# Partial Differential Equations

## An Introduction

G. hellwig

# AUTHOR'S PREFACE

This book is intended to give an introduction to the field of partial differential equations. The presentation is intentionally not too brief so that graduate students should be able to read it without serious difficulty. In addition to requiring a thorough knowledge of differential and integral calculus as well as of the theory of ordinary differential equations, it presupposes a few results from complex variables, and, in its last part, a few from functional analysis and real variables.

The goals of the book necessitated a careful selection of material. Of course, in the framework of this "guidebook," problems that today stand in the foreground of scientific development could only be taken into consideration peripherally. However, the author hopes that his efforts to present some of these can be felt.

In Part I, simple examples are treated, namely, the wave, potential, and heat equations. There the Gauss integral theorem in $R_n$ appears as an important tool. Part II deals with the normal forms and characteristic manifolds for partial differential equations of the second order and for systems of partial differential equations of the first order in more than one unknown function. Here normal forms are given that can be obtained by very elementary means. In Part III questions of uniqueness for various initial-value and boundary-value problems are discussed, by means of the maximum-minimum principle and the energy-integral method, respectively. Since such considerations are much simpler than questions of existence, they are treated first; dealing with them first often brings with it the right point of view for the questions of existence which are to be treated in the following two parts. Different means of proof are purposely selected each time in order to provide the reader with at least a modest insight into the variety of methods. In Part IV, the method of successive iteration and the use of the characteristic relations are discussed for hyperbolic equations and systems, while the Laplace transform calculus is used for initial- and boundary-value problems in hyperbolic and parabolic equations. For boundary-value problems in elliptic equations the theory of weak solutions, together with an extended version of Weyl's lemma, is used. The delicate question about the assumption of the boundary values is treated by means of a new method due to E. Wienholtz, which—though not published so far—he has kindly made available for this book. The last part deals with questions of existence for elliptic equations and systems, using simple tools from functional analysis. It outlines Schauder's technique of proof and the treatment of the eigenvalue problem, and concludes with an introduction to the boundary-value problems for elliptic systems of the first order in two unknown functions. It is interesting here that for such poblems the Fredholm alternative does not hold. A part on singular problems, which ad been planned, has been postponed for the time being.

Partial differential equations of the first order in one unknown function have not been included in the book, since their theory can be reduced to the theory of ordinary differential equations; therefore their treatment perhaps belongs in a textbook on ordinary differential equations.

A few exercises are scattered through the text, among which the more difficult ones are indicated by an asterisk. Their solutions are given at the end of the book. References to the literature have been kept brief intentionally, since a small textbook is not in a position to provide a survey of the enormous wealth of literature in this field. Fortunately there are excellent summarizing reports that should be accessible to the reader after reading this book.

Formulas are numbered by section; for example, by (IV-3.19) we mean Formula 19 of Chapter 3 in Part IV. In references to places in the same part, the number of the part is omitted.

In particular, I must express my gratitude to my revered teacher Professor Haack for important stimulations and suggestions. Our joint investigations and numerous seminars on this subject have had much influence upon this book. My stay at the Institute of Mathematical Sciences, New York University, during the academic year 1954–55 has been another influencing factor. For many new points of view I have to thank Professors L. Asgeirsson, L. Bers, R. Courant, K. O. Friedrichs, F. John, P. Lax, and L. Nirenberg.

I also wish to express my cordial gratitude to my coworker Dr. E. Wienholtz for many hints and valuable suggestions which made many presentations clearer.

Further, I have to thank my secretary Mrs. L. Schröder for her cooperation in the preparation of the manuscript, Mr. H. Drucks for offering help with the proof reading and preparing the index, Mr. K.-H. Diener and Dr. K. Jörgens for many useful remarks, and Mr. H. Zehle for drawing the original figures.

Last, but not least, I must thank the editor of this series of books, Professor G. Köthe, who encouraged me to write this book, and to the publishers for their patient compliance with my wishes.

G. HELLWIG

*Berlin—Charlottenburg, July 1959.*

# TRANSLATOR'S NOTE

In line with the common practice in partial differential equations, coordinates of vectors and points have been given lower indices. This is practiced uniformly except in Section II-2.6, where concepts and notations from differential geometry are used.

With the exception of the above and a few other inconsequential changes in notation, no change has been made in Professor Hellwig's text.

E. GERLACH

# PREFACE TO THE SECOND EDITION

After the English edition had been out of print for some time and since the original edition is now also no longer available, publishers and author decided to reprint the English edition.

It contains numerous additions as compared with the German edition, and it conforms in its notation better to the present standard.

For an introductory course into the whole subject particularly Part 1 and Part 3 seem especially suitable. These two parts take into account the various types of partial differential equations and are independent of Part 2.

Precisely the uniqueness questions which are dealt with in Part 3 give the proper perspective concerning the existence problems without being encumbered with the usual difficulties of the latter.

In remembrance of our joint efforts in this field this edition will be dedicated to my honoured teacher Professor Dr. Dr. h. c. W. Haack on the occasion of his 75th birthday in April 1977.

Aachen, April 1977

G. HELLWIG

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## *Part 4 [illegible]xistence*

**Mathematische Leitfäden**

Herausgegeben von
em. o. Prof. Dr. phil. Dr. h.c. G. Köthe, Universität Frankfurt/M., und
o. Prof. Dr. rer. nat. G. Trautmann, Universität Kaiserslautern

# Partial Differential Equations

## An Introduction

By Dr. rer. nat. Günter Hellwig
o. Professor at the Technische Hochschule Aachen

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Prof. Dr. rer. nat. Günter Hellwig

Born 1926. Studied from 1945 until 1949 mathematics and physics at the Universität Göttingen, 1949 degree in mathematics. Then assistant, senior assistant, and later "docent" at the Technische Universität Berlin, 1951 Dr. rer. nat. and 1952 "habilitation". 1955/56 Research Associate at the Institute of Mathematical Sciences (later the Courant Institute) of New York University. 1958 full professor and director of the Mathematisches Institut at the Technische Universität Berlin and permanent lecturer at the 1. Mathematisches Institut of the Freie Universität Berlin. Since 1966 full professor at the Technische Hochschule Aachen. Working fields: Partial differential equations and spectral theory of differential operators in Hilbert spaces.

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# 1
# INTRODUCTION

## 1.1 Definitions

A relation of the form

$$F(x_1, x_2, \ldots, x_n, u, u_{x_1}, u_{x_2}, \ldots, u_{x_n}, u_{x_1x_1}, u_{x_1x_2}, \ldots, u_{x_nx_n}) = 0 \tag{1.1}$$

where $n > 1$, will be called a *partial differential equation of the second order*. Here (1.1) is considered in a suitable domain $\mathfrak{D}$ of the *n-dimensional* space $R_n$ in the independent variables $x_1, x_2, \ldots, x_n$. We look for functions $u = u(x_1, x_2, \ldots, x_n)$ which satisfy (1.1) identically in $\mathfrak{D}$. Such functions $u$ are called solutions of (1.1).

In (1.1), the partial derivatives of $u$ are denoted in abbreviated form by indices; that is,

$$u_{x_i} \equiv \frac{\partial u}{\partial x_i}, \qquad u_{x_ix_k} \equiv \frac{\partial^2 u}{\partial x_i\, \partial x_k}. \tag{1.2}$$

The expression (1.1) is said to be of the *second order* because the highest partial derivatives which appear are of the second order.

If $n = 1$, then (1.1) becomes an *ordinary differential equation* of the second order:

$$F(x_1, u, u', u'') = 0 \quad \text{where} \quad u' \equiv \frac{du}{dx_1}, \qquad u'' = \frac{d^2u}{dx_1^2}.$$

In general, even an equation of this form has infinitely many solutions $u = u(x_1)$.

From this infinity of possible solutions we attempt to single out a unique one by introducing suitable additional conditions. *Initial conditions* often serve the purpose; these arbitrarily prescribe the value of $u$ and its first derivative at a point $a$:

$$u(a) = u_0, \qquad u'(a) = u_1.$$

Frequently, *boundary conditions* also suffice; these arbitrarily prescribe the value of $u$ at two points $a$ and $b$:

$$u(a) = u_0, \qquad u(b) = u_1.$$

It is readily apparent that the analogous problem for the expression (1.1) is substantially more difficult. For the time being, discussion will be restricted to the simplest representatives of (1.1).

To find representatives of (1.1) that are not only simple, but also typical and important, we look to mathematical physics; many of the problems in this field reduce to partial differential equations. Before starting in this direction, we provide some mathematical tools.

## 1.2 The Gauss Integral Theorem

Let *n-dimensional space* consisting of points $P$: $(x_1, x_2, \ldots, x_n)$ be denoted by $R_n$. Points in $R_n$ will in most cases be represented in vectorial form, where

$$x = (x_1, x_2, \ldots, x_n) \tag{1.3}$$

denotes the vector representing the point $P$. The inner product of two vectors $x, y$ and the absolute value or modulus $|x|$ of $x$ are given by

$$(x,y) = \sum_{i=1}^{n} x_i y_i, \qquad |x| = (x,x)^{1/2} = \sqrt{\sum_{i=1}^{n} (x_i)^2}, \tag{1.4}$$

respectively. In particular, $|x - y|$ is then the distance of the points $x$ and $y$ from one another.

In the following we briefly collect some formulas which can be found in any textbook on differential and integral calculus. Here it suffices to consider merely $R_2$ or $R_3$ and sufficiently simple domains $\mathfrak{D}$.

By a domain $\mathfrak{D}$ we understand an *open connected point set* in $R_n$. By $\bar{\mathfrak{D}}$ we denote the closure of $\mathfrak{D}$; by $\dot{\mathfrak{D}}$, the set of boundary points of $\mathfrak{D}$. Then $\bar{\mathfrak{D}} = \mathfrak{D} + \dot{\mathfrak{D}}$. As the simplest example for $\mathfrak{D}$, we mention the ball $S$ with center $a = (a_1, a_2, \ldots, a_n)$ and radius $r$. In this case $S$: $|x - a| < r$, $\dot{S}$: $|x - a| = r$, and $\bar{S}$: $|x - a| \leqslant r$. When $n = 2$, we call $S$ a disk.

Frequently it is necessary to form *volume integrals* over $\mathfrak{D}$ and surface integrals over $\dot{\mathfrak{D}}$. The *volume element* is denoted by $dx = dx_1\, dx_2 \cdots dx_n$ ($dx$ is not a vector); $dS$ denotes the surface element. For a function $u(x_1, x_2, \ldots, x_n)$ defined on $\mathfrak{D}$ or $\dot{\mathfrak{D}}$, respectively, we briefly write $u(x)$. We write $u \in C^0$ in $\mathfrak{D}$ if $u$ is continuous in $\mathfrak{D}$; $u \in C^j$ in $\mathfrak{D}$ if $u(x)$ is $j$ times continuously differentiable in all its variables, including all mixed derivatives up to the $j$th order. The notations $u(x) \in C^j$ in $\bar{\mathfrak{D}}$ and $u(x) \in C^j$ on $\dot{\mathfrak{D}}$ are analogous; in the latter, we suppose that $\dot{\mathfrak{D}}$ is described by a parametric representation and consider $u(x)$ as a function of the parameters. The integrals are of the form

$$\int_{\mathfrak{D}} u(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \cdots dx_n \equiv \int_{\mathfrak{D}} u(x)\, dx \quad \text{or} \quad \int_{\dot{\mathfrak{D}}} u(x)\, dS, \tag{1.5}$$

respectively.

A domain $\mathfrak{D}$ will be called a normal domain if it is bounded and simply connected and if it admits the application of the Gauss integral theorem; that is, if on $\dot{\mathfrak{D}}$ there is a vector field $\nu(x)$, where

$$\nu(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_n(x)) \quad \text{and} \quad (\nu,\nu) = 1 \tag{1.6}$$

such that

$$\int_{\mathfrak{D}} u_{x_i}(x)\, dx = \int_{\dot{\mathfrak{D}}} u(x)\nu_i(x)\, dS; \qquad i = 1, 2, \ldots, n, \tag{1.7}$$

for all $u(x) \in C^1$ in $\bar{\mathfrak{D}}$. If this is the case, the $\nu_i(x)$ are such that, at the points $x \in \dot{\mathfrak{D}}$ where $\bar{\mathfrak{D}}$ possesses an outer normal, the vector $\nu(x)$ of (1.6) coincides with this outer normal. We commonly omit explicit statement of the independent variable in $\nu$ and $\nu_i$. In the case of $R_2$, $\dot{\mathfrak{D}}$ is a closed curve and $dS$ is to be interpreted as $ds$, where $s$ is the arc length on $\dot{\mathfrak{D}}$. Note that under the stated assumptions we can choose either $\mathfrak{D}$ or $\bar{\mathfrak{D}}$ for the domain of integration in the volume integral in (1.7). Whenever the limits of integration are given explicitly, we indicate the boundary of $\mathfrak{D}$.

Nearly every domain $\mathfrak{D}$ occurring in this book is such a normal domain; thus we always neglect pointing out this fact and make special mention of the exceptions only. The explicit assumptions that must be made about $\dot{\mathfrak{D}}$ so that $\mathfrak{D}$ will be a normal domain are discussed in textbooks on differential and integral calculus. Hypotheses particularly suited to our purposes may be found in O. D. Kellogg[1] and Cl. Müller[2].

If $\dot{\mathfrak{D}}$ is the union of several pairwise disjoint closed components

$$\dot{\mathfrak{D}} = \sum_{j=1}^{N} \dot{\mathfrak{D}}_j$$

so that $\mathfrak{D}$ is not simply connected, then the Gauss integral theorem holds too, provided that every $\mathfrak{D}_j$ is a normal domain. Instead of relation (1.7), we then have

$$\int_{\mathfrak{D}} u_{x_i}(x)\, dx = \sum_{j=1}^{N} \int_{\dot{\mathfrak{D}}_j} u(x)\nu_i^j(x)\, dS^j, \tag{1.8}$$

where $\nu^j$ is the outer normal of $\mathfrak{D}$ on $\dot{\mathfrak{D}}_j$, and $dS^j$ is the surface element corresponding to $\dot{\mathfrak{D}}_j$.

If $\dot{\mathfrak{D}}$ is the unit sphere in $R_n$: $|x| = 1$, we denote its $dS$ by $d\omega$. For this surface,

$$\omega_n = \int_{|x|=1} d\omega = \frac{2[\Gamma(\frac{1}{2})]^n}{\Gamma(\frac{1}{2}n)} = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{1.9}$$

where $\Gamma(z)$ is the gamma function. Of course, we have the values $\omega_3 = 4\pi$ and $\omega_2 = 2\pi$. Finally, the volume of the unit ball $|x| \leqslant 1$ has the value $\omega_n/n$ which can be seen as follows: If, for $\mathfrak{D}$ in (1.7), we choose the unit ball $|x| \leqslant 1$ in $R_n$ and set $u(x) = x_i$, we obtain

$$\int_{|x|\leqslant 1} dx = \int_{|x|=1} x_i\nu_i\, d\omega.$$

Since $\nu_i = x_i$ here, summation yields

$$n\int_{|x|\leqslant 1} dx = \int_{|x|=1} \sum_{i=1}^{n} x_i\nu_i\, d\omega = \int_{|x|=1} \sum_{i=1}^{n} (x_i)^2\, d\omega = \int_{|x|=1} d\omega = \omega_n.$$

If, in (1.7), we merely suppose that $u(x) \in C^0$ in $\mathfrak{D}$ and $u(x) \in C^1$ in $\mathfrak{D}$, then, in general, $\int_{\mathfrak{D}} u_{x_i}(x)\, dx$ will be an improper integral, because $u_{x_i}$ may become infinite on $\dot{\mathfrak{D}}$. If, however, the existence of the integral $\int_{\mathfrak{D}} u_{x_i}(x)\, dx$ is required, (1.7) remains correct. See O. D. Kellogg[3].

## 1.3 Vector Fields

If the components $u^i(x)$ of a vector field $u(x) = (u^1(x), u^2(x), \ldots, u^n(x))$ belong to $C^1$ in $\mathfrak{D}$, then the *divergence* of $u$ is defined by

$$\operatorname{div} u = \sum_{i=1}^{n} u^i_{x_i}(x), \tag{1.10}$$

so that under use of (1.7) the Gauss integral theorem can also be written in the form

$$\int_{\mathfrak{D}} \operatorname{div} u(x)\, dx = \int_{\dot{\mathfrak{D}}} (u,\nu)\, dS. \tag{1.11}$$

By forming the *gradient* of a function $u(x) \in C^1$, the vector field

$$\operatorname{grad} u(x) = (u_{x_1}, u_{x_2}, \ldots, u_{x_n}) \tag{1.12}$$

is obtained.

Now for two vector fields, $u$, $v$ in $R_3$, there are the notions of *vector product* $u \times v$, and of forming the *rotation* (*curl*) rot $u$:

$$u \times v = (u^2v^3 - u^3v^2, u^3v^1 - u^1v^3, u^1v^2 - u^2v^1), \tag{1.13}$$

$$\operatorname{rot} u = (u^3_{x_2} - u^2_{x_3}, u^1_{x_3} - u^3_{x_1}, u^2_{x_1} - u^1_{x_2}). \tag{1.14}$$

## 1.4 The Green Formulas

By the *directional derivative* $u_\nu$ of $u(x)$ in direction of the outer normal $\nu$ we understand

$$u_\nu = \sum_{i=1}^{n} \nu_i u_{x_i}. \tag{1.15}$$

Further, we put

$$\Delta_n u = \sum_{i=1}^{n} u_{x_i x_i}.$$

If we assume $u(x) \in C^1$ in $\bar{\mathfrak{D}}$; $v(x) \in C^2$ in $\bar{\mathfrak{D}}$, we obtain the *first Green formula:*

$$\int_{\mathfrak{D}} u\, \Delta_n v\, dx = \int_{\dot{\mathfrak{D}}} uv_\nu\, dS - \int_{\mathfrak{D}} \sum_{i=1}^{n} u_{x_i} v_{x_i}\, dx \equiv \int_{\dot{\mathfrak{D}}} uv_\nu\, dS - \int_{\mathfrak{D}} (\operatorname{grad} u, \operatorname{grad} v)\, dx. \tag{1.16}$$

Indeed, we have

$$\int_{\mathfrak{D}} u\, \Delta_n v\, dx = \int_{\mathfrak{D}} \sum_{i=1}^{n} \{(uv_{x_i})_{x_i} - u_{x_i} v_{x_i}\}\, dx.$$

If we apply formula (1.7) to the middle term here and consider (1.15), then (1.16) follows immediately.

According to the remark at the end of Section 1.2, it would suffice to assume $u(x)$, $v(x) \in C^1$ in $\bar{\mathfrak{D}}$, $v(x) \in C^2$ in $\mathfrak{D}$, and the existence of $\int_{\mathfrak{D}} u\, \Delta_n v\, dx$.

By interchanging $u$ and $v$ in (1.16) and then subtracting the new formula, we obtain the *second Green formula:*

$$\int_{\mathfrak{D}} (u\, \Delta_n v - v\, \Delta_n u)\, dx = \int_{\dot{\mathfrak{D}}} (uv_\nu - vu_\nu)\, dS. \tag{1.17}$$

If $\dot{\mathfrak{D}}$ is the union of several components, then using the hypotheses made for (1.8) we find

$$\int_{\mathfrak{D}} (u\, \Delta_n v - v\, \Delta_n u)\, dx = \sum_{j=1}^{N} \int_{\dot{\mathfrak{D}}_j} (uv_{\nu^j} - vu_{\nu^j})\, dS^j. \tag{1.17a}$$

Here we have to suppose $u(x)$, $v(x) \in C^2$ in $\bar{\mathfrak{D}}$, or the corresponding weaker conditions. Setting $v \equiv 1$ in (1.17), we obtain the special case

$$\int_{\mathfrak{D}} \Delta_n u\, dx = \int_{\dot{\mathfrak{D}}} u_\nu\, dS. \tag{1.18}$$

Here we have to suppose $u(x) \varepsilon C^2$ in $\bar{\mathfrak{D}}$, or, in the weakened form, $u(x) \varepsilon C^1$ in $\bar{\mathfrak{D}}$, $u(x) \varepsilon C^2$ in $\mathfrak{D}$, and the existence of $\int_{\mathfrak{D}} \Delta_n u \, dx$.

## 1.5 The Maxwell Equations

We begin with the study of partial differential equations arising in mathematical physics. As is well known, part of the theory of electrodynamics can be deduced from the *Maxwell equations*. In $R_3$, the vector fields $E$ and $H$, representing the electric and magnetic fields, which also depend on a parameter $t$ (that is, time) are investigated:

$$E(x,t) = (E^1(x,t), E^2(x,t), E^3(x,t)), \qquad E^i(x,t) = E^i(x_1, x_2, x_3, t);$$
$$H(x,t) = (H^1(x,t), H^2(x,t), H^3(x,t)), \qquad H^i(x,t) = H^i(x_1, x_2, x_3, t).$$

We assume they are free of sources: div $E$ = div $H = 0$. In the volume element $dx$, the electric and the magnetic energy is given by

$$du_{\text{el}} = \frac{\epsilon}{8\pi} (E,E)\, dx, \; du_{\text{magn}} = \frac{\mu}{8\pi} (H,H)\, dx;$$

the energy flow in the time $dt$, through the surface element of $\dot{\mathfrak{D}}$ to the outside amounts to $(c/4\pi)(E \times H, \nu)\, dS\, dt$. Further, in the time $dt$, the electric energy $\sigma(E,E)\, dx\, dt$ in $dx$ will be transformed into heat. Here $\epsilon$, $\mu$, $\sigma$, $c$ are nonnegative physical constants.

The law of the conservation of energy tells us that the decrease in time of the energy in $\mathfrak{D}$ equals the sum of the energy converted into heat in $\mathfrak{D}$ and of the energy flowing out through $\dot{\mathfrak{D}}$:

$$-\frac{1}{8\pi}\frac{d}{dt}\int_{\mathfrak{D}} \{\epsilon(E,E) + \mu(H,H)\}\, dx = \sigma \int_{\mathfrak{D}} (E,E)\, dx + \frac{c}{4\pi}\int_{\dot{\mathfrak{D}}} (E \times H,\nu)\, dS. \quad (1.19)$$

Using (1.11) we have $\int_{\dot{\mathfrak{D}}} (E \times H, \; \nu)\, dS = \int_{\mathfrak{D}} \text{div } E \times H\, dx$. Further, div $E \times H$ = $(H, \text{rot } E) - (E, \text{rot } H)$ is a well-known vector identity. If we substitute this in (1.19), we see that (1.19) holds for every $\mathfrak{D}$ whenever the equations

$$\left.\begin{aligned} \text{rot } E &= -\frac{\mu}{c} H_t, \\ \text{rot } H &= \frac{\epsilon}{c} E_t + \frac{4\pi\sigma}{c} E, \end{aligned}\right\} \quad \text{where} \quad \text{div } E = \text{div } H = 0, \qquad (1.20)$$

are satisfied. Relation (1.20) is a special case of the Maxwell equations. It is a system of eight partial differential equations of the first order in the six unknown functions $E^1$, $E^2$, . . . , $H^3$.

The mathematical problem consists of finding solutions $E$, $H$ of (1.20) for all times $t$ which for $t = 0$ agree with given vector fields $E_0(x)$, $H_0(x)$. Therefore the *initial conditions* for (1.20) read

$$E(x,0) = E_0(x), \qquad H(x,0) = H_0(x) \quad \text{where} \quad \text{div } E_0 = \text{div } H_0 = 0. \qquad (1.21)$$

At first glance (1.20) might seem to be overdetermined, since the system contains more equations than unknown functions. However, the last two scalar equations are satisfied for all $t$ as soon as they hold for $t = 0$, which is the case according to (1.21). They can therefore be omitted; namely, if we form the divergence of the second equation (1.20), it follows that

$$0 = \text{div rot } H = \frac{\epsilon}{c} f_t + \frac{4\pi\sigma}{c} f,$$

where div $E(x,t) = f(x,t)$. Integration over $t$ gives

$$f(x,t) = C(x)\, e^{-(4\pi\sigma/\epsilon)t}$$

with arbitrary $C(x)$. Now $f(x,0) = 0$, so that $C(x) = 0$, which proves that div $E = 0$ for all values of $t$. Analogously it can be shown that div $H = 0$.

If we make use of the vector identity

$$\text{rot rot } E = \text{grad div } E - \Delta_3 E,$$

and of grad div $E = 0$ (since div $E = 0$), then, when the rotation is formed, the two vector equations in (1.20) go over into

$$\Delta_3 E = \frac{\epsilon\mu}{c^2} E_{tt} + \frac{4\pi\sigma\mu}{c^2} E_t, \qquad \Delta_3 H = \frac{\epsilon\mu}{c^2} H_{tt} + \frac{4\pi\sigma\mu}{c^2} H_t. \tag{1.22}$$

Here, of course, $\Delta_3 E$ stands for $(\Delta_3 E^1, \Delta_3 E^2, \Delta_3 E^3)$. Equations (1.22) are six partial differential equations of the second order for six unknown functions. As initial conditions—for the time being—we have to require conditions (1.21); however, $E_t(x,0)$ and $H_t(x,0)$ can be determined by use of (1.20). Therefore, for (1.22) we obtain the initial conditions

$$\begin{aligned} E(x,0) &= E_0(x), \qquad E_t(x,0) = E_1(x), \quad \text{where} \quad E_1(x) = \frac{c}{\epsilon} \text{rot } H_0 - \frac{4\pi\sigma}{\epsilon} E_0; \\ H(x,0) &= H_0(x), \qquad H_t(x,0) = H_1(x), \quad \text{where} \quad H_1(x) = -\frac{c}{\mu} \text{rot } E_0. \end{aligned} \tag{1.23}$$

Obviously we can solve the problem (1.22), (1.23) if we can merely solve the initial-value problem for an equation of the second order

$$\Delta_3 u = \frac{\epsilon\mu}{c^2} u_{tt} + \frac{4\pi\sigma\mu}{c^2} u_t \quad \text{where} \quad u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \tag{1.24}$$

For $\sigma = 0$, (1.24) is called the *wave equation* in three space dimensions and one time dimension, and for $\sigma \neq 0$ it is called the *telegraph equation*.

## 1.6 The Equations of Gas Dynamics

In $R_3$ we consider a compressible medium (gas) which is in motion, and whose pressure $p$, density $\rho$, and velocity vector $v = (v^1, v^2, v^3)$ are functions of $x$ and $t$. If we neglect viscous friction, heat conduction, and exterior forces, then *Euler's equation of motion* yields

$$\rho \frac{dv}{dt} = -\text{ grad } p, \tag{1.25}$$

and the theorem of the conservation of mass gives the *equation of continuity*

$$\rho_t + \text{div}(\rho v) = 0. \tag{1.26}$$