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Introduction.

These notes contain an introduction to the theory of linear algebraic groups over an algebraically closed ground field. They lead in a straightforward manner to the basic results about reductive groups.

The main difference with the existing introductory texts on this subject (e.g. those of Borel and Humphreys) lies in the treatment of the prerequisites from algebraic geometry and commutative algebra. These texts assume a number of such prerequisites, whereas I have tried to give proofs of everything. I have also tried to limit as much as possible the commutative algebra. For example, the use of the concept of normality has been avoided.

Moreover, in the later chapters most of the facts about root systems which are needed are proved, in an ad hoc manner. Exception must be made for the results on classification of root systems which are used in the last two chapters.

The exercises contain additional material. Sometimes use is made later on of the results contained in the easier exercises.

Except for a brief discussion in chapter 3 of groups over finite fields, these notes do not contain material about algebraic groups over non algebraically closed ground fields. An adequate treatment of such material would probably require another volume of the same size.

The notes had their origin in a course on linear algebraic groups, given at the University of Notre Dame in the fall of 1978. This course covered most of the material contained in the first ten chapters. I have added two chapters, with a treatment of the uniqueness and existence theorems for reductive groups.

I am grateful to my colleagues at the University of Notre Dame, in particular O.T. O'Meara and W.J. Wong, for their invitation to give a course on algebraic groups. I am also grateful to Hassan Azad, for making a first draft of the notes and to F.D. Veldkamp, for a thorough critical reading of the manuscript. Finally I want to thank Renske Kuipers for the efficient preparation of the manuscript.

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Contents.

1. Some algebraic geometry.

1.1. The Zariski topology.

1.1.1. Let k be an algebraically closed field, and put $V = k^n$. The elements of the polynomial algebra $S = k[T_1,\dots,T_n]$ (abbreviated to $k[T]$) can be viewed as k-valued functions on V. We say that $v \in V$ is a zero of $f \in k[T]$ if $f(v) = 0$. Moreover, v is a zero of the ideal I of S if $f(v) = 0$ for all $f \in I$. We denote by $V(I)$ the set of zeros of the ideal I. If X is any subset of V, let $I(X) \subset S$ be the ideal formed by the $f \in S$ vanishing on X.

Recall that the radical $\sqrt{I}$ of the ideal I is the ideal of all $f \in S$ such that $f^n \in I$ for some integer $n \geq 1$. A radical ideal is one equal to its radical. It is obvious that all $I(X)$ are radical ideals.

We shall need Hilbert's Nullstellensatz, in two (equivalent) formulations.

1.1.2. Theorem ("Nullstellensatz") (i) *If I is a proper ideal of S then* $V(I) \neq \phi$;

(ii) *For any ideal I of S we have* $I(V(I)) = \sqrt{I}$.

For a proof see, for example [26, Ch. X, §2]. We also give a proof in the appendix to this chapter.

1.1.3. Zariski topology on V.

The function $I \mapsto V(I)$ on ideals of S has the following properties:

(a) $V(\{0\}) = V$, $V(S) = \phi$;

(b) *If* $I \subset J$ *then* $V(J) \subset V(I)$;

(c) $V(I \cap J) = V(I) \cup V(J)$;

(d) If $(I_\alpha)_{\alpha\in A}$ is a family of ideals and $I = \sum_{\alpha\in A} I_\alpha$ their sum then $V(I) = \bigcap_{\alpha\in A} V(I_\alpha)$.

The proof of these properties is left to the reader (hint for (c): use that $IJ \subset I \cap J$).

It follows from (a), (c) and (d) that there is a topology on V whose closed sets are the $V(I)$, I running through the ideals of S. This is the Zariski topology. The induced topology on a subset X of V is the Zariski-topology of X. A closed set in V is called an algebraic set.

1.1.4. Exercises. (1) Let $V = k$. The proper algebraic sets are the finite ones.

(2) Let X be any subset of V. Its Zariski closure is $V(I(X))$.

(3) The map I defines an order-reversing bijection of the family of Zariski-closed subset of V onto the family of radical ideals of S, its inverse is V.

(4) The Euclidean topology on $\mathbb{C}^n$ is finer than the Zariski topology.

1.1.5. Proposition. Let $X \subset V$ be an algebraic set.

(i) The Zariski topology of X is T_1, i.e. points are closed;

(ii) Any family of closed subsets of X contains a minimal one;

(iii) If $X_1 \supset X_2 \supset \dots$ is a descending sequence of closed subsets of X there is h such that $X_i = X_h$ for $i > h$;

(iv) Any open covering of X has a finite subcovering.

If $x = (x_1, \dots, x_n) \in X$ then x is the zero of the ideal of S generated by $T_1 - x_1, \dots, T_n - x_n$. This implies (i).

(ii) and (iii) follow from the fact that S is a noetherian ring

[26, Ch. VI, §1], using 1.1.4(3).

To establish (iii) we formulate it in terms of closed sets. We then have to show: if $(I_\alpha)_{\alpha\in A}$ is a family of ideals such that $\bigcap_{\alpha\in A} V(I_\alpha) = \phi$, then already a finite intersection of some $V(I_\alpha)$ is empty. Now using properties (a), (d) of 1.1.3 and 1.1.4(3) we have $\sum_{\alpha\in A} I_\alpha = S$. Hence there are finitely many of the I_α, say $I_1, I_2, \ldots, I_h$ such that $I_1 + \ldots + I_h = S$. Then $\bigcap_{i=1}^{h} V(I_i) = \phi$.

A topological space X with the property (ii) is called noetherian. Notice that (ii) and (iii) are equivalent properties (compare the corresponding properties in noetherian rings, cf. [26, p. 142]). X is quasi-compact if it has the property of (iv).

1.1.6. Exercise. A closed subset of a noetherian space (with the induced topology) is noetherian.

1.2. Irreducibility of topological spaces.

1.2.1. A topological space X is reducible if it is the union of two proper closed subsets. Otherwise X is irreducible. A subset $A \subset X$ is irreducible if it is irreducible for the induced topology.

Notice the following fact: X is irreducible if and only if any two non-empty open subsets of X have a non-empty intersection.

1.2.2. Exercise. An irreducible Hausdorff space is reduced to a point.

1.2.3. Lemma. Let X be a topological space.

(i) $A \subset X$ _is_ _irreducible_ _if_ _and_ _only_ _if_ _its_ _closure_ $\bar{A}$ _is_ _irreducible_;

(ii) _Let_ $f: X \to Y$ _be_ _a_ _continuous_ _map_. _If_ X _is_ _irreducible_ _then_ _so_ _is_ fX.

Let A be irreducible. If $\bar{A}$ is the union of two closed subsets A_1 and A_2, then A is the union of the closed subsets $A \cap A_1$ and $A \cap A_2$, whence (say) $A \cap A_1 = A$, and $A \subset A_1$, $\bar{A} \subset A_1$. Hence $\bar{A} = A_1$. So $\bar{A}$ is irreducible.

Conversely, if $\bar{A}$ is irreducible, and if A is the union of two closed subsets $A \cap B_1$, $A \cap B_2$, where B_1, B_2 are closed in X, then $\bar{A} \subset B_1 \cup B_2$. So $\bar{A} \cap B_1 = \bar{A}$ (say), whence $A \cap B_1 = A$. The irreducibility of A follows.

The proof of (ii) is easy and can be omitted.

1.2.4. _Proposition_. _Let_ X _be_ _a_ _noetherian_ _topological_ _space_.

(i) X _is_ _a_ _union_ _of_ _finitely_ _many_ _irreducible_ _closed_ _subsets_, _say_ $X = X_1 \cup \ldots \cup X_s$;

(ii) _If_ _there_ _are_ _no_ _inclusions_ _among_ _the_ X_i, _they_ _are_ _uniquely_ _determined_, _up_ _to_ _order_.

Recall that a noetherian space is one with the property of 1.1.5(ii). If (i) is false, the noetherian property and 1.1.6 show that there is a minimal closed subset A of X which is _not_ a finite union of irreducible closed subsets. Then A must be reducible, so A is a union of two proper closed subsets. But these do have the property in question, and a contradiction emerges. This establishes (i).

To prove (ii), assume there are no inclusions among the X_i, and let $X = Y_1 \cup \ldots \cup Y_t$ be a second decomposition with the same properties. Then $X_i = \bigcup_j (X_i \cap Y_j)$, and by the irreducibility there is a function $f: \{1,...,s\} \to \{1,...,t\}$ with $X_i \subset Y_{f(i)}$. Similarly, there is $g: \{1,...,t\} \to \{1,...,s\}$ with $Y_j \subset Y_{g(j)}$. Since $X_i \subset X_{g(f(i))}$, we have $g \circ f = \mathrm{id}$, also $f \circ g = \mathrm{id}$. This implies (ii).

The X_i are called the (irreducible) <u>components</u> of X.
We now return to the Zariski topology on $V = k^n$.

1.2.5. <u>Proposition</u>. <u>A closed subset X of V is irreducible if and only if $I(X)$ is a prime ideal.</u>
Let X be irreducible and let $f,g \in S$ be such that $fg \in I(X)$. Then $X = (X \cap V(fS)) \cup (X \cap V(gS))$ and the irreducibility of X implies that $X \subset V(fS)$, say, i.e. $f \in J(X)$. So $J(X)$ is prime. Conversely, let $I(X)$ be prime, and let $X = V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$. If $X \neq V(I_1)$, there is $f \in I_1$ with $f \notin I(X)$. Since $fg \in I(X)$ for all $g \in I_2$, it follows from the fact that $I(X)$ is prime that $I_2 \subset I(X)$, whence $X = V(I_2)$. So X is irreducible.

1.2.6. <u>Exercise</u>. (1) Let X be a noetherian space. The components of X are the maximal irreducible closed subsets of X.
(2) Any radical ideal I of S is an intersection $I = P_1 \cap ... \cap P_s$ of finitely many prime ideals. If there are no inclusions among the P_i they are uniquely determined, up to order.

1.3. <u>Affine</u> k-<u>algebras</u>.

1.3.1. We now turn to more intrinsic descriptions of algebraic sets. Let $X \subset V$ be one. The restrictions to X of the polynomial functions of S form a k-algebra, denoted by $k[X]$, which clearly is isomorphic to $S/I(X)$. The following properties of $k[X]$ are obvious:

(a) $k[X]$ is a commutative k-algebra of finite type, i.e. there is a finite subset $\{x_1,\dots,x_r\}$ of $k[X]$ such that $k[X] = k[x_1,\dots,x_r]$;

(b) $k[X]$ is reduced, i.e. 0 is the only nilpotent element of $k[X]$. A k-algebra A with the properties of (a) and (b) is called an affine k-algebra. If A is an affine k-algebra there is an r and an algebraic subset X of k^r such that $A \simeq k[X]$. For $A \simeq k[T_1,\dots,T_r]/I$, where I is the kernel of the homomorphism $k[T_1,\dots,T_r] \to A$ sending T to x_i, this is a radical ideal. We call $k[X]$ the affine algebra of X.

1.3.2. We next show that the set X together with its Zariski topology is determined by its affine algebra $k[X]$. First observe that (by 1.2.5) X is irreducible if and only if $k[X]$ is an integral domain. If I is an ideal in $k[X]$ let $V_X(I)$ be the set of the $x \in X$ such that $f(x) = 0$ for all $f \in I$. If Y is a subset of X, let $I_X(Y)$ be the ideal of the $f \in k[X]$ such that $f(x) = 0$ for all $x \in Y$.

A being an affine algebra, let Max (A) be the set of its maximal ideals. If X is as before, and $x \in X$, denote by M_x the ideal of all $f \in k[X]$ vanishing in x. Then M_x is a maximal ideal (since $k[X]/M_x$ is the field k).

1.3.3. Proposition. (i) The map $x \mapsto M_x$ defines a bijection of X onto Max(k[X]), moreover $x \in V_X(I)$ if and only if $I \subset M_x$;

(ii) The closed sets of X are the $V_X(I)$, I running through the ideals of k[X].

Since $k[X] \simeq S/I(X)$, the maximal ideals of k[X] correspond to the maximal ideals of S containing $I(X)$. Let M be a maximal ideal of S. Then 1.1.4(3) and 1.1.5(ii) imply that M is the set of all $f \in S$ vanishing in some $x \in k^n$. From this the first point of (i) follows, and the second point is obvious.

(ii) is a direct consequence of the definition of the Zariski topology of X.

1.3.4. Exercises. (1) For any ideal I of k[X] we have $I_X(V_X(I)) = \sqrt{I}$; for any subset Y of X we have $V_X(I_X(Y)) = \overline{Y}$.

(2) The map I_X defines an order-reversing bijection of the family of Zariski-closed subsets of X onto the family of radical ideals of k[X], its inverse is V_X.

(3) Let A be an affine k-algebra. Define a bijection of Max(A) onto the set of k-algebra homomorphisms $A \to k$.

From 1.3.3 we see that the algebra k[X] completely determines X and its Zariski topology.

1.3.5. We also have to consider locally defined functions on X. For this we need special open subsets of X, which we now define.

If $f \in k[X]$, put

$$D(f) = \{x \in X | f(x) \neq 0\}.$$

Clearly, this is an open subset, viz. the complement of $V_X(fk[X])$. It is also clear that

$$D(fg) = D(f) \cap D(g), \quad D(f^n) = D(f) \quad (n \geq 1).$$

We call the D(f) principal open subsets of X.

1.3.6. Lemma. (i) If $f,g \in k[X]$ and $D(f) \subset D(g)$ then $f^n \in gk[X]$ for some $n \geq 1$;

(ii) The D(f) form a basis of the topology of X.

Using 1.1.4(3) we see that $D(f) \subset D(g)$ if and only if $\sqrt{fk[X]} \subset \sqrt{gk[X]}$, which implies (i).

To prove (ii) one has to show that any closed set is an intersection of sets of the form $V_X(fk[X])$, which is obvious from the definitions.

1.4. Regular functions, ringed spaces.

1.4.1. The notations are as before. Let $x \in X$. A k-valued function defined in a neighbourhood U of x is called regular in x if there are $g,h \in k[X]$ such that $h(x) \neq 0$ and such that there is an open neighbourhood $V \subset U$ of x with $h(y) \neq 0$ and $f(y) = g(y)h(y)^{-1}$ for all $y \in V$.

A function f defined in a non-empty open subset U of X is regular if it is regular in all points of U (so for each $x \in X$ there exist g_x, h_x with the properties stated above, they may depend on x). We denote by $\mathcal{O}_X(U)$ or $\mathcal{O}(U)$ the k-algebra of regular functions in U.

The following properties are obvious:

(A) If V is a non-empty open subset of the open set U, the restriction map of functions defines a k-algebra homomorphism $\mathcal{O}(U) \to \mathcal{O}(V)$;

(B) Let $U = \bigcup_{\alpha \in A} U_\alpha$ be an open covering of the open set U. Suppose that for each $\alpha \in A$ we are given $f_\alpha \in \mathcal{O}(U_\alpha)$ such that

f_α and f_β restrict to the same function on $U_\alpha \cap U_\beta$ $(\alpha,\beta \in A)$. Then there is $f \in \mathcal{O}(U)$ whose restriction to U_α is f_α, for all $\alpha \in A$.

1.4.2. Sheaves of functions.

Let X be a topological space, and suppose for each non-empty open subset U of X a k-algebra $\mathcal{O}(U)$ is given such that (A) and (B) hold. Then $\mathcal{O}$ is a sheaf of k-valued functions on X (we shall not need the general notion of a sheaf on a topological space). A pair $(X,\mathcal{O})$ of a topological space and a sheaf of functions is called a ringed space.

Let $(X,\mathcal{O})$ be a ringed space, let Y be a subset of X. We define the induced ringed space $(Y,\mathcal{O}|Y)$ as follows. Y is provided with the induced topology. If U is an open subset of Y, then $(\mathcal{O}|Y)(U)$ consists of the functions f on U with the following property: there exists an open covering $U \subset \bigcup_{\alpha\in A} U_\alpha$ by open sets of X and for each $\alpha \in A$ an element $f_\alpha \in \mathcal{O}(U_\alpha)$ whose restriction to $U_\alpha \cap U$ coincides with that of f.

Then $\mathcal{O}|Y$ is a sheaf of functions on Y. The proof of this fact is left to the reader. If Y is open in X, then $(\mathcal{O}|Y)(U)=\mathcal{O}(U)$ for all open $U \subset Y$.

1.4.3. Affine algebraic varieties.

The ringed spaces $(X,\mathcal{O}_X)$ of 1.4.1. are the affine algebraic varieties over k. In the sequel we shall usually drop the $\mathcal{O}_X$, and speak of an algebraic variety X,... . In this case, denote by $\mathcal{O}_{x,X}$ the k-algebra of functions regular in $x \in X$. By definition these are functions defined and regular in some open neighbourhood of x, two such functions being identified if they coincide in some neighbourhood of x (a formal definition