Elementary Theory of Electric and Magnetic Fields

Warren B. Cheston

Elementary Theory of Electric and Magnetic Fields

Warren B. Cheston

Professor of Physics
University of Minnesota
Minneapolis, Minnesota

John Wiley & Sons, Inc. New York · London · Sydney

Library of Congress Catalog Card Number: 64–17135
Printed in the United States of America

Preface

The theory of electric and magnetic fields is one aspect of classical physics. An author who chooses such a subject for a text can add little if anything to the fund of knowledge physicists possess of the structure of the physical world. Yet old theories, and physical theories age exceedingly rapidly in the twentieth-century world, bear re-examination from time to time, not with the intent of modification but rather of recasting in a form more suitable to the present era. In other words, a contemporary text on a classical subject is an attempt at style. This is a text written for students of physics who exist in the 1960's and who will shortly join the ranks of practicing physical scientists whose view of the physical world is particularly characteristic of the 1960's. It is a text written for a full year's course for upper division or senior college students; it presupposes some exposure to electricity and magnetism in a general physics course. It also presupposes a facility with mathematical concepts which ten years ago could only be assumed of beginning graduate students. Above all, as its title suggests, this is a text in fields, and little if any attention is paid to circuits and devices. There is some attempt, particularly in the final chapters, to discuss subjects of contemporary interest such as plasma physics, but serious students of these subjects will find the material in this text a limited introduction.

The underlying theme of the first half of the text, before the introduction of the Maxwell–Lorentz theory of electromagnetism, is the one-to-one correspondence between electrostatic and magnetostatic (steady-state) phenomena. For students who have read the chapters on electricity and magnetism, the transition to electromagnetism should be seen as an obvious outcome of the basic similarity between the separate theories of electricity and magnetism. Chapters 2–4 concern themselves with a discussion of the electrostatics of charges residing in vacuum or upon conducting surfaces. This discussion is based on Coulomb's law of force between two fixed charged particles. Chapter 5 is a discussion of the modifications brought about by the presence of nonconducting material media, and a macroscopic theory of the behavior of material media in the presence of static charges is developed. Chapters 6–8 concern themselves with a discussion of the magnetic effects produced by slowly moving charges in approximately uniform motion. The discussion of magnetostatics is based on the empirical law of force between slowly moving charged particles, and the discussion parallels that of electrostatics. In Chapter 9 the effect of material media on steady-state magnetic phenomena is discussed, and a macroscopic theory of magnetism is introduced. Chapters 10 and 11 serve as an introduction to the Maxwell–Lorentz equations of electromagnetism, and emphasis is placed in these chapters on electromagnetic waves in free space and guided waves. Chapter 12 is a brief discussion of the macroscopic Maxwell–Lorentz equations which are suitable for the discussion of material media, and the behavior of electromagnetic waves in conductors and insulators is discussed. In Chapter 13 the radiation of electromagnetic waves from systems of charges and currents is introduced. Chapter 14 concerns itself with two main topics: the motion of single charges in electric and magnetic fields and the behavior of an ensemble of charged particles, the latter being a brief discussion of magnetohydrodynamics and plasma physics. The final chapter, Chapter 15, is an abbreviated discussion of those aspects of solid-state physics which are relevant to a text on electricity and magnetism. Some of the subjects treated in this last chapter are electric and magnetic susceptibilities, ferromagnetism, and superconductivity. The book begins, Chapter 1, with a very condensed treatment of the salient mathematical techniques to be used throughout the text. It is assumed that this chapter will serve as a reference to the student and that the lecturer will begin his discussion with Chapter 2. The text is supplemented by a mathematical appendix consisting of a list of useful formulas and identities, and an appendix in which the various systems of units are commented upon.

Since the text aims at adopting a style consistent with the era in which it is written, attention should be called to a few of the topics of a strictly contemporary character. Chapter 6 concludes with a comparison between electrostatics and magnetostatics as to the behavior of the quantities related in the two theories under time reversal and space inversions; Chapter 9 contains a discussion of the Meissner effect and superconductivity; Chapter 12 discusses a simplified version of Kramers' classical dispersion theory; Chapter 14 concerns itself to a large extent with a subject of contemporary interest, namely, the physics of plasmas; Chapter 15 attempts to present a modern point of view concerning the behavior of real materials in electric and magnetic fields.

A text does not get written without the participation, some of it unintentional, of a large number of people. Included among these are the students in the classes to which I have lectured during the past ten years, and my colleagues at the University of Minnesota who have been the subjects of my many questions concerning certain aspects of electricity and magnetism which appeared to me to be subject to particularly murky discussions in the existing texts. Finally, grateful acknowledgment is offered to the quantity and high quality of the work of Mrs. Joyce Fay and Mrs. Kay Kirwin who prepared the manuscript for publication.

Warren B. Cheston

University of Minnesota
June 1964

Contents

1 Mathematical and Mechanical Preliminaries

In a discussion of electric and magnetic fields, the physical phenomena under investigation concern the interaction of charged objects and current-carrying elements among themselves. Such a discussion necessarily presumes a certain prior knowledge of the behavior of objects under the application of forces. The general ideas of analytical mechanics form the background to this discussion. In addition, a familiarity with the analytical tools employed in physics is essential. These analytical tools are the logic and techniques of mathematics, particularly those of the calculus, vector analysis, and vector calculus. A text in electric and magnetic fields cannot encompass a detailed discussion of analytical mechanics or mathematics, although certain ideas from these areas will be introduced as needed. Such mechanical or mathematical features will be handled in a rather cursory fashion. Nevertheless the discussion will be sufficiently complete to enable the student to understand the particular physical phenomenon under consideration. It is assumed that the student will be sufficiently curious to explore these ideas and techniques further in texts devoted entirely to the discussion of analytical mechanics or mathematics.

Although many mathematical techniques will be introduced as the physical ideas are developed, certain basic techniques will be set forth as a preliminary to the main discussion.

1.1 Vector Analysis

For the analytical description of certain physical entities, it is necessary to assign one number to represent the entity. For example, one number suffices to represent the mass of an object. An entity which can be described by giving one number is said to be a *scalar*. A scalar quantity may be restricted to take on positive values only (i.e., mass, temperature on the Kelvin scale, length, etc.) or it may assume both positive and negative values (i.e., mechanical energy; temperature on the centigrade scale, electric charge, etc.). Since all physical quantities are of necessity real, scalar quantities are represented by real numbers.

A given scalar quantity may depend in some way on the value of another scalar quantity. For example, the temperature of an object may depend on the time at which it is measured. Here, the temperature would be represented by a *scalar function* of the time.

There are many physical quantities for which it is necessary to specify more than one number in order to determine the quantity completely. For example, to locate the position of a point on a plane, or on any physical surface, it is necessary to specify two numbers. To locate the position of a point in space, it is necessary to specify three numbers. Many of the physical entities of this type can be represented by *vectors*. In general, the vectors used to represent physical quantities are specified by three numbers.

All the vectors dealt with in this text represent physical quantities which have a *magnitude* and an *orientation* defined with respect to some arbitrarily fixed direction in space. Since it is customary to represent a vector by a directed line segment (see Fig. 1.1), the magnitude of a vector is represented by the length of the directed line segment. To define the direction of a vector, a set of three mutually perpendicular straight lines—a set of coordinate axes—is erected, and the orientation of the directed line segment with respect to these coordinate axes represents the direction of the vector. Any vector can be represented by the projection of the vector along the three coordinate axes. These projections are numerically equal to the magnitude of the vector multiplied by the *direction cosines*.

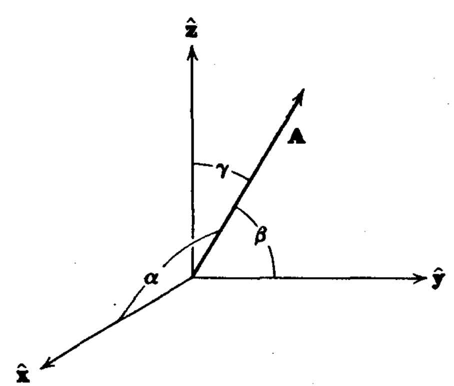

Fig. 1.1 A vector quantity **A** is represented by a directed line-segment. The orientation of the vector is given by the cosines of the angles between the vector and the x, y, z coordinate axes.

of the vector with respect to the coordinate axes (see Fig. 1.1). These projections are called the *components* of the vector and are a triad of numbers.

The three coordinate axes are customarily labelled the x-axis, the y-axis, and the z-axis, respectively. In this text, the axes (x, y, z) are always labelled in the right-hand sense. The components of the vector are consequently referred to as the x, y, and z components of the vector. This designation of a vector is called its *Cartesian representation*. An arbitrary vector **A** is therefore represented by three numbers (A_x, A_y, A_z). The magnitude of the vector is written as

$$|\mathbf{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{1.1}$$

and the orientation of the vector is specified by the three direction cosines as

$$\cos\alpha \equiv A_x/A, \qquad \cos\beta \equiv A_y/A, \qquad \cos\gamma = A_z/A \tag{1.2}$$

with the restriction implied by eq. 1.1, namely

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

A very useful concept is that of a *unit vector*. A unit vector is one whose magnitude is 1 (dimensionless). It is usually represented by affixing a "karat" symbol on top of the vector:

$$|\hat{\mathbf{A}}| \equiv 1; \qquad \hat{A}_x = \cos\alpha, \quad \hat{A}_y = \cos\beta, \quad \hat{A}_z = \cos\gamma. \tag{1.3}$$

The Cartesian components of a unit vector are simply the direction cosines of the vector. Any vector **A** can be written in terms of its magnitude A and the unit vector $\hat{\mathbf{A}}$ which is parallel to it; for example,

$$\mathbf{A} = A\hat{\mathbf{A}}. \tag{1.4}$$

A vector can be represented in terms of its components (A_x, A_y, A_z) and unit vectors along the coordinate axes $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$. This representation can be obtained from the above and is written as

$$\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}. \tag{1.5}$$

Often several vector quantities are considered simultaneously. If the vectors are expressed in the same units or dimensions, the "sum" of the vectors is a well-defined concept. Consider three vectors **A**, **B**, **C**. The sum of the vectors is also a vector and is uniquely defined as

$$\mathbf{A} + \mathbf{B} + \mathbf{C} \equiv \mathbf{D} = D_x\hat{\mathbf{x}} + D_y\hat{\mathbf{y}} + D_z\hat{\mathbf{z}} \tag{1.6}$$

where

$$D_x \equiv A_x + B_x + C_x$$
$$D_y \equiv A_y + B_y + C_y$$
$$D_z \equiv A_z + B_z + C_z$$

From eq. 1.6, it is obvious that vector addition is an *associative operation*, that is,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}).$$

It is also a *commutative operation*, that is,

$$\mathbf{A} + \mathbf{C} + \mathbf{B} = \mathbf{B} + \mathbf{C} + \mathbf{A} = \cdots.$$

Sometimes it is necessary to know the relative orientation of two vectors **A** and **B** (see Fig. 1.2). The relative orientation of **A** and **B** is specified by the angle between them. Since ϑ is restricted to lie between 0 and 2π radians (or alternatively between $-\pi$ and π radians), it is necessary to specify both the sine and the cosine of ϑ to determine it completely. The cosine of the relative orientation of two vectors is involved in the "scalar" or "inner" product of two vectors. The scalar product of the vectors **A** and **B** is defined as

$$\mathbf{A} \cdot \mathbf{B} \equiv |\mathbf{A}|\,|\mathbf{B}| \cos \vartheta. \tag{1.7}$$

It is quite obvious from the definition of a scalar product that it is a commutative operation:

$$\mathbf{A} \cdot \mathbf{B} \equiv \mathbf{B} \cdot \mathbf{A}. \tag{1.8}$$

The scalar product of **A** and **B** can be written in terms of the Cartesian components of these vectors

$$\mathbf{A} \cdot \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \cdot (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}).$$

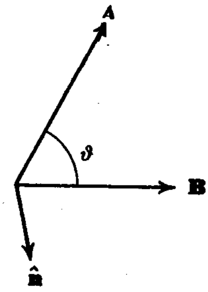

Fig. 1.2 The relative orientation of two vectors **A** and **B** is given by the angle ϑ between them. The angle ϑ in turn is determined by specifying its cosine and sine. The cosine of ϑ is involved in the scalar product of **A** and **B**; the sine of ϑ is involved in the vector product of **A** and **B**. The unit vector $\hat{\mathbf{n}}$ is perpendicular to the plane formed by the vectors **A** and **B**.

Since the unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are mutually orthogonal (i.e., $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$), it follows that

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z. \tag{1.9}$$

Use can be made of the concept of a scalar product to arrive at an expression for the Cartesian components of a vector alternative to that already employed. Taking cognizance of eqs. 1.2 and 1.9, it follows that

$$A_x = \mathbf{A} \cdot \hat{\mathbf{x}}, A_y = \mathbf{A} \cdot \hat{\mathbf{y}}, A_z = \mathbf{A} \cdot \hat{\mathbf{z}} \tag{1.10}$$

The sine of the relative orientation angle of two vectors is involved in the *vector product* of two vectors. This vector product is given the symbol $\mathbf{A} \times \mathbf{B}$ and is defined as

$$\mathbf{A} \times \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}| \sin \vartheta \hat{\mathbf{n}} \tag{1.11}$$

where $\hat{\mathbf{n}}$ = unit vector perpendicular to the plane defined by **A** and **B**. The direction of $\hat{\mathbf{n}}$ is arbitrarily chosen so that **A**, **B**, and $\hat{\mathbf{n}}$ form a right-handed triad of vectors. It is evident that the vector product of two vectors is a noncommutative operation. In fact,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1.12}$$

as can be seen directly from eq. 1.11. The vector product can be written in terms of the Cartesian components of the vectors involved and the unit vectors parallel to the Cartesian coordinate axes.

$$\mathbf{A} \times \mathbf{B} = (A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}) \times (B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}})$$

using such relations as

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$$
$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0,$$

it follows after some computation that

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\hat{\mathbf{x}} + (A_zB_x - A_xB_z)\hat{\mathbf{y}} + (A_xB_y - A_yB_x)\hat{\mathbf{z}}. \tag{1.13}$$

There are many vector identities of some usefulness, some of which are listed below. They can all be developed in a straightforward manner by Cartesian expansion of the expressions involved.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \tag{1.14}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{1.15}$$

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}) \tag{1.16}$$

1.2 Coordinate Systems

All physical phenomena take place in space. Whether the phenomena under consideration are mechanical, electrical, etc., they take place at a given time at a given point in space. If one wishes to describe a point in space analytically, it is necessary to relate the point in some way to a given reference point. It is evident that it is not only necessary to specify "how far" the point under consideration is from the reference point but also in what "direction." It is clear therefore that it is possible to represent a point in space by a vector, called a *position vector* and given the symbol **r**. The position vector (or simply the position) can be given by the three Cartesian components of **r** defined by

$$\mathbf{r} = r_x\hat{\mathbf{x}} + r_y\hat{\mathbf{y}} + r_z\hat{\mathbf{z}}.$$

Without any danger of misinterpretation, a more common notation will be adopted:

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \tag{1.17}$$

where $x \equiv (\mathbf{r} \cdot \hat{\mathbf{x}}), \quad y \equiv (\mathbf{r} \cdot \hat{\mathbf{y}}), \quad z \equiv (\mathbf{r} \cdot \hat{\mathbf{z}}).$

The Cartesian axes to which **r** is referred is a set of axes whose origin is located at the reference point, but whose orientation is arbitrary but fixed *ab initio*.

Most physical quantities to be considered in this text fall into two classes: (1) scalar point functions, and (2) vector point functions. A scalar point function is a scalar function of the vector variable **r**. A scalar point function is sometimes called a *scalar field*. The electrostatic potential function $\varphi(\mathbf{r})$ is such a scalar field. A vector point function is a vector function of the variable **r**; for example, at every point in space there exists a given value for the vector quantity under consideration. A vector point function is sometimes referred to as a *vector field*. The electric field $\mathbf{E}(\mathbf{r})$ and magnetic field $\mathbf{B}(\mathbf{r})$ are vector fields.

A vector point function can also be decomposed into its components; the value of the vector function **G** at **r** may be written as

$$\mathbf{G}(\mathbf{r}) = G_x(\mathbf{r})\hat{\mathbf{x}} + G_y(\mathbf{r})\hat{\mathbf{y}} + G_z(\mathbf{r})\hat{\mathbf{z}} \tag{1.18}$$

where

$$G_x(\mathbf{r}) \equiv \hat{\mathbf{x}} \cdot \mathbf{G}(\mathbf{r}), \quad G_y(\mathbf{r}) \equiv \hat{\mathbf{y}} \cdot \mathbf{G}(\mathbf{r}), \quad G_z(\mathbf{r}) = \hat{\mathbf{z}} \cdot \mathbf{G}(\mathbf{r}).$$

Equation 1.18 may be interpreted as follows. At every point in space **r** is imbedded a set of Cartesian axes. The value of the vector point function at **r** is a *position vector* in this Cartesian space; the projections of **G** on the coordinate axes are the Cartesian components of the vector **G** in this Cartesian space.

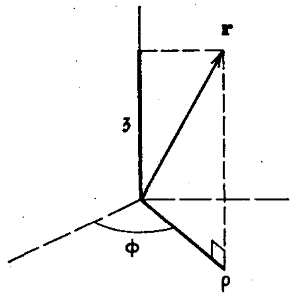

Fig. 1.3 In cylindrical coordinates, the position vector **r** is specified by the triad of numbers (ρ, ϕ, z).

Coordinate systems other than Cartesian may conveniently represent a vector point function. The most widely used coordinate systems other than Cartesian are (1) cylindrical, and (2) spherical polar. *Cylindrical coordinates* are defined as follows (see Fig. 1.3). One of the three components of **r** is the projection of **r** on the Cartesian z-axis or the so-called *azimuthal* axis. This axis is designated in cylindrical coordinates by a unit vector $\hat{\mathbf{z}}$. The other cylindrical components of **r** are obtained by projecting **r** onto the plane perpendicular to the azimuthal axis (the x-y plane). The magnitude of this projection is given the symbol ρ. A unit vector from the origin parallel to the projection of **r** is given the symbol $\hat{\boldsymbol{\rho}}$. $\hat{\boldsymbol{\rho}}$ is obviously perpendicular to $\hat{\mathbf{z}}$. The third orthogonal direction is defined by the unit vector $\hat{\boldsymbol{\phi}}$ which is perpendicular to both $\hat{\boldsymbol{\rho}}$ and $\hat{\mathbf{z}}$. It is defined in such a way that ($\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{z}}$) form a right-handed orthogonal system. It is obvious that $\hat{\boldsymbol{\phi}}$ lies in the Cartesian x-y plane. The position vector **r** can be written in cylindrical coordinates as

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}} \tag{1.19}$$

where $\rho \equiv (\mathbf{r} \cdot \hat{\boldsymbol{\rho}})$ and $z \equiv (\mathbf{r} \cdot \hat{\mathbf{z}})$.

($\hat{\boldsymbol{\phi}} \cdot \mathbf{r} = 0$ for *all* **r**.) The position vector **r** is not defined by the numbers ρ and z alone; a third number is needed and that number is taken to be the angle ϕ that $\hat{\boldsymbol{\rho}}$ makes with the Cartesian x-axis. Evidently $\hat{\boldsymbol{\rho}}$ is a function of ϕ. Then **r** is specified by giving the triad (ρ, ϕ, z). A vector point function can also be written in cylindrical coordinates, i.e.,

$$\mathbf{G}(\mathbf{r}) = \mathbf{G}(\rho, \phi, z).$$

and

$$\mathbf{G}(\mathbf{r}) = G_\rho(\mathbf{r})\hat{\boldsymbol{\rho}} + G_\phi(\mathbf{r})\hat{\boldsymbol{\phi}} + G_z(\mathbf{r})\hat{\mathbf{z}} \tag{1.20}$$

with $G_\rho \equiv \hat{\boldsymbol{\rho}} \cdot \mathbf{G}, \quad G_\phi \equiv \hat{\boldsymbol{\phi}} \cdot \mathbf{G}, \quad G_z \equiv \hat{\mathbf{z}} \cdot \mathbf{G}.$