Advances in Geometric Programming

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Advances in Geometric Programming

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Preface

In 1961, C. Zener, then Director of Science at Westinghouse Corporation, and a member of the U.S. National Academy of Sciences who has made important contributions to physics and engineering, published a short article in the *Proceedings of the National Academy of Sciences* entitled "A Mathematical Aid in Optimizing Engineering Design." In this article Zener considered the problem of finding an optimal engineering design that can often be expressed as the problem of minimizing a numerical cost function, termed a "generalized polynomial," consisting of a sum of terms, where each term is a product of a positive constant and the design variables, raised to arbitrary powers. He observed that if the number of terms exceeds the number of variables by one, the optimal values of the design variables can be easily found by solving a set of linear equations. Furthermore, certain invariances of the relative contribution of each term to the total cost can be deduced.

The mathematical intricacies in Zener's method soon raised the curiosity of R. J. Duffin, the distinguished mathematician from Carnegie-Mellon University who joined forces with Zener in laying the rigorous mathematical foundations of optimizing generalized polynomials. Interestingly, the investigation of optimality conditions and properties of the optimal solutions in such problems were carried out by Duffin and Zener with the aid of inequalities, rather than the more common approach of the Kuhn–Tucker theory. One of the inequalities that they found useful in studying the optimality properties of generalized polynomials is the classical inequality between the arithmetic and geometric means. Because of this inequality, and some more general ones, called "geometric inequalities," Duffin coined the term "geometric programming" for the problem of optimizing generalized polynomials. In 1963, E. L. Peterson, a student of Duffin, started to work on developing the theory of constrained geometric programming problems. Thus a new branch of optimization was born.

The significance of the theory developed by Duffin, Peterson, and Zener was recognized very early by D. J. Wilde, then Professor of Chemical Engineering at Stanford University, who was equally interested in optimizing engineering design and in the theoretical aspects of optimization. He was fascinated by the simplicity and the potential usefulness of geometric programming and urged his two doctoral students M. Avriel and U. Passy to devote parts of their dissertations to geometric programming. These were completed in 1966 and contained topics that served as a kernel from which many future developments have sprouted. In the book *Geometric Programming*, published in 1967, Duffin, Peterson, and Zener collected their pioneering work. Their book was instrumental in inspiring continued research on theory, computational aspects, and applications.

Since the mid-60's, geometric programming has gradually developed into an important branch of nonlinear optimization. The developments include first of all significant extensions of the type of problems that were considered 10 years ago as geometric programs. Also, two-sided relationships with convex, generalized convex, and nonconvex programming, separable programming, conjugate functions, and Lagrangian duality were established. Numerical solution methods and their convergence properties were studied. Subsequently, computer software that can handle large constrained problems were developed. Applications of geometric programming to more and more problems of engineering optimization and design and problems from many other diverse areas were demonstrated.

In recognition of the important role of geometric programming in optimization, the *Journal of Optimization Theory and Applications* devoted two special issues to this subject. The majority of works appearing in this volume were first published there. In order to make this book as self-contained as possible, three earlier works on geometric programming are also reprinted here. These include (i) the 1968 paper of M. Hamala relating geometric programming duality to conjugate duality, which was originally published as a research report; (ii) the comprehensive survey paper of E. L. Peterson that appeared in 1976 in *SIAM Review*, and on which his other articles in this book are based; and (iii) the 1975 paper of M. Avriel, R. Dembo, and U. Passy that appeared in the *International Journal of Numerical Methods in Engineering*, describing the GGP algorithm that serves as a prototype of several condensation and linearization-based methods. The permission granted to reprint these works is gratefully acknowledged. (For a more detailed description of the articles in this volume, see the Introduction.)

I wish to express my appreciation to Professor Angelo Miele, Editor-in-Chief of the *Journal of Optimization Theory and Applications*, and Series Editor of the *Mathematical Concepts and Methods in Science and Engineering* texts and monographs, for inviting me to edit the two special issues of *JOTA* and also this book. Thanks are also due to Mrs. Batya Maayan for assisting me in these projects.

Haifa Mordecai Avriel

Contents

Introduction

M. AVRIEL

Geometric programming is probably still considered by many applied mathematicians and operations researchers as a technique for optimizing posynomials (generalized polynomials). The objective of this book is to bring to the attention of interested readers some of the advances made in recent years in geometric programming and related fields which reflect the greatly widened scope of this branch of nonlinear optimization. The advances are in three major categories: analysis, computations, and applications. The papers appearing in the book can be also classified accordingly.

The first two papers are introductory and set the stage for the analysis part of the book. The paper by Hamala presents the reader with the basic notions of convex functions, convex programming, and conjugate duality. Next, it introduces a class of primal–dual program pairs whose optimality properties are derived, and it is shown that primal–dual geometric programs are a special case of this class. Whereas the original development of duality in geometric programming was obtained with the aid of inequalities, Hamala derives analogous results by conjugate duality.

The second introductory paper is an extensive survey by Peterson in which he defines a quite general class of problems termed generalized geometric programs. For such programs he derives a comprehensive theory of optimality and duality, related to the convex analysis of Fenchel and Rockafellar.

An important result is that several well-known classes of optimization problems, possessing certain separability properties, can be reformulated as generalized geometric programs. These are amenable then to a unified treatment by optimality analysis. The next four papers, all written by Peterson, elaborate on the topics introduced in his survey and provide proofs of the results.

The theory of generalized geometric programming is carried further to infinite dimensional spaces in the paper by Jefferson and Scott. They consider convex optimal control problems, derive the appropriate dual problem, and discuss the optimality properties of the primal–dual pair.

In the next paper, Abrams and Wu investigate primal–dual pairs of generalized geometric programs in which one program, say the dual, does not have feasible interior points. In this case the primal problem has an unattained infimum or an unbounded optimal solution set. It is shown that after finitely many restrictions of the dual problem to an affine set and projections of the primal problem onto a subspace, a pair of problems results that has feasible interior points, bounded optimal solution sets, and the same value of the infima as the original pair of programs.

Lidor and Wilde study in the next paper extensions of ordinary (prototype) geometric programs in which some of the primal variables may appear also as exponents or in logarithms. These are called transcendental programs and they resemble in many ways ordinary geometric programs, although, due to lack of convexity, they can have local minima that are nonglobal. Corresponding dual programs are derived that are not "pure" duals, in the sense that they contain also primal variables. Avriel and Passy showed in their doctoral work the mathematical identity between the so-called "chemical equilibrium" problem of reacting species in an ideal system and the dual program of an ordinary geometric program. Lidor and Wilde demonstrate in this paper a relationship between dual transcendental programs and the chemical equilibrium problem in nonideal systems. This paper also concludes the analytic part of this book.

The numerical solution of geometric programs is a subject of great interest, and many specialized algorithms have been developed for this purpose. One of the basic questions concerns solving the primal or the dual program. In case of ordinary geometric programs the dual has a concave objective function and linear constraints, whereas in the primal program the constraints are usually nonlinear. It seems, therefore, logical to solve ordinary geometric programs via the dual. In practice, however, this approach is not always advisable because, if some dual variables must vanish in the optimal solution, the gradient of the objective function becomes unbounded and causes considerable numerical difficulties. Also, since programs more general than ordinary geometric programs either do not have a dual or the dual does not offer apparent computational advantages, several primal-based algorithms have been derived. Most of these use condensation, a technique for combining a sum of positive terms into a single term that enables a nonconvex program to be approximated by a convex one. Further condensation can result in the approximation of a convex program by a linear program. A representative member of such a class of methods is the GGP algorithm described in the paper by Avriel, Dembo, and Passy, which opens the computational part of the book. The main purpose (including this paper in the book is to make this volume more self-contain as this paper describes many details mentioned in the other computationally oriented works in this book.

The next three papers mainly deal with comparisons of various algorithms for the solution of geometric programming problems. The first paper by Dembo surveys primal-based and dual-based methods for ordinary (prototype) and generalized geometric programs. Special codes for geometric programs are compared among themselves and also with general-purpose algorithms by solving a set of test problems.

Sarma, Martens, Reklaitis, and Rijckaert continue the comparisons for ordinary geometric programs by testing various primal-based and dual-based algorithms. Their main conclusion is that dual-based methods do not offer any significant computational advantages, except in special cases.

Rijckaert and Martens performed a more extensive comparison by using 17 algorithms on 24 test problems. They also report that primal-based methods are generally superior to dual-based ones and that methods using condensation are the fastest and most robust.

The development of procedures for the practical evaluation and comparison of numerical methods and computer software is one of the most interesting problems of optimization awaiting a satisfactory solution. To illustrate this point, in the paper of Rijckaert and Martens it is shown that computer codes specially written for geometric programs clearly perform better than a general-purpose code included in their study. On the other hand, Ratner, Lasdon, and Jain, who developed an excellent and very efficiently programmed computer code that implements the generalized reduced gradient (GRG) method, report that their general-purpose nonlinear programming code can perform just as well as the special-purpose codes when applied to geometric programs. Another interesting observation is that the hitherto generally accepted "standard time" defined by Colville is an inadequate means of compensating for different computing environments. Clearly, much more research is needed in this area.

Although, judging from the above comparisons, the role of duality from a computational standpoint may have been overemphasized, there are several computational aspects of duality that deserve attention. Dembo's paper focuses on the interpretation of Lagrange multipliers that correspond to the constraints of a dual geometric program. He also analyzes the question of subsidiary problems that are needed when the exponents of the primal variables are linearly dependent.

The next two papers offer conceptual algorithms with limited computational experience. In the first paper, Ecker, Gochet, and Smeers present a modified reduced gradient algorithm for solving the dual problem of ordinary geometric programming. The difficulties encountered by other methods that attempt to solve dual geometric programs, such as the Beck and Ecker convex simplex method, are taken into consideration in the development of the new algorithm.

Methods discussed in the previous papers can in the best case find only local minima of nonconvex generalized geometric programs. Such local minima need not be global. Passy proposes here an implicit enumeration method for finding global solutions of nonconvex generalized geometric programs. His method can also be applied to a larger class of nonconvex programs.

The next two papers by Mancini and Wilde explore and demonstrate the possibilities of using interval arithmetic (an extension of ordinary arithmetic in which the basic elements are closed intervals) in geometric programming. The first paper applies interval arithmetic to geometric programs in which either the number of terms appearing in the primal problem exceeds the number of variables by two (one degree of difficulty) or the primal problem is unidimensional. The solution is based on an interval-arithmetic version of Newton's method. The second paper applies interval arithmetic to more general dual geometric programs in order to verify the existence and uniqueness of solutions and to compute error bounds on them.

Next there are two papers in this book that deal with engineering design applications of geometric programming. The usefulness of geometric programming is again demonstrated in this type of application. First, Avriel and Barrett consider a structural design problem of optimizing the geometry of certain wood beams. Second, Ecker and Wiebking formulate and solve the problem of optimally designing a cooling tower used in dissipating the waste heat of steam electrical plants. An interesting aspect of both engineering design papers is that existing theory and computational methods of geometric programming require all constraints to be inequalities. If the design problem has equality constraints, they must be expressed as inequalities in the geometric programming formulation—often an unnecessarily complicating feature. This is yet another unexplored area of geometric programming.

The book is concluded with a classified bibliography of publications in geometric programming, compiled by Rijckaert and Martens.

1

Geometric Programming in Terms of Conjugate Functions[1]

M. HAMALA[2]

Abstract. Our purpose is to show that the duality of geometric programming is a special case of Rockafellar's general theory of duality, and to construct a class of dual programs, which can be considered as a generalization of the usual geometric duality.

1. Introduction

The main purpose of this paper is (i) to show that the geometric duality of Duffin, Peterson, and Zener (Ref. 1) is a special case of Rockafellar's general theory of duality (Ref. 2), and (ii) to construct a general class of dual programs, which can be considered as a generalization of the geometric duality (Ref. 1), and which enable analogous procedures such as geometric programming to be developed.

The generalization proposed here is different from the one given in Ref. 1: the generalized geometric dual program in Ref. 1 is derived from geometric inequalities and in general is not convex. (Provided that the appropriate geometric inequalities are available, the nonconvexity is the main disadvantage of that approach.) The generalized dual program proposed here is derived from the properties of conjugate functions and it is convex. The only disadvantage of this approach is that the dual program has more variables than the primal one. But in many cases the number of variables can be reduced, e.g., in the case of classical geometric programs or linear programs.

It is worth noting that if G^* is the conjugate function of a differentiable convex function of n variables, and $\lambda > 0$, then

$$\sum_{i=1}^{n} x_i y_i \leqq \lambda G(x) + \lambda G^*(y/\lambda),$$

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and if $\lambda(y) \geq 0$ is a homogeneous function such that $\lambda(\nabla G(x)) = 1$, then

$$\sum x_i y_i \leqq \lambda(y) G(x) + \lambda(y) G^*(y/\lambda(y)) = \lambda(y) G(x) - F(y)$$

is a geometric inequality in the sense defined in Ref. 1.

Now we see that, to apply the approach given in Ref. 1, we need the explicit form of the function λ. In our approach λ is considered simply as an additional new variable.

For the sake of an easier exposition, it is convenient to restate briefly the main ideas of Rockafellar's theory given in Ref. 2. In the first part the necessary prerequisites are developed and in the second part Rockafellar's results are presented. In the third part the special case of Rockafellar's dual programs—the general geometric dual programs—is studied and two theorems are given analogous to those in Ref. 1.

Finally some concrete examples are discussed.

2. Basic Concepts

2.1. Convex Functions

Definition 2.1. The set $C \subset R^n$ is said to be convex if

$$\forall_{x_1, x_2 \in C} \; \forall_{0 \leqq \lambda \leqq 1} : \lambda x_1 + (1-\lambda) x_2 \in C.$$

Proposition 2.1. A nonempty convex set $C \subset R^n$ has a nonempty relative interior r int C (see Ref. 3, p. 16).

Definition 2.2. A real-valued function f defined on a nonempty convex set $C \subset R^n$ is said to be *convex on C* if

$$\forall_{x_1, x_2 \in C} \; \forall_{0 \leq \lambda \leq 1} : f(\lambda x_1 + (1-\lambda) x_2) \leqq \lambda f(x_1) + (1-\lambda) f(x_2).$$

Proposition 2.2. If f is convex on C and $x_1 \in r$ int C, then

$$\exists_{x^* \in R^n} \; \forall_{x \in C} : f(x) - f(x_1) \geqq (x - x_1) x^*.$$

[This result follows from the existence of a nonvertical supporting hyperplane (see Ref. 4, p. 398) and the Hahn–Banach theorem (see Ref. 5, p. 28).]