Differential and Integral Equations and Their Applications

Series Editor: A.D. Polyanin

Mezhlum A. Sumbatyan
Antonio Scalia

EQUATIONS OF MATHEMATICAL DIFFRACTION THEORY

CHAPMAN & HALL/CRC

EQUATIONS OF MATHEMATICAL DIFFRACTION THEORY

Mezhlum A. Sumbatyan
Antonio Scalia

CHAPMAN & HALL/CRC

A CRC Press Company
Boca Raton London New York Washington, D.C.

Library of Congress Cataloging-in-Publication Data

Sumbatyan, Mezhlum A.
Equations of mathematical diffraction theory / Mezhlum A. Sumbatyan, Antonio Scalia.
p. cm.
Includes bibliographical references and index.
ISBN 0-415-30849-6 (alk. paper)
1. Diffraction—Mathematics. I. Scalia, A. II. Title.

QC415.S95 2004
535'.42'0151—dc22 2004051957

Direct all inquiries to CRC Press LLC, 2000 N.W. Corporate Blvd., Boca Raton, Florida 33431.

International Standard Book Number 0-415-30849-6
Library of Congress Card Number 2004051957
Printed in the United States of America 1 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

EQUATIONS OF MATHEMATICAL DIFFRACTION THEORY

PREFACE

The connection between heuristic and strictly formal methods is seemingly one of the most interesting and debatable questions in modern mathematics. Each of these two different approaches, whose foundations were laid by Socrates and Aristotle, respectively, and in the new history are reflected in discussions and written papers by Descartes, Leibnitz and Bacon, has its own intrinsic merits and restrictions. Moreover, a large number of discoveries in science were made owing to a combination of strict and heuristic methods of investigation.

Apparently, one of the brightest examples in modern mathematical physics is diffraction theory, where the combination of the two approaches would lead so efficiently to such impressive results. Many important and interesting solutions and even some classical theories appeared from heuristic ideas, and the most impressive example was given by Kirchhoff's physical diffraction theory, which is based upon a clear "light and shadow" concept for diffracted wave fields. Subsequently, many of these heuristic results were rigorously substantiated and proved as theorems. On the other hand, unsuccessful attempts to prove some other heuristic ideas caused significant progress in the development of formal methods that yielded correct solutions, different sometimes from those prompted by someone's intuition.

The above specific features have affected the style of presentation of the book. Each section deals with a discussion of heuristic ideas, which as a rule are substantiated (or disproved) with the use of rigorous mathematical methods. Due to limited volume of the book, at some places we give only a brief sketch of the substantiation, referring the reader to the original literature for more details.

One more specific feature of the presented material is connected with the rapid progress in computer technology over the last 20 years, which has significantly changed our viewpoint on what could be accepted as efficient methods of investigation. Only recently, expansion of unknown functions into series in terms of special functions, when a problem reduced to infinite system of linear algebraic equations with respect to coefficients of the expansion, was regarded as a standard method. Such a "semi-analytical" approach was efficient 15–20 years ago, when the evaluation of regularity of the obtained infinite systems seemed to be very important, since this could guarantee accuracy of a solution by retaining only few first equations, which was acceptable for first-generations computers. Nowadays, when there is not much difference between 10×10 and 500×500 systems even for home personal computers, such a viewpoint looks archaic, since the time required to convert the system to a form appropriate for "fast computations" is much greater than that for "slow computations" based on modern direct numerical methods like boundary element method and finite element method. Apparently, it should be agreed that in the cases where direct numerical techniques provide reliable results in an acceptable computational time, they should be regarded as most efficient for the problem in question. It is very important to recognize the cases where one has *a priori* to reject direct numerical methods. These are listed below.

1°. Problems where an exact analytical solution or a good approximation to it can be obtained. Diffraction theory shows many examples of this kind.

2°. Studying dynamic processes with high frequencies. Here, one has to take at least

10 nodes per wavelength to obtain reliable results by any direct numerical method. As the wavelength decreases (i.e., the frequency increases) within a given frequency range, the total number of nodes increases very rapidly, which results in too large algebraic systems. An impressive example is given by room acoustics. Suppose a sound wave of frequency $f = 2\,\text{kHz}$, whose wavelength in air is 17 cm, propagates in a 17-m long room. For reasonable numerical accuracy, one should hence take at least 1000 nodes along the room length. If the room has a width of 8 m and a height of 5.1 m, one has to consider $1000 \times 500 \times 300 \approx 10^8$ finite-element nodes and perform complex-valued arithmetic. This cannot be implemented even on the most powerful super computers. Here, a reasonable criterion for acceptability of a numerical approach is its implementability on a PC or similar computer. So, obtaining solutions to such high-frequency problems in exact formulation by direct numerical techniques does not seem to be feasible in the visible future.

3°. Studying phenomena of complex qualitative nature. Since direct numerical methods provide only numbers, which are usually tabulated and plotted, it is often very difficult to extract such complex qualitative effects from numerous tables and graphs. Instead, it is preferable to construct an approximate analytical solution, from which qualitative effects may be extracted explicitly.

4°. Cases where an exact analytical solution has been obtained but its representation is inapplicable to practice for specific calculations. An example of this kind is considered in Section 6.1. In such interesting cases, one should look for an alternative approach, which is often the construction of an approximate solution that would be more appropriate for fast computations than the exact analytical solution obtained.

The above situations are not widespread but, when met, are very difficult to cope with efficiently, especially if the researcher does not have sufficient experience in tackling them. This prompted us to conclude each section with a special subsection titled "Helpful Remarks," which may help the reader to build up his or her own less formal conception and allow the creation of a more complete picture of the issue under consideration.

Note that the application of numerical methods in regular cases is well described in the classical literature. For this reason, we only consider numerical methods for some irregular operator problems; see Chapter 9.

To summarize, the main purpose of the present book is to show the close connection between heuristic and rigorous methods in mathematical diffraction theory. We focus on differential and integral equations that can easily be utilized in practical applications.

Such an approach is accounted for by the choice of our potential readers. The book presents clear and elegant methods and is aimed at graduate and post-graduate students, so that they could quickly examine the state of the art in a specific field of interest. At the same time, researchers with considerable expertise in dealing with diffraction theory will hopefully discover that the time of clear explicit solutions in unsolved complex problems has not passed yet—this is demonstrated by the authors' original results in Sections 4.5, 4.6, 5.4–5.7, and 6.3–6.6 as well as in many sections of Chapters 7–9. Furthermore, we hope that an experienced reader will be able to discover for him- or herself new helpful methods, both analytical and numerical.

The reader will see in what follows that we prefer to rely upon classical results of the founders of modern science unlike a rather widespread (mistaken) point of view that only very complicated recent "abstract" theories can provide further progress in contemporary science. We strongly recommend the younger reader to operate with classical mathematical theories, and the present book will demonstrate that the fruitful ideas of Hilbert, Cauchy, Fourier, Abel, Poisson, Weyl, Riemann, Green, Kirchhoff, Rayleigh, Helmholtz, Neumann, and others can guide the reader very efficiently around present-day problems in diffraction theory. It should also be stressed that we tried to avoid too formal presentation, since we believe that wielding thorough knowledge in any mathematical theory implies applying it effectively and successfully to practice rather than operating with the formal apparatus of the theory.

Due to its limited volume, any monograph cannot cover all important questions, and the present book is no exception. For example, the reader will not find here transient problems at all. The presentation is confined to boundary problems for elliptic operators only, and only those with constant coefficients (except Section 3.6). Moreover, the main focus is on methods that provide solutions without too cumbersome mathematical manipulations. For example, the reader will not find the structure of the wave field in the "semi-shadow" zone in diffraction by convex obstacles, and in the method of "edge waves" in diffraction from linear segments, the reader will only find the leading high-frequency asymptotic term, which is constructed by a simple and elegant technique.

The sections, displayed formulas, and figures are enumerated independently within each chapter with the chapter number in front.

The book is intended for the reader familiar with fundamentals of real, complex-valued, and functional analysis within a standard course on calculus in the first three years of any university program of mathematical, physical, or engineering departments.

The style and content of this book have been influenced by the authors' friends, teachers, and colleagues, Alexander Vatulyan (Rostov State University, Russia), Mauro Fabrizio (University of Bologna, Italy), and Dorin Iesan (University of Iasi, Romania).

The authors are grateful to Alexander Manzhirov and Alexei Zhurov for their helpful discussions and comments.

The first author is thankful to his wife, Angela Sumbatyan, and to his daughters, Laura, Carina, and Angelica, who assisted and inspired him in the writing of the book.

M. A. Sumbatyan
A. Scalia

AUTHORS

Mezhlum A. Sumbatyan, Ph.D., D.Sc., is a noted scientist in the field of wave dynamics, diffraction theory, and ill-posed problems.

Mezhlum Sumbatyan graduated from the Faculty of Mechanics and Mathematics, Rostov State University, Russia, in 1969 and received his Ph.D. degree in 1980 at the Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow. His Ph.D. thesis was devoted to asymptotic methods for solving integral equations arising in some problems of mechanics and acoustics with mixed boundary conditions. In 1995, Professor Sumbatyan received his Doctor of Sciences degree; his D.Sc. thesis was dedicated to direct and inverse problems of diffraction theory, with applications to reconstruction of obstacles by ultrasonic techniques.

In 1985–2000, Mezhlum Sumbatyan worked in the Research Institute of Mechanics and Applied Mathematics of the Rostov State University. Since 2001, Professor Sumbatyan has been a member of the staff of the Faculty of Mechanics and Mathematics, Rostov State University.

Professor Sumbatyan has made important contributions to new methods in the theory and analytical methods applied to direct and inverse diffraction problems. He is an author of more than 120 scientific publications, a member of the Russian Acoustical Society, and a member of the Acoustical Society of America.

Address: Faculty of Mechanics and Mathematics
Zorge Street 5
344090 Rostov-on-Don, Russia

E-mail: sumbat@math.rsu.ru

Antonio Scalia, Ph.D., D.Sc., is a prominent scientist in the field of mathematical physics.

Antonio Scalia graduated from the Faculty of Mathematics, Physics and Natural Sciences, University of Catania, Italy, in 1972 and received his Ph.D. degree in 1980 at the University of Catania. His thesis was devoted to solvability and uniqueness in mathematical formulation of mechanical and acoustical problems for media with micro-structure.

Since 1989, Professor Scalia has been a member of the staff of the Department of Mathematics and Informatics at the University of Catania as an Associate Professor, and since 2001, as a Full Professor.

Professor Scalia has great expertise in the theory of viscoelasticity, electromagnetism, and acoustics. He has significant achievements in the mixture theory in solids, with applications to wave processes in elastic and acoustic media. Recently, within the theory of porous media, he devised a new approach to the analysis of wave propagation in elastic solids with pores. He explicitly studied a characteristic equation of the porous composite media applied to the layered geometry. Professor Scalia is an author of nearly 100 scientific publications.

Address: Dipartimento di Matematica e Informatica
Universitá di Catania, Viale A. Doria n. 6
95125 Catania, Italy

E-mail: scalia@dmi.unict.it

CONTENTS

Chapter 1

Some Preliminaries from Analysis and the Theory of Wave Processes

1.1. Fourier Transform, Line Integrals of Complex-Valued Integrands, and Series in Residues

Let a function $f(x)$ be integrable on the real axis: $f(x) \in L_1(-\infty, \infty)$. Then its Fourier transform $F(s)$ is defined as

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx, \qquad F(s) \in L_1(-\infty, \infty), \tag{1.1}$$

and in the case when $f(x)$ is continuous, the following inversion formula is valid:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds = \frac{1}{2\pi} \lim_{a\to\infty} \int_{-a}^{a} F(s)\, e^{-isx}\, ds, \quad x \in (-\infty, \infty). \tag{1.2}$$

The function $f(x)$ will be called an *original* and the function $F(s)$ the Fourier *image* of $f(x)$. The fact that the original and the image are related by formulas (1.1) and (1.2) will be denoted $f(x) \Longrightarrow F(s)$.

Many important and helpful properties of the Fourier transform are well known (e.g., see Titchmarsh, 1948; Bremermann, 1965). We will use only the following two of them:

1°. *Fourier image of the derivative.* Let $f(x) \Longrightarrow F(s)$ and $f^{(n)}(x) \in L_1(-\infty, \infty)$, then

$$f^{(n)}(x) \Longrightarrow (-is)^n\, F(s). \tag{1.3}$$

2°. *Fourier image of the convolution.* Let $f(x) \in L_1(-\infty, \infty)$, $g(x) \in L_1(-\infty, \infty)$, and $f(x) \Longrightarrow F(s)$, $g(x) \Longrightarrow G(s)$. Then the convolution of $f(x)$ and $g(x)$ is given by

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi \in L_1(-\infty, \infty), \quad \text{and} \quad h(x) \Longrightarrow F(s)\, G(s). \tag{1.4}$$

The first property (1.3) can be obtained by the direct differentiation of Eq. (1.2), and the second property (1.4) follows from a change of variable when applying the Fourier transform to Eq. (1.4).

The Fourier transform can also be defined for functions from the Hilbert space L_2: $f(x) \in L_2(-\infty, \infty)$. The classical *Plancherel theorem* asserts the existence of a Fourier transform $F(s)$ (Wiener, 1934):

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, ds = \lim_{a\to\infty} \int_{-a}^{a} F(s)\, e^{isx}\, ds, \quad x \in (-\infty, \infty). \tag{1.5}$$

The convergence here is implied in the mean-square sense, i.e., as a convergence in L_2. In this case, $F(s) \in L_2(-\infty, \infty)$, and the inverse Fourier transform is valid in the same sense,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds = \frac{1}{2\pi} \lim_{a\to\infty} \int_{-a}^{a} F(s)\, e^{-isx}\, ds, \quad x \in (-\infty, \infty), \tag{1.6}$$

of mean-square convergence. For L_2 functions $f(x), g(x) \in L_2(-\infty, \infty)$ the *Parseval identity* states that if $f(x) \Longrightarrow F(s)$ and $g(x) \Longrightarrow G(s)$, then

$$\int_{-\infty}^{\infty} f(x)\,\overline{g}(x)\,dx = \int_{-\infty}^{\infty} F(s)\,\overline{G}(s)\,ds, \quad \text{in particular,} \quad \int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(s)|^2\,ds, \tag{1.7}$$

where the bar over a symbol denotes a complex conjugate. The convolution theorem also remains valid in L_2.

Let $H(D)$ denote a set of complex-valued analytic functions $f(z)$ of the complex variable $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$ defined over a domain D: $f(z) \in H(D)$, $z \in D$. Recall that this implies that $f(z)$ is analytic and single-valued together with all its derivatives: $f^{(n)}(z) \in H(D)$, $\forall n = 0, 1, 2, \ldots$ (see Markushevich, 1963). Then the Cauchy theorem declares that the value of the line integral

$$I(\Gamma) = \int_{z_A}^{z_B} f(z)\,dz, \qquad z_A, z_B \in D, \tag{1.8}$$

along a curve $\Gamma \subset D$ of finite length with endpoints z_A, z_B is the same for any Γ, no matter how Γ connects z_A and z_B. This is equivalent to the statement that $I(\Gamma) = 0$ for any closed contour $\Gamma \subset D$ of finite length.

It is clear from the previous consideration that $I(\Gamma)$ in Eq. (1.8) is contour dependent only in the case when $f(z)$ has singular points in D. In the present book, we will consider only *poles* and *branching points* out of the whole variety of singular points.

A point $z_0 \in D$ is a pole of the function $f(z)$ if and only if z_0 is a zero of $g(z) = 1/f(z)$, i.e., $g(z_0) = 0$. The multiplicity n of the zero z_0 of $g(z)$ is, at the same time, the multiplicity of the pole z_0 of $f(z)$. It can be proved that the leading term in the Laurent series of the function $f(z)$ in a neighborhood of z_0 is $(z - z_0)^{-n}$, i.e.,

$$f(z) = \sum_{m=-n}^{\infty} a_m (z - z_0)^m, \tag{1.9}$$

where the coefficient a_{-1} is called the *residue* of the function $f(z)$ at the pole z_0 and denoted $a_{-1} = \mathrm{Res}\,[f(z), z_0]$. If $n = 1$ in Eq. (1.9), then a_{-1} is the leading coefficient in the Laurent expansion, and such a pole is called a *simple pole*. There is quite a simple way to calculate the residue at a simple pole z_0:

$$\mathrm{Res}\,[f(z),\ z_0] = \frac{h(z_0)}{g'(z_0)} \qquad \text{if} \quad f(z) = \frac{h(z)}{g(z)}, \tag{1.10}$$

which is very efficient with any natural fractional decomposition (as in the case $\tan z = \sin z/\cos z$).

Residues at poles play a key role in the calculation of integrals in the complex plane. This fact is represented by the *Cauchy integral formula* valid for any closed contour $\Gamma \subset D$ traced counterclockwise

$$\int_{\Gamma} f(z)\,dz = 2\pi i \sum_m \mathrm{Res}[f(z), z_m], \tag{1.11}$$

where the residues are taken at all poles z_m (of arbitrary multiplicity) inside Γ.

This formula is very helpful for the calculation of integrals of the type (1.8). Quite often $I(\Gamma)$ in (1.8) may be easily calculated for a certain simple path $I(\Gamma_s)$ passing through the endpoints z_A, z_B. Then the difference between $I(\Gamma)$ and $I(\Gamma_s)$, is equal to the sum of the residues at the poles between Γ and Γ_s, taken with appropriate sign.

This strategy can be applied to integrals written along infinite lines also. In particular, let $f(z)$ in the Fourier transform (1.1) be analytic in a finite-width strip $|\operatorname{Im}(z)| \le \delta$. Then the integration contour $\Gamma = (-\infty, \infty)$ may be arbitrarily shifted up or down within this strip. Indeed, if the integral (1.1) is finite under the integration along the initial path $(-\infty, \infty)$, this implies that $f(z) \to 0$ as $\operatorname{Re}(z) \to \infty$. Consequently, the integral of the same integrand (1.1) along any closed contour $(-\infty, \infty) \cup (\infty, \infty+i\varepsilon) \cup (\infty+i\varepsilon, -\infty+i\varepsilon) \cup (-\infty+i\varepsilon, -\infty)$, $|\varepsilon| < \delta$, is zero. Since the two integrals over far finite vertical intervals vanish, because $f(z)$ decays in a far-zone, this proves our simple statement. In the case when there is a number of poles between the real axis and the line $\operatorname{Im}(z) = \varepsilon$ parallel to it, it is evident that the same shift of the contour is possible if we add the residues at these poles. Sometimes this technique permits explicit calculation of Fourier transforms.

In order to shift the integration contour Γ more up (or down), outside of a finite-width strip, we need to apply the following

LEMMA (JORDAN). *Let*

$$I_R = \int_{C_R} f(z)\, e^{isz}\, dz, \tag{1.12}$$

where $f(z)$ is analytic everywhere in the upper half-plane $\operatorname{Im}(z) \ge 0$, except perhaps a finite number of poles; $\operatorname{Re}(s) > 0$; $f(z) \to 0$ as $z \to \infty$ uniformly over $0 \le \arg(z) \le \pi$; and C_R is an upper semi-circle of radius R: $|z| = R$, $\operatorname{Im}(z) \ge 0$. Then $I_R \to 0$ as $R \to \infty$.

The proof of this lemma is simple and can be found in the classical literature.

Corollary. Under the same conditions, the Fourier transform $F(s)$ of (1.1) can be explicitly expressed as

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = 2\pi i \sum_{\operatorname{Im}(z_m)>0} \operatorname{Res}\,[f(z),\ z_m]\, e^{isz_m}. \tag{1.13}$$

This result directly follows from the Cauchy integral formula (1.11) if you apply it to the function $f(z)\exp(isz)$ along the contour $\Gamma = (-R, R) \cup C_R$, with $R \to \infty$.

LEMMA (GENERALIZED JORDAN LEMMA). *Let $f(z)$ have a countable set of poles z_m, $m = 1, 2, \ldots$, $\operatorname{Im}(z_m) > 0$, $z_m \to \infty$, $m \to \infty$; and $f(z)$ vanishes uniformly on semi-circles C_{R_m} of radius R_m as $R_m \to \infty$; and each C_{R_m} passes somewhere between z_m and z_{m+1}. Then for any s such that $\operatorname{Re}(s) > 0$, we have*

$$I_{R_m} = \int_{C_{R_m}} f(z)\, e^{isz}\, dz \to 0 \quad as \quad m \to \infty. \tag{1.14}$$

The proof of this less known result repeats the one for the classical Jordan lemma.

Corollary. Under the same conditions, the Fourier transform can be explicitly calculated as an infinite series:

$$F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx = 2\pi i \sum_{m=1}^{\infty} \operatorname{Res}\,[f(z),\ z_m]\, e^{isz_m}. \tag{1.15}$$

This corollary is very helpful when $f(z)$ is *meromorphic*, i.e., is the ratio of two entire functions: $f(z) = h(z)/g(z)$. Recall that *entire* functions are defined as analytic over the whole complex plane, so countable sets of zeros and poles of any meromorphic function $f(z)$ are given by zeros of the entire functions $h(z)$ and $g(z)$, and both the resulting sets are finite if and only if $f(z)$ is rational.

This is clearly demonstrated by the function $f(z) = \tanh(z)/z$, where $h(z) = \sinh(z)/z$ and $g(z) = \cosh(z)$. It is also clear that in this example the set of upper semi-circles C_{R_m}, $m = 1, 2, \ldots$, can be determined from the condition $\tanh(iR_m) = 0 \sim \tan(R_m) = 0 \sim R_m = \pi m$, which causes respective semi-circles to pass through the imaginary points $iR_m = \pi mi$ (see Fig. 1.1).

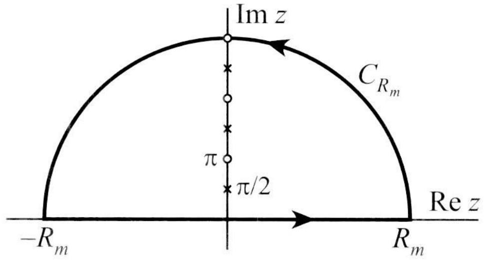

Figure 1.1. Alternate poles and zeros of a meromorphic function

Very often we will encounter below, in diffraction problems, some functions of a complex-valued argument that have branching points and hence are not single-valued. A typical representative here is the root square difference

$$\gamma(z) = \sqrt{z^2 - k^2}, \tag{1.16}$$

with a certain constant positive parameter $k > 0$ (see Mittra and Lee, 1971). Usually, in order to operate with a single-valued function, one has to arrange some cuts that become boundaries between different branches. For the root square difference (1.16) there are two branching points: $z = k$ and $z = -k$, and it is quite natural to make such cuts that allow operating with the arithmetic value of the root square difference, i.e., the branch with $\mathrm{Re}(z) \geq 0$. This can be provided by the cuts shown in Fig. 1.2, one of which passes totally in the upper half-plane $\mathrm{Im}(z) > 0$ and the other in the lower half-plane $\mathrm{Im}(z) < 0$. Note that for real z, $\gamma(z) = \sqrt{z^2 - k^2} \geq 0$ if $|z| \geq k$, and $\gamma(z) = -i\sqrt{k^2 - z^2}$ if $|z| \leq k$.

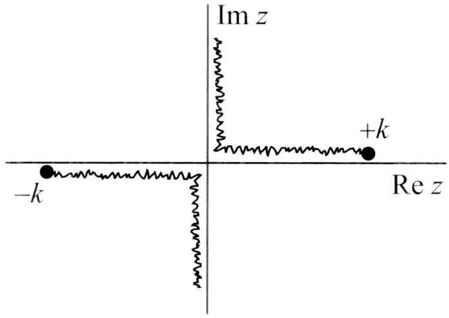

Figure 1.2. Cuts in the complex plane z making the function $\gamma(z) = \sqrt{z^2 - k^2}$ single-valued

For example, with such cuts the integral representation of the Hankel function (Abramowitz and Stegun, 1965)

$$\int_{-\infty}^{\infty} \frac{\exp(ixs)}{\gamma(s)}\, ds = 2\int_0^{\infty} \frac{\cos(xs)}{\gamma(s)}\, ds = \pi i H_0^{(1)}(k|x|) \tag{1.17}$$

implies integration along the real axis, when the integration path lies between the two cuts. Note that the singularities $s = \pm k$ are integrable in the classical sense.

It should be noted that branching functions, like the root square difference (1.16), generally are not analytic. However, some combinations of such functions can yield analytic and even entire functions, as can be seen by the example of the function $\sin[b\gamma(z)]/\gamma(z)$ (b is constant). It is certainly an entire function, since it can be represented by a Taylor series,

$$\frac{\sin[b\gamma(z)]}{\gamma(z)} = \sum_{m=0}^{\infty} \frac{(z^2 - k^2)^m\, b^{2m+1}}{(2m+1)!}(-1)^m, \tag{1.18}$$

that is analytic and convergent for all finite z.

Helpful remarks

1°. Interestingly, quite often the "shortest way" between two real points "lies" in the complex plane. To illustrate this, let us consider the following integral over an interval of the real axis and with real integrands:

$$J_1 = \int_0^{\infty} \frac{\cos(ax)}{x^2 + b^2}\, dx \qquad (a \geq 0,\ b > 0). \tag{1.19}$$

With the help of the Jordan lemma, we obtain

$$J_1 = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iax}\, dx}{x^2 + b^2} = \pi i \operatorname{Res}\left[\frac{e^{iaz}}{z^2 + b^2},\ ib\right] = \pi i \frac{e^{-ab}}{2bi} = \frac{\pi}{2b} e^{-ab}, \tag{1.20}$$

where the residue at the simple pole $z = ib$ has been calculated with the method described above.

2°. The same approach is applicable to a meromorphic function if you use the generalized Jordan lemma $(a, b > 0)$:

$$\begin{aligned} J_2 &= \int_0^{\infty} \frac{\tanh(bx)}{x}\cos(ax)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{iax}\frac{\sinh(bx)}{x\cosh(bx)}\, dx \\ &= \pi i \sum_{m=1}^{\infty} \operatorname{Res}\left[e^{iaz}\frac{\sinh(bz)}{z\cosh(bz)},\ \frac{\pi i}{b}\left(m - \frac{1}{2}\right)\right] = \pi i \sum_{m=1}^{\infty} \frac{e^{-\pi a(m-1/2)/b}}{\pi(m - 1/2)\, i} \\ &= \sum_{m=1}^{\infty} \frac{e^{-\pi a(m-1/2)/b}}{m - 1/2} = \ln\frac{1 + e^{-\pi a/2b}}{1 - e^{-\pi a/2b}} = \ln\left[\coth\left(\frac{\pi a}{4b}\right)\right], \end{aligned} \tag{1.21}$$

where, in order to calculate the residues at simple poles, we have put $f(z) = h(z)/g(z)$ with entire functions $h(z) = e^{iaz}\sinh(bz)/z$ and $g(z) = \cosh(bz)$. The following tabulated series has also been taken into account here (Gradshteyn and Ryzhik, 1994):

$$\sum_{m=0}^{\infty} \frac{x^{2m+1}}{2m+1} = \frac{1}{2}\ln\frac{1+x}{1-x}, \qquad (|x| < 1). \tag{1.22}$$

3°. Another remarkable phenomenon is related to the question why the Fourier transform, which (as a rule) converts real-valued functions to complex-valued, is so helpful when solving real boundary value problems. The answer can be seen from property 1° of the Fourier transform, since any derivative of an unknown function is converted to the same image with a factor containing Fourier parameter s. Therefore, if you solve any boundary value problem in a domain where the Cartesian coordinate x varies from $-\infty$ to ∞, then the application of the Fourier transform will allow you to reduce the dimension of the problem by 1. This can reduce ordinary differential equations to algebraic ones, and a partial differential equation in two variables to an ordinary differential equation.

Property 2° of the Fourier transform allows you to solve integral equations with convolution kernels explicitly. Both techniques will be demonstrated in detail below.

1.2. Convolution Integral Equations and the Wiener–Hopf Method

Generally, a convolution integral equation has the following form:

$$\alpha\,\varphi(x) + \int_a^b K(x-\xi)\,\varphi(\xi)\,d\xi = f(x), \qquad a < x < b, \tag{1.23}$$

which is evidently an equation of the second kind. In the case $\alpha = 0$ it becomes an equation of the first kind. The function $K(x)$ is a (known) kernel of the equation, and $f(x)$ is a (known) right-hand side. The function $\varphi(x)$ is unknown and is to be determined from Eq. (1.23). There is a special, unique case when equation (1.23) generally admits exact analytical solution. This is the case of $b = \infty$, where we get the Wiener–Hopf equation. In this case, a can be made equal to zero by a linear change of variable and only the first-kind equation ($\alpha = 0$) will be important to us in this case. The solution of this equation is based upon some evident properties of the Fourier transform in the complex plane (see Bremermann, 1965; Mittra and Lee, 1971; Noble, 1958):

1°. If $|f(x)| \le A\,e^{\tau_- x}$, $x \to +\infty$, then the function

$$F_+(s) = \int_0^\infty f(x)\,e^{isx}\,dx \tag{1.24}$$

is analytic in the upper half-plane $\mathrm{Im}(s) > \tau_-$.

2°. If $|f(x)| \le B\,e^{\tau_+ x}$, $x \to -\infty$, then the function

$$F_-(s) = \int_{-\infty}^0 f(x)\,e^{isx}\,dx \tag{1.25}$$

is analytic in the lower half-plane $\mathrm{Im}(s) < \tau_+$.

3°. If both properties 1° and 2° are satisfied and $\tau_+ > \tau_-$, then the full Fourier transform

$$F(s) = \int_{-\infty}^\infty f(x)\,e^{isx}\,dx \tag{1.26}$$

is analytic in the strip $\tau_- < \mathrm{Im}(s) < \tau_+$, and the inverse Fourier transform may be calculated as follows:

$$f(x) = \frac{1}{2\pi}\int_{-\infty+i\tau}^{\infty+i\tau} F(s)\,e^{-isx}\,ds, \qquad \tau_- < \tau < \tau_+. \tag{1.27}$$

It is obvious from the previous section that you may arbitrarily deform the infinite integration contour Γ in (1.27) within the marked strip, if necessary.

Now we are ready to apply the Wiener–Hopf method to the equation

$$\int_0^\infty K(x-\xi)\,\varphi(\xi)\,d\xi = f(x), \qquad 0 < x < \infty. \tag{1.28}$$

Equation (1.28) is equivalent to

$$\int_{-\infty}^\infty K(x-\xi)\,\varphi_+(\xi)\,d\xi = f_+(x) + f_-(x), \qquad |x| < \infty, \tag{1.29}$$

where

$$\varphi_+(x) = \begin{cases} \varphi(x), & x > 0, \\ 0, & x < 0, \end{cases} \qquad f_+(x) = \begin{cases} f(x), & x > 0, \\ 0, & x < 0, \end{cases} \tag{1.30}$$