in high speed flow or due to the weight of the fluid in atmospheric flows.

The density is the mass per unit volume. For gases, changes in density are connected to changes in the pressure and temperature through the ideal gas equation (11.1). However, for liquids very substantial changes in pressure are necessary to alter the density, so that water (in the liquid phase) is often treated as an incompressible (constant density) fluid. The properties of air and water for different values of pressure, temperature and density are shown in Tables 11.1 and 11.2, respectively.

For fluids in motion, the concept of thermodynamic equilibrium must be given a local interpretation so that equations like (11.1) are still valid. But now the properties are functions of position and time, i.e.

$$p = p(x, y, z, t) , \quad \rho = \rho(x, y, z, t) \quad \text{and} \quad T = T(x, y, z, t) .$$

In addition, it is necessary to describe the motion uniquely. Here we use the Eulerian description. That is, the values of the velocities and the thermodynamics

**Table 11.1.** Properties of air at atmospheric pressure

Temperature, $T$ [K]	Density, $\rho$ [kg/m <sup>3</sup> ]	Dynamic viscos., $\mu \times 10^5$ [kg/ms]	Thermal conduc., $k$ [W/mK]	Thermal diffus., $\alpha \times 10^5$ [m <sup>2</sup> /s]	Prandtl number, Pr	Specific heat ratio, $\gamma$
100	3.6010	0.6924	0.00925	0.2501	0.770	1.39
300	1.1774	1.983	0.02624	2.216	0.708	1.40
500	0.7048	2.671	0.04038	5.564	0.680	1.39
900	0.3925	3.899	0.06279	14.271	0.696	1.34
1900	0.1858	6.290	0.11700	48.110	0.704	1.28

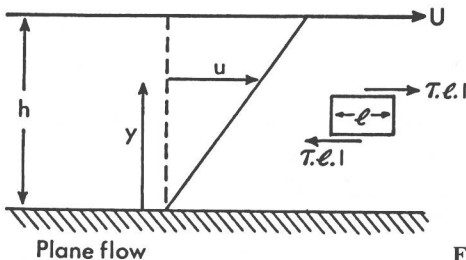
**Table 11.2.** Properties of water for saturated conditions

Temperature, $T$ [°C]	Pressure, $p$ [kPa]	Density, $\rho$ [kg/m <sup>3</sup> ]	Dynamic viscos., $\mu \times 10^5$ [kg/ms]	Thermal conduc., $k$ [W/mK]	Thermal diffus., $\alpha \times 10^5$ [m <sup>2</sup> /s]	Prandtl number Pr
0.01	0.611	1002.28	179.2	0.552	0.01308	13.6
40	7.384	994.59	65.44	0.628	0.01512	4.34
100	101.35	960.63	28.24	0.680	0.01680	1.74
200	1553.8	866.76	13.87	0.665	0.01706	0.937
300	8581.0	714.26	9.64	0.540	0.01324	1.019

properties are given at fixed locations  $(x, y, z, t)$  in the space-time domain. The alternative Lagrangian description follows individual fluid particles treating their position and thermodynamic properties as dependent variables. The connection between the Eulerian and the Lagrangian representation is discussed by von Schwind (1980, p. 22).

For fluids in motion, the ability to transmit a shear force introduces the property of dynamic viscosity. Consider the motion of a plane surface with velocity  $U$  parallel to a second stationary plane surface (Fig. 11.1).

Fluid adjacent to the upper surface moves with the velocity  $U$  and exerts a resisting force on the plate of  $\tau A$ , where  $A$  is the surface area of the upper plate and  $\tau$  is the shear stress. For a given element in the fluid two shear forces  $(\tau \cdot l \cdot 1)$  are felt, to the right at the top and to the left at the bottom. The fluid adjacent to the bottom


**Fig. 11.1.** Plane flow parallel to a stationary surface

surface exerts a drag force  $\tau A$  on the lower fixed plate. It is found empirically that the shear stress is directly proportional to the velocity gradient,  $\partial u / \partial y$ , i.e.

$$\tau = \mu \frac{\partial u}{\partial y}, \quad (11.3)$$

where the constant of proportionality,  $\mu$ , is the (dynamic) viscosity. Viscosity is measured in kg/ms. For this example, the shear stress,  $\tau$ , is constant; it follows that the velocity distribution is given by

$$\frac{u}{U} = \frac{y}{h} \quad (11.4)$$

The relationship (11.3) defines Newtonian fluids. Flows involving air or water satisfy (11.3). Non-Newtonian fluids, which do not satisfy (11.3), are described by Tanner (1985).

The viscosity of gases like air is, to a close approximation, a function of temperature alone (for normal temperatures and pressures). For air the viscosity increases approximately like  $T^{0.76}$ , where  $T$  is the absolute temperature. Typical values are given in Table 11.1. The viscosity of liquids like water is a weak function of pressure but a strong function of temperature. In contrast to the behaviour for gases, the viscosity of liquids typically decreases rapidly with increasing temperature. Representative values for water are given in Table 11.2.

For flow problems involving temperature changes, Fourier's law indicates that the local rate of heat transfer is a linear function of the local temperature gradient, i.e.

$$\dot{Q}_i = -k \frac{\partial T}{\partial x_i}, \quad (11.5)$$

where  $\dot{Q}_i$  is the rate of the heat transfer per unit area in the  $x_i$  direction and  $k$  is the thermal conductivity. The similarity in structure between (11.5) and (11.3) is noteworthy. If the temperature of the two plates in Fig. 11.1 were different, (11.5) indicates that there would be a heat transfer through the fluid, given by

$$\dot{Q}_y = -k \frac{\partial T}{\partial y}. \quad (11.6)$$

The thermal conductivity is measured in W/mK. Like viscosity, the thermal conductivity increases with temperature for gases. However, for liquids such as water the thermal conductivity rises slightly with temperature in the range  $0^\circ$ – $100^\circ\text{C}$  at a pressure of one atmosphere. Typical values for the thermal conductivities of air and water are shown in Tables 11.1 and 11.2.

Because of the way that viscosity and thermal conductivity appear in the momentum (11.31) and energy (11.38) equations it is convenient to define the

kinematic viscosity  $\nu$  and the thermal diffusivity  $\alpha$  by

$$\nu = \frac{\mu}{\rho} \quad \text{and} \quad \alpha = \frac{k}{\rho c_p},$$

where  $c_p$  is the specific heat at constant pressure. Both  $\nu$  and  $\alpha$  are diffusivities, controlling the diffusion of momentum (or vorticity) and heat, respectively. Both  $\nu$  and  $\alpha$  are measured in  $\text{m}^2/\text{s}$ . For gases like air both  $\nu$  and  $\alpha$  increase with temperature (Table 11.1). For liquids like water the kinematic viscosity  $\nu$  falls rapidly with increasing temperature but the thermal diffusivity  $\alpha$  increases slightly (Table 11.2).

For a discussion of the fluid properties, particularly in relation to the underlying molecular behaviour, the reader is referred to Lighthill (1963) or Batchelor (1967, pp. 1–60). Eckert and Drake (1972) provide a tabulation of the properties of common fluids.

## 11.2 Equations of Motion

The general technique for obtaining the equations governing fluid motion is to consider a small control volume through which the fluid moves, and to require that mass and energy are conserved, and that the rate of change of the three components of linear momentum are equal to the corresponding components of the applied force. This produces five equations which, when combined with an equation of state, provide sufficient information for the determination of six variables:  $p, \rho, T, u, v, w$  typically. For flows associated with combustion and some geophysical flows, more than one species will be present. Each new species requires an additional (species conservation) equation. For some flow problems not all six variables will be involved and less than six equations will be required.

### 11.2.1 Continuity Equation

For an arbitrary control volume  $V$  fixed in space and time (Fig. 11.2), conservation of mass requires that the rate of change of mass within the control volume is equal to the mass flux crossing the surface  $S$  of  $V$ , i.e.

$$\frac{d}{dt} \int_V \rho \, dV = - \int_S \rho \mathbf{v} \cdot \mathbf{n} \, dS, \quad (11.7)$$

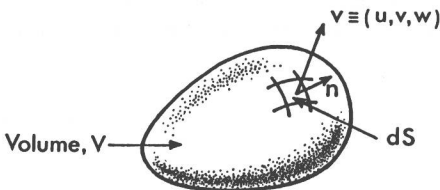


Fig. 11.2. Conservation of mass

where  $\mathbf{n}$  is the unit (outward) normal vector. Using the Gauss (divergence) theorem (Gustafson 1980, p. 35), the surface integral may be replaced by a volume integral. Then (11.7) becomes

$$\int_V \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right] dV = 0 \quad (11.8)$$

where  $\nabla \cdot (\rho \mathbf{v}) \equiv \text{div } \rho \mathbf{v}$ . Since (11.8) is valid for any size of  $V$ , it implies that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (11.9)$$

which is the mass-conservation or continuity equation. In Cartesian coordinates (11.9) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \quad (11.10)$$

It is convenient to collect all the density terms together and to write (11.10) as

$$\frac{1}{\rho} \left( \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right) + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0, \quad (11.11)$$

or

$$\frac{1}{\rho} \left( \frac{D\rho}{Dt} \right) + \mathcal{D} = 0, \quad (11.12)$$

where  $D/Dt$  is called the time derivative following the motion or the material derivative and  $\mathcal{D}$  is called the dilatation. For flows of constant density (e.g. incompressible flow), (11.12) reduces to

$$\mathcal{D} \equiv \nabla \cdot \mathbf{v} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (11.13)$$

for both steady and unsteady flow.

### 11.2.2 Momentum Equations: Inviscid Flow

Newton's second law of motion states that the time rate of change of linear momentum is equal to the sum of the forces acting. For a small element of fluid treated as a closed system (i.e. no flow across its boundaries) Newton's second law is

$$\frac{d}{dt} \int \rho \mathbf{v} dV_{cs} = \sum \mathbf{F}, \quad (11.14)$$

where subscript  $cs$  denotes a closed system.

For a control volume  $V$  fixed in space and time with flow allowed to occur across the boundaries (Fig. 11.3), the following connection with the closed system is available (Streeter and Wylie 1979, p. 93):

$$\frac{d}{dt} \int_V \rho \mathbf{v} dV_{cs} = \int_V \frac{\partial}{\partial t} (\rho \mathbf{v}) dV + \int_S \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n}) dS . \quad (11.15)$$

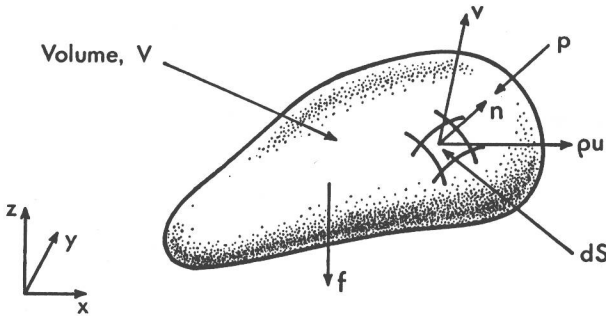


Fig. 11.3. Control volume geometry for Euler's equations

In (11.15)  $\rho \mathbf{v}$  is the momentum and  $\mathbf{v} \cdot \mathbf{n}$  is the velocity normal to the surface of the control volume. By the Gauss theorem, (11.15) becomes

$$\frac{d}{dt} \int_V \rho \mathbf{v} dV_{cs} = \int_V \left( \frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) \right) dV . \quad (11.16a)$$

By expanding (11.16a) and making use of (11.9), one obtains

$$\frac{d}{dt} \int_V \rho \mathbf{v} dV_{cs} = \int_V \rho \frac{D\mathbf{v}}{Dt} dV , \quad (11.16b)$$

where  $D\mathbf{v}/Dt = \partial \mathbf{v} / \partial t + \mathbf{v} \cdot \nabla \mathbf{v}$ .  $D\mathbf{v}/Dt$  is the total time rate of change of  $\mathbf{v}$  or the acceleration.

Thus, (11.14) becomes

$$\int_V \rho \frac{D\mathbf{v}}{Dt} dV = \Sigma \mathbf{F} , \quad (11.17)$$

i.e. "mass  $\times$  acceleration = force".

Contributions to the summation  $\Sigma \mathbf{F}$  come from forces acting at the surface of the control volume (surface forces) and throughout the volume (volume or body forces). The most common volume force is the force due to gravity and this is the only volume force considered here. The nature of the surface forces depends on whether the fluid viscosity is taken into account or not. Initially, an inviscid fluid will be assumed, in which case the only surface force is due to the pressure, which acts normal to the surface. Thus, the right-hand side of (11.17) can be written

$$\Sigma \mathbf{F} = \int_V \rho \mathbf{f} dV - \int_S p \mathbf{n} dS , \quad (11.18)$$

