21 世纪高等院校教材

Functions of a Complex Variable and Integral Transforms

（复变函数与积分变换）

盖云英 邢宇明 编

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内 容 简 介

本书是一本用于同名课程双语教学的英文教材. 编者参考多本有关的经典原著英文教材, 按照国家教育部对本课程的基本要求, 结合多年的教学实践编撰而成. 内容分两部分, 共 8 章. 第 1 ~ 6 章为复变函数部分, 包括 complex numbers and functions of a complex variable (复数与复变函数), analytic functions (解析函数), complex integrals(复积分), series(级数), residues (留数), conformal mappings(保形映射). 第 7 章和第 8 章是积分变换部分, 包括 Fourier transform(傅里叶变换)和 Laplace transform(拉普拉斯变换). 书中各章节都安排了足够量的例题, 在每章后也安排了大量精选的习题, 并按大纲的要求及难易程度分为 A、B 两类.

本书既可作为理工科大学同名课程的双语教材, 也可供有关工程技术人员参考.

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出版说明

为了尽快与世界先进科技接轨, 培养出能够走出国门的世界一流人才, 以应变国际间的交流与竞争, 双语教学已成为中国高等教育课程改革的一个发展趋势. 近几年来, 我们在哈尔滨工业大学实验学院进行了本课程的双语教学尝试. 按照国家教育部对本课程的基本要求, 参考多本国外优秀原版英文教材, 形成了备课笔记, 并编写了讲义. 经过几年的教学实践和不断完善, 编成了本教材.

本课程属于工程数学范畴, 是理工科院校学生继工科数学分析课程之后的又一门数学基础课. 通过对本课程的学习, 不仅能学到复变函数与积分变换中的基本理论及工程技术中常用的数学方法, 同时还可以巩固和复习工科数学分析的基础知识, 为学习有关的后续课程和进一步扩大数学知识面奠定了基础.

在编写过程中, 注意到国外同类的原版英文教材同我国大学数学课的教学内容与教学要求的差异, 严格按照国家教育部对本课程的基本要求进行选材. 由于是英文教材, 我们尽量使用原版教材中通用的用语、术语等. 鉴于数学教学的特点, 我们精选了大量的例题和习题, 以备学生巩固所学内容, 提高分析问题、解决问题的能力. 习题分为 A、B 两类. 对 A 类习题, 读者应该独立完成, 而 B 类习题是为那些学有余力的学生准备的. 完成本教材的全部教学内容大约需要 48 学时.

本教材得到科学出版社、哈尔滨工业大学数学系的热情鼓励和大力支持, 同时请到美国西雅图大学数学系丁树森教授审稿, 并提出宝贵意见, 在此一并表示感谢!

由于编者水平有限, 书中的缺点和疏漏在所难免, 恳请广大读者批评指正.

编　者

2006 年 9 月于哈尔滨工业大学

Preface

The present book is meant as a text for the undergraduates in the science or engineering who are taking a course on *Functions of a Complex Variable and Integral Transforms* in English in China. Before taking the course, the students have completed at a course in *Mathematical Analysis for Engineering*. The full book is suitable for a one-semester course (48 hours).

In this text, the students will find abundant motivation, examples, problems and applications. Each section includes examples and problems to help the student master the material as it is presented.

The book essentially decomposes in two parts.

The first part, Chapter 1 through 6, is Functions of a Complex Variable. In this part, the theory of analytic functions of a complex variable will be introduced.

The functions of a complex variable (i.e. the complex analysis) was developed in the nineteenth century, mainly by Augustion Cauchy (1789~1857), later his theory was made more rigorous and extended by such mathematicians as Peter Dirichlet (1805~1859), Karl Weierstrass (1815~1897), and Georg Friedrich Riemann (1826~1866).

Complex analysis has become an indispensable and standard tool of the working mathematician, physicist and engineer. Neglect of it can prove to be a severe handicap in most areas of research and application involving mathematical ideas and techniques.

The second part, Chapter 7 through 8, is integral transforms, Fourier transform and Laplace transform.

Transforms that named for Jean Baptiste Joseph Fourier(1768~1830) and Pierre Simon Laplace(1749~1827), are well known as providing techniques for solving problems in linear system those transforms play an important part in the theory of many branches of science. While they may be regarded as purely mathematical functional, as is customary in the treatment of other transforms, they also assume in many fields just as definite a physical meaning as the functions from which they stem.

Contents

Chapter 1
Complex Numbers and Functions of a Complex Variable

1.1 Complex numbers and its four fundamental operations

1. Introduction to complex numbers

As early as the sixteenth century Girolamo Cardano (Italian, 1501~1576) considered quadratic (and cubic) equations such as $x^2 + 2x + 2 = 0$, which is satisfied by no real number x, for example $-1 \pm \sqrt{-1}$. Cardano noticed that if these "complex numbers" were treated as ordinary numbers with the multiplication rule that $\sqrt{-1}\cdot\sqrt{-1} = -1$, they did indeed solve the equations.

The important expression $\sqrt{-1}$ is now given the widely accepted designation $\mathrm{i} = \sqrt{-1}$. (This convention is not followed by the electrical engineers who prefer the symbol $\mathrm{j} = \sqrt{-1}$ since they wish to reserve the symbol i for electric current.)

It is customary to denote a complex number:

$$z = x + \mathrm{i}y.$$

The real numbers x and y are known as the *real and imaginary parts* of z, respectively, and we write

$$\mathrm{Re}z = x, \qquad \mathrm{Im}z = y.$$

Two complex numbers are *equal* whenever they have the same real parts and the same imaginary parts, i.e. $z_1 = z_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

In what sense are these complex numbers an extension of the real ? We have already said that if a is a real, we also write to stand for $a+0\mathrm{i}$. In other words, we are this regarding the real numbers as those complex numbers $a+b\mathrm{i}$, where $b=0$. If in the expression $a+b\mathrm{i}$ the term $a=0$. We call a *pure imaginary number.*

2. Four fundamental operations

The addition and multiplication of complex numbers are the same as for real numbers.

$$(x_1+\mathrm{i}y_1)\pm(x_2+\mathrm{i}y_2)=(x_1\pm x_2)+\mathrm{i}(y_1\pm y_2),$$

$$(x_1+\mathrm{i}y_1)(x_2+\mathrm{i}y_2)=(x_1x_2-y_1y_2)+\mathrm{i}(x_1y_2+x_2y_1).$$

Observe that the right-hand sides of these equations can be obtained by formally manipulating the terms on the left as if they involved only real numbers and by replacing i^2 by -1 when it occurs. If z and w are complex numbers with $w\neq 0$, then the symbol z/w means zw^{-1}. We call z/w the *quotient* of z by w. Thus $z^{-1}=\dfrac{1}{z}$.

If $x_2+\mathrm{i}y_2\neq 0$,

$$\frac{x_1+\mathrm{i}y_1}{x_2+\mathrm{i}y_2}=\frac{(x_1+\mathrm{i}y_1)(x_2-\mathrm{i}y_2)}{(x_2+\mathrm{i}y_2)(x_2-\mathrm{i}y_2)}=\frac{(x_1x_2+y_1y_2)+\mathrm{i}(x_2y_1-x_1y_2)}{x_2^2+y_2^2}.$$

In short, all the usual algebraic rules for manipulating real numbers, fractions, polynomial, and so on hold in complex analysis.

Formally, the system of complex numbers is a *field.*

The crucial rules for a field, stated here for reference only, are

additive rules:

(i) $z+w=w+z$,

(ii) $z+(w+s)=(z+w)+s$,

(iii) $z+0=z$,

(iv) $z+(-z)=0$.

multiplication rules:

(i) $zw=wz$,

(ii) $(zw)s=z(ws)$,

(iii) $1z=z$,

(iv) $z(z^{-1})=1$ for $z\neq 0$.

distributive law:

$z(w+s) = zw + zs$.

Theorem 1.1.1. The complex numbers form a field.

If the usual ordering properties for real are to hold, then such an ordering is impossible for complex.

1.2 Geometric representation of complex numbers

1. Complex numbers are represented by the points and vectors in the plane

A complex number may be thought of geometrically as a (two-dimensional) vector and pictured as an arrow from the origin to the point in $\mathbb{R}^2$ given by the complex number

$$(x, y) = x + y\mathrm{i}.$$

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the plane called the *complex plane* $\mathbb{C}$ with rectangular coordinates x and y (Figure 1.1). Because the points $(x, 0) \in \mathbb{R}^2$ correspond to real numbers, the horizontal or x axis is called the *real axis*, the vertical axis (y axis) is called the *imaginary axis*.

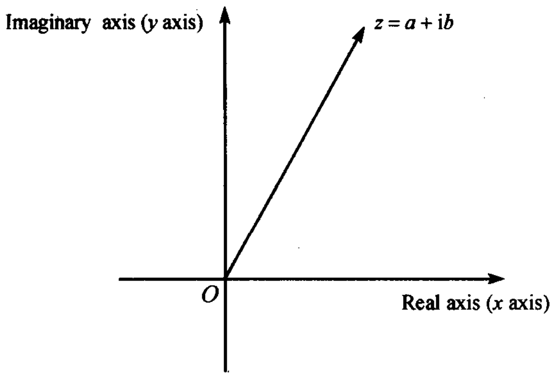

Figure 1.1 vector representation of complex numbers

2. Modulus and argument

The length of the vector $(a, b) \equiv a + ib$ is defined as $r = \sqrt{a^2 + b^2}$ and suppose that the vector makes an angle θ with the positive direction of the real axis, where $-\pi < \theta \leqslant \pi$ (Figure 1.2). Thus $\tan\theta = b/a$. Since $a = r\cos\theta$ and $b = r\sin\theta$, we thus have $a + b\mathrm{i} = r\cos\theta + (r\sin\theta)\mathrm{i} = r(\cos\theta + \mathrm{i}\sin\theta)$. This way writing the complex number is called the *polar coordinate representation.*

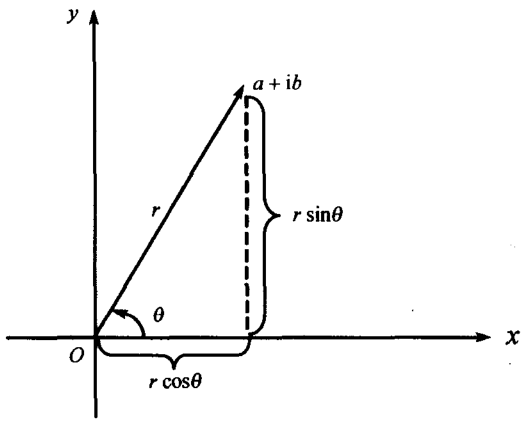

Figure 1.2 polar coordinate representation of complex numbers

The length of the vector $z = a + ib$ is denoted $|z|$ and is called the *norm*, or *modulus*, or *absolute value* of z. The angle θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π. Each value of θ is called the *argument* or *amplitude* of the complex number z, and the set of all such values is denoted by Arg$z$. The *principal value* of θ is denoted $\theta = \arg z$, where $-\pi < \theta \leqslant \pi$. If $z = 0$, the coordinate θ is undefined; and so it is always understood that $z \neq 0$ whenever Arg$z$ is discussed.

$$\text{Arg}z = \arg z + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \cdots, \qquad -\pi < \arg z \leqslant \pi.$$

We have

$$\arg z = \begin{cases} \arctan \dfrac{y}{x}, & z \in \text{I or IV}, \\ \arctan \dfrac{y}{x} + \pi, & z \in \text{II}, \\ \arctan \dfrac{y}{x} - \pi, & z \in \text{III}. \end{cases}$$

1.3 Complex conjugates

1. Definition and properties

If $z = a + ib$, then $\bar{z}$, the *complex conjugate* of z, is defined by $\bar{z} = a - ib$ (Figure 1.3).

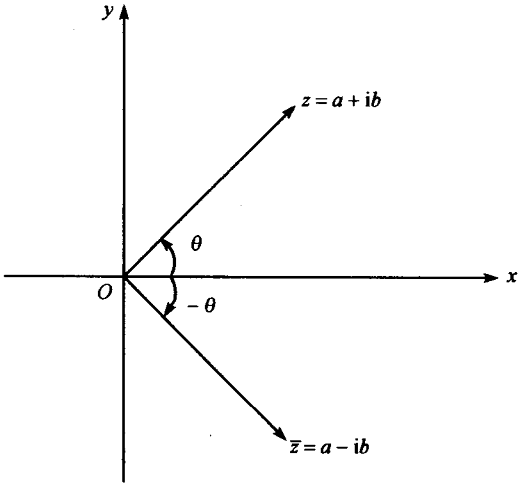

Figure 1.3 complex conjugation

Theorem 1.3.1.

(i) $\overline{z+z'} = \overline{z} + \overline{z}'$.

(ii) $\overline{zz'} = \overline{z} \cdot \overline{z}'$.

(iii) $\overline{z/z'} = \overline{z}/\overline{z}'$ for $z' \neq 0$.

(iv) $z\overline{z} = |z|^2$, and hence $z \neq 0$, we have $z^{-1} = \overline{z}/|z|^2$.

(v) $z = \overline{z}$ if and only if z is real.

(vi) $\mathrm{Re}z = \dfrac{z+\overline{z}}{2}$ and $\mathrm{Im}z = \dfrac{z-\overline{z}}{2\mathrm{i}}$.

(vii) $\overline{\overline{z}} = z$.

Proof. We will omit it.

2. Triangle inequality

We turn now to the triangle inequality, which provides an upper bound for the modulus of the sum of two complex numbers and

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

This important inequality is geometrically evident in Figure 1.4, since it is merely a statement that the length of one side of a triangle is less than or equal to the sum of the lengths of the other two sides. And we have

Theorem 1.3.2.

(i) $|zz'| = |z| \cdot |z'|$.

(ii) If $z' \neq 0$, then $|z/z'| = |z|/|z'|$.

(iii) $-|z| \leqslant \text{Re}z \leqslant |z|$ and $-|z| \leqslant \text{Im}z \leqslant |z|$, that is, $|\text{Re}z| \leqslant |z|$ and $|\text{Im}z| \leqslant |z|$.

(iv) $|\overline{z}| = |z|$.

(v) $|z + z'| \leqslant |z| + |z'|$.

(vi) $|z - z'| \geqslant ||z| - |z'||$.

(vii) $|z_1 w_1 + \cdots + z_n w_n| \leqslant \sqrt{|z_1|^2 + \cdots + |z_n|^2} \cdot \sqrt{|w_1|^2 + \cdots + |w_n|^2}$.

Proof. We will omit it.

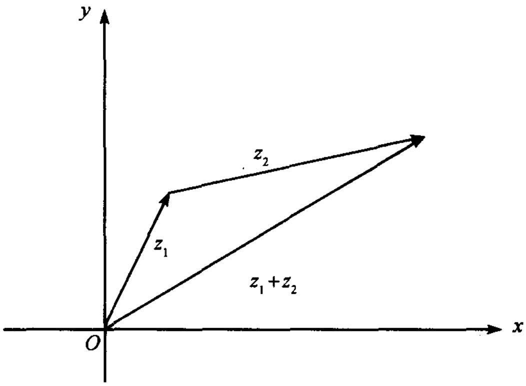

Figure 1.4 triangle inequality

1.4 Powers and roots

Polar representation of complex numbers simplifies the task of describing geometrically the product of two complex numbers.

Let $z_1 = r_1(\cos\theta_1 + \mathrm{i}\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + \mathrm{i}\sin\theta_2)$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2([\cos\theta_1 \cdot \cos\theta_2 - \sin\theta_1 \cdot \sin\theta_2] \\ &\quad + \mathrm{i}[\cos\theta_1 \cdot \sin\theta_2 + \cos\theta_2 \cdot \sin\theta_1]) \\ &= r_1 r_2[\cos(\theta_1 + \theta_2) + \mathrm{i}\sin(\theta_1 + \theta_2)]. \end{aligned}$$

Theorem 1.4.1. $|z_1 z_2| = |z_1| \cdot |z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$.

As a result of the preceding discussion, the second equality in Theorem 1.4.1 should be written as $\arg z_1 z_2 = \arg z_1 + \arg z_2 (\text{mod } 2\pi)$. "mod 2π" means that the left and right sides of the equation agree after addition of a multiple of 2π to the right side.

Theorem 1.4.2(De Moivre formula). If $z = r(\cos\theta + \mathrm{i}\sin\theta)$ and n is a positive integer, then $z^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta)$.

Theorem 1.4.3. Let w be a given (nonzero) complex number with polar representation $w = r(\cos\theta + \mathrm{i}\sin\theta)$, then the nth roots of w are given by the n complex numbers

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + \mathrm{i}\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \quad k = 0, 1, \cdots, n-1.$$

Example 1.4.1. Solve $z^3 = -1$ for z.
Solution.

$$z = \sqrt[3]{-1} = \sqrt[3]{|-1|}\left(\cos\frac{\pi + 2k\pi}{3} + \mathrm{i}\sin\frac{\pi + 2k\pi}{3}\right)$$

$$= \begin{cases} \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3} \\ \cos\pi + \mathrm{i}\sin\pi \\ \cos\frac{5\pi}{3} + \mathrm{i}\sin\frac{5\pi}{3} \end{cases} = \begin{cases} \frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, & k = 0, \\ -1, & k = 1, \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}, & k = 2. \end{cases}$$

Example 1.4.2. Find all roots of $\sqrt{-\mathrm{i}}$, and write in the triangle form.
Solution. Let $z = \sqrt{-\mathrm{i}}$, then we have

$$z = r(\cos\theta + \mathrm{i}\sin\theta) = -\mathrm{i} = \cos\frac{3}{2}\pi + \mathrm{i}\sin\frac{3}{2}\pi,$$

$$z^2 = r^2(\cos 2\theta + \mathrm{i}\sin 2\theta), \quad r^2 = 1, \ 2\theta = \frac{3}{2}\pi + 2k\pi,$$

hence

$$r = 1, \ \theta = \frac{3}{4}\pi + k\pi, \ k = 0, 1,$$

$$\sqrt{-\mathrm{i}} = \begin{cases} \cos\frac{3}{4}\pi + \mathrm{i}\sin\frac{3}{4}\pi, \\ \cos\frac{7}{4}\pi + \mathrm{i}\sin\frac{7}{4}\pi. \end{cases}$$

1.5 Riemann sphere and infinity

For some purposes it is convenient to introduce a point "∞" in addition to the points $z \in \mathbb{C}$. The complex plane together with this point is called the *extended* complex plane. To visualize the point at infinity, one can think of the complex plane as passing through the south pole S of a unit sphere at $z = 0$ (Figure 1.5). To each point z in the plane there corresponds exactly one point Q on the surface of the sphere. The point Q is determined by the intersection of the line through the point z and the north pole N of the sphere with that surface. In like manner, to each point on the surface of the sphere, other than the north pole N, there corresponds exactly one point z in the plane. By letting the north pole N of the sphere corresponds to the point at infinity, we obtain a one to one correspondence between the point of the sphere and the points of the extended complex plane. The sphere is known as the *Riemann sphere.*

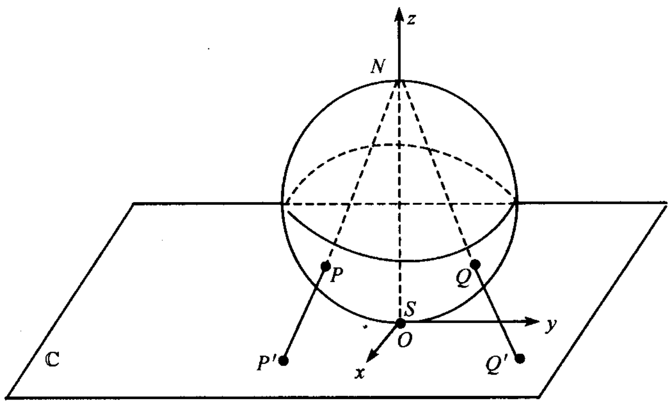

Figure 1.5 complex sphere

Formally we add a symbol "∞" to $\mathbb{C}$ to obtain the *extended complex plane* $\overline{\mathbb{C}}$ and define operations with ∞ by the "rules"

$$
\begin{aligned}
&a+\infty=\infty+a=\infty,\\
&\infty+\infty=\infty,\\
&a\cdot\infty=\infty\cdot a=\infty,\\
&\infty\cdot\infty=\infty,\\
&\frac{a}{0}=\infty\ (a\neq 0),\qquad \frac{a}{\infty}=0.
\end{aligned}
$$

1.6 Complex number sets

1. Fundamental concepts

The first concept is a δ- *neighborhood* of a given point z_0. It consists of all points of z lying inside but not on a circle centered at z_0 and a specified positive radius δ: $N_\delta(z_0)=\{z\,||z-z_0|<\delta\}$. When the value of δ is understood or is immaterial in the discussion, it is often referred to as just a neighborhood. Occasionally, it is convenient to speak of a *deleted δ- neighborhood* of a given point z_0: $\{z|\, 0 < |z - z_0| < \delta\}$.

A point z_0 is said to be an *interior point* of the set E whenever there is some neighborhood of z_0 that contains only points of E. If there exists $N_\delta(z_0) \subset E$, it is called an *exterior point* of E when there exists a neighborhood of it containing no points of E. If z_0 is neither of these, it is a *boundary point* of E. A boundary point is, therefore, a point all of whose neighborhoods contain points in E and points not in E. The totality of all boundary points is called the *boundary* of E,denoted by ∂E.

A set E is *open* if and only if each of its points is an interior point of E.

2. Domain and curve

An open set S is *connected* if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end to end, that lies entirely in S.

An open set that is connected is called a *domain.*

A curve $\Gamma : z = z(t) = x(t) + \mathrm{i}y(t)$ $(\alpha \leqslant t \leqslant \beta)$ if $x(t),\ y(t) \in C[\alpha, \beta]$, then Γ is continuous and if $t_1 \neq t_2 \Rightarrow z(t_1) \neq z(t_2)$, then Γ is called a *simple curve.*

If $z'(t) = x'(t) + \mathrm{i}y'(t) \neq 0$ and $x'(t),\ y'(t) \in C[\alpha, \beta]$, Γ is called a *smooth curve.* Finite smooth curves are called a *piecewise smooth curve.*

A domain $D \subset \mathbb{C}$ is called the *simply connected* if and only if for every simply closed curve γ in D, the inside of γ also lies in D, or else it is called the *multiple connected domain.*

1.7 Functions of a complex variable

1. Definition and geometry significance

Let $G \subset \mathbb{C}$ be a set of complex numbers. A *function* f defined on G is a rule that assigns to each z in G a complex number w. The number w is called the *value* of f at z and is denoted by $f(z)$. That is, $w = f(z)$. G is the *domain of definition* of f.

It must be emphasized that both a domain of definition and a rule are needed in order for a function to be well defined. When the domain of definition is not mentioned, we agree that the largest possible set is be taken.