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Nonlinear Waves and Diffusion Processes

Dedicated to

Professor P L Sachdeva

Preface

The present book is the proceedings of UGC-DSA National Conference, "Nonlinear Waves And Diffusion" held at School of Mathematics, Madurai Kamaraj University, Madurai, India. The major aim of the conference was to provide an opportunity to scientists and engineers to meet and discuss recent developments in the field of nonlinear wave propagation and diffusion phenomena.

The conference was attended by a large number of academicians, research scholars and students from various colleges, universities and institutions. I am indebted to many for their ingenuity, creativity and resourcefulness.

I thank Dr. P. Maruthamuthu, Vice-Chancellor, Dr. P. K. Ponnuswamy, Dr. P. V. Ramakrishnan and Dr. G. Arivarignan for providing the necessary facilities to conduct the conference.

I record my sincere thanks to Mrs. J. K. Subashini, Mr. M. Senthil Kumaran and Mr. R. Asokan for their support in bringing out this book. I would like to thank my students Ms. G. Sivagami and Mr. S. Padmasekaran for their cooperation during the preperation of the manuscript.

I am grateful to my mother B. Velammal for her encouragement. I feel indebted to my wife M. Bala Raja Seevili for patiently listening to my rumblings when I encountered numerous obstacles and bore the major share of the responsibility of bringing up our daughter, M. Keerthana Sivaali, and our son, M. Kiran Vasistha.

Madurai

B. Mayil Vaganan

Contents

Exponential and algebraic solutions describing unsteady rectilinear flows of non-homentropic, perfect gas

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Abstract: Equations of motion of a perfect, non-homentropic gas in Lagrangian form are subjected to the separation of variables and the direct method to obtain a new exponential solution, besides recovering the algebraic solution of Steketee (1979,1976,1972).

1. Introduction

The equations of the conservation of mass and the momentum of the unsteady rectilinear motion of a gas in the Lagrangian coordinates are (Stanyukovich (1960))

$$\frac{\partial V}{\partial t} - \frac{\partial u}{\partial h} = 0, \tag{1.1}$$

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial h} = 0, \tag{1.2}$$

where V, u, p, t and h denote the specific volume, the velocity, the pressure, the time and the Lagrangian coordinate, respectively.
For the non-homentropic gas, the equation of state and the conservation of energy lead to

$$pV^{\gamma} = e^{S/c_v} = B(h), \text{say}. \tag{1.3}$$

We later show that (1.3) is nothing but the Poisson relation

$$pV^{\gamma} = e^{S/c_v} = B(h) = \bar{B}h^{n(\gamma+1)}. \tag{1.4}$$

In (1.3) S stands for the entropy per unit mass of the gas, $\gamma = c_p/c_v$, a constant and we assume that $\gamma > 1$. We neglect the effects of viscosity and heat conduction.
The cartesian coordinate x and the Lagrangian coordinate h are related through

$$dh = \rho\, dx, \tag{1.5}$$

where $\rho(= 1/V)$ is the density of the gas.
Equation (1.1) is identically satisfied if we introduce $E(h,t)$ as

$$u = E_t \quad \text{and} \quad V = E_h. \tag{1.6}$$

It follows from the very definition of E that $E = x$.
Inserting (1.3) and (1.6) into (1.2) we obtain the following equation governing

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$E(h,t)$:

$$E_{tt} + [B(h)E_h^{-\gamma}]_h = 0, \tag{1.7}$$

or

$$E_{tt} - \gamma B(h)E_h^{-\gamma-1}E_{hh} + B'(h)E_h^{-\gamma} = 0. \tag{1.8}$$

In general $B(h)$ is not arbitrary as in the case of the flow behind the non-uniformly travelling shock waves in the wave-interaction problems. Therefore we seek to identify the forms of $B(h)$ for which the system of equations (1.1)-(1.3) or (1.7) admits exact solutions.
It is a fact that the sound speed a is given by $a^2 = \gamma p/\rho$. In a series of papers Steketee (1979, 1976, 1972) obtained self-similar solution of the system of equations (1.1)-(1.3) as

$$\begin{aligned} V(h,t) &= V_0 h^{n-1+\kappa} t^{1-\kappa}, & (1.9)\\ u(h,t) &= u_0 h^{n+\kappa} t^{-\kappa} + U, & (1.10)\\ p(h,t) &= p_0 h^{n+1+\kappa} t^{-1-\kappa}, & (1.11)\\ a(h,t) &= a_0 h^{n+\kappa} t^{-\kappa}, & (1.12) \end{aligned}$$

where $\kappa = \frac{\gamma-1}{\gamma+1}$, by writing

$$\begin{aligned} V(h,t) &= h^n \bar{V}(\eta), \quad p(h,t) = h^n \bar{p}(\eta),\\ u(h,t) &= h^n \bar{u}(\eta), \quad B(h) = \bar{B} h^{n\gamma+n}, \end{aligned}$$

$$\text{with}\eta = \frac{t}{h}, \text{thesimilarityvariable.} \tag{1.13}$$

into (1.1)-(1.3) and solving the resulting system of ordinary differential equations

$$\begin{aligned} \bar{V}' - n\bar{u} + \eta\bar{u}' &= 0, & (1.14)\\ \bar{u}' + n\bar{p} - \eta\bar{p}' &= 0, & (1.15)\\ \bar{p}\bar{V}^\gamma - \bar{B} &= 0. & (1.16) \end{aligned}$$

Solution of (1.14)-(1.16) is

$$\begin{aligned} \bar{V} &= V_0 \eta^{1-\kappa}, & (1.17)\\ \bar{u} &= u_0 \eta^{-\kappa}, & (1.18)\\ \bar{p} &= p_0 \eta^{-1-\kappa}, & (1.19)\\ \bar{a} &= a_0 \eta^{-\kappa}. & (1.20) \end{aligned}$$

Here we obtain new solution of the system (1.1)-(1.3) or (1.7) besides recovering the solution (1.9) - (1.12) of Steketee by applying varigated techniques such as separation of variables (Ames,(1972)), the direct method of Clarkson and Kruskal (1989).
The rest of the paper is organised as follows. In section 2 we apply the method of separation of variables to (1.7) with a view to derive its solutions. And the direct method is employed to (1.7) in section 3. The results and discussions of the present paper are set forth in section 4.

2. Method of separation of variables

Assume

$$E(h,t) = T(t)H(h) \quad \text{and} \quad B(h) = bh^m. \tag{2.1}$$

Substituting (2.1) into (1.7) yields

$$\frac{T''}{T^{-\gamma}} = -\frac{[B(h)(H')^{-\gamma}]'}{H} = \lambda, \text{say}, \tag{2.2}$$

where λ is the constant of proportionality. From (2.2), we have

$$T''T^{\gamma} = \lambda, \tag{2.3}$$

$$[B(h)(H')^{-\gamma}]' = -\lambda H. \tag{2.4}$$

2.1 $B(h)$ is algebraic.

A solution of (2.3)-(2.4) is

$$T(t) = \left(\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right)^{\frac{1}{1+\gamma}} t^{\frac{2}{1+\gamma}}, \tag{2.5}$$

$$H(h) = ch^{\frac{\gamma+m-1}{\gamma+1}}, \tag{2.6}$$

provided that $B(h) = bh^m$ where

$$b = \frac{-\lambda c^{1+\gamma}(\gamma+m-1)^{\gamma}}{(2\gamma+m)(1+\gamma)^{\gamma-1}}. \tag{2.7}$$

Here c is an arbitrary constant. Therefore $B(h)$ is

$$B(h) = \frac{-\lambda c^{1+\gamma}(\gamma+m-1)^{\gamma}}{(2\gamma+m)(1+\gamma)^{\gamma-1}} h^m. \tag{2.8}$$

Hence the solution $E(h,t)$ is

$$E(h,t) = c\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right]^{\frac{1}{1+\gamma}} t^{\frac{2}{1+\gamma}} h^{\frac{\gamma+m-1}{1+\gamma}}. \tag{2.9}$$

Here λ, c and m are arbitrary constants. Substituting (2.9) into (1.6), (1.3) and $a^2 = \gamma p/\rho$, we obtain the volume, velocity, pressure and the sound speed:

$$V(h,t) = \frac{c(\gamma+m-1)}{1+\gamma}\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right]^{\frac{1}{1+\gamma}} t^{\frac{2}{1+\gamma}} h^{\frac{m-2}{1+\gamma}}, \tag{2.10}$$

$$u(h,t) = \frac{2c}{1+\gamma}\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right]^{\frac{1}{1+\gamma}} t^{\frac{1-\gamma}{1+\gamma}} h^{\frac{\gamma+m-1}{1+\gamma}} + U, \tag{2.11}$$

$$p(h,t) = \frac{c(\gamma-1)}{2\gamma+m}\left[\frac{\lambda(1+\gamma)^{1-\gamma}2^{\gamma}}{1-\gamma}\right]^{\frac{1}{1+\gamma}} t^{\frac{-2\gamma}{1+\gamma}} h^{\frac{2\gamma+m}{1+\gamma}}, \tag{2.12}$$

$$a(h,t) = \left[\frac{c^2\gamma(\gamma-1)(\gamma+m-1)}{2\gamma+m}\right]^{\frac{1}{2}} \left[\frac{\lambda(1+\gamma)^{1-\gamma}2^{(\gamma-1)/2}}{(1-\gamma)}\right]^{\frac{1}{1+\gamma}} t^{\frac{1-\gamma}{1+\gamma}} h^{\frac{2\gamma+2m-2}{2(1+\gamma}}. \tag{2.13}$$

The two solutions, namely, (1.9)-(1.12) of Steketee (1979) and (2.10)-(2.13) obtained in the presenet work are one and the same if n and m are related through
$n(\gamma+1)=m$, and

$$\begin{aligned} V_0 &= \frac{\gamma+m-1}{2}u_0, \\ u_0 &= \frac{2c}{1+\gamma}\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right]^{\frac{1}{1+\gamma}}, \\ p_0 &= \frac{\gamma-1}{2\gamma+m}u_0, \\ a_0 &= \left[\frac{\gamma(\gamma-1)(\gamma+m-1)}{2(2\gamma+m)}\right]^{\frac{1}{2}} u_0. \end{aligned}$$

2.2 $B(h)$ is exponential.

Writing $T(t)=a_1t^{a_2}$, $H(h)=c_1e^{qh}$ and $B(h)=b_1e^{m_1h}$ into (2.3)-(2.4), we find that

$$T(t) = \left(\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right)^{\frac{1}{1+\gamma}} t^{\frac{2}{1+\gamma}}, \tag{2.14}$$

$$H(h) = c_1e^{\frac{m_1h}{1+\gamma}}, \tag{2.15}$$

provided that

$$b_1=-\lambda c_1^{\gamma}\left(\frac{m_1}{1+\gamma}\right)^{\gamma-1} \tag{2.16}$$

Hence the solution of (1.7) in this case is

$$E(h,t) = c_1\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right]^{\frac{1}{(1+\gamma)}} t^{\frac{2}{(1+\gamma)}}\exp\left(\frac{m_1h}{1+\gamma}\right), \tag{2.17}$$

$$B(h) = b_1e^{m_1h}. \tag{2.18}$$

Here λ, c_1 and m_1 are free constants. From (2.17), (1.3), (1.6) and $a^2=\frac{\gamma p}{\rho}$, we get the volume, velocity, pressure and the sound speed:

$$V(h,t) = \frac{m_1c_1}{1+\gamma}\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right]^{\frac{1}{1+\gamma}} t^{\frac{2}{1+\gamma}}e^{\frac{m_1h}{1+\gamma}}, \tag{2.19}$$

$$u(h,t) = \frac{2c_1}{1+\gamma}\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right]^{\frac{1}{1+\gamma}} t^{\frac{1-\gamma}{1+\gamma}}e^{\frac{m_1h}{1+\gamma}}+U, \tag{2.20}$$

$$p(h,t) = \frac{2c_1(\gamma-1)}{m_1(1+\gamma)}\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right]^{\frac{1}{1+\gamma}} t^{\frac{-2\gamma}{1+\gamma}}e^{\frac{m_1h}{1+\gamma}}, \tag{2.21}$$

$$a(h,t) = \frac{c_1}{1+\gamma}[2\gamma(\gamma-1)]^{1/2}\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right]^{\frac{1}{1+\gamma}} t^{\frac{1-\gamma}{1+\gamma}}e^{\frac{m_1}{1+\gamma}h}. \tag{2.22}$$

The solutions (2.19)-(2.22) when $B(h)$ is given by(2.18) of (1.7) is new.The coefficients attached with V, u, p and a are given by the following relations.

$$\begin{aligned} V_0 &= \frac{m_1 u_0}{2}, \\ u_0 &= \frac{2c_1}{1+\gamma}\left[\frac{\lambda(1+\gamma)^2}{2(1-\gamma)}\right], \\ p_0 &= \frac{\gamma-1}{m_1}u_0, \\ a_0 &= \left[\frac{\gamma(\gamma-1)}{2}\right]^{\frac{1}{2}} u_0. \end{aligned} \qquad (2.23)$$

2.3 $B(h)$ arbitrary.

Case (2.3.1): $\lambda = 0$

When $\lambda = 0$, the solution of equations (2.3) and (2.4)

$$T = a_1 t + b_1 \qquad (2.24)$$

$$H = c_1 + c\int B^{1/\gamma} dh.. \qquad (2.25)$$

Hence the solution of (1.7) is

$$E(h,t) = [a_1 t + b_1]\left[c_1 + c\int B^{1/\gamma} dh\right]. \qquad (2.26)$$

Case (2.3.2): $\lambda = -k^2 < 0$

From (2.3) and (2.4) we have

$$T'' = -k^2 T^{-\gamma} \qquad (2.27)$$

$$[B(h)(H')^{-\gamma}]' = k^2 H. \qquad (2.28)$$

Multiplying equation (2.27) by T' and integrating yields

$$(T')^2 = \left(\frac{-2k^2}{1-\gamma}\right) T^{1-\gamma} + c_*. \qquad (2.29)$$

The cases $c_* = 0$ and $c_* \neq 0$ have to be treated seperately.

Case (2.3.2(a)):

If $c_* = 0$, then the solution of (2.29) is

$$T = \left[\frac{k^2(\gamma+1)^2}{2(\gamma-1)}\right]^{\frac{1}{\gamma+1}} t^{\frac{2}{\gamma+1}}. \qquad (2.30)$$

Case:(2.3.2(b))

If $c_* \neq 0$ and $\lambda = -k^2$, then the integration of (2.29) yields the equation

$$T^{3(\gamma-1)} - \frac{3A}{c_*}T^{2(\gamma-1)} + \frac{4A^3}{{c_*}^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2 = 0, \quad A = \frac{2\lambda}{1-\gamma} \qquad (2.31)$$

The solution of the equation (2.31) when $\lambda \neq 0$ is

$$
\begin{aligned}
T \;=\; & \Bigg[\frac{1}{6}\Bigg[-108\left(\frac{-32k^6}{(1-\gamma)^3c_*^3}-\frac{9c_*}{4}(\gamma-1)^2(t+m)^2-8a^3\right) \\
& +12\left(\frac{-32k^6}{(1-\gamma)^3c_*^3}-\frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)^{\frac{1}{2}}\Bigg(81\Bigg(\frac{-32k^6}{(1-\gamma)^3c_*^3} \\
& -\frac{9c_*}{4}(\gamma-1)^2(t+m)^2\Bigg)+12a^3\Bigg)^{\frac{1}{2}}\Bigg]^{\frac{1}{3}}+\frac{12a^2}{3} \\
& \Bigg[-108\left(\frac{-32k^6}{(1-\gamma)^3c_*^3}-\frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)-8a^3 \\
& +12\left(\frac{-32k^6}{(1-\gamma)^3c_*^3}-\frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)^{\frac{1}{2}} \\
& \left(81\left(\frac{-32k^6}{(1-\gamma)^3c_*^3}-\frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)\right. \\
& \left.+12a^3\right)^{\frac{1}{2}}\Bigg]^{\frac{-1}{3}}-\frac{a}{3}\Bigg]^{\frac{1}{\gamma-1}},
\end{aligned}
\tag{2.32}
$$

where $a = \frac{6\lambda}{(\gamma-1)c_*}$.
Rewriting (2.28), we have

$$
H'' - \frac{B'H'}{\gamma B} + \frac{k^2(H')^{\gamma+1}H}{\gamma B} = 0. \tag{2.33}
$$

To solve equation (2.33) for $k \neq 0$, assume $H(h) = K(s)$ where $s = s(h)$. Therefore equation (2.33) takes the form

$$
K'' + \left(\frac{s_{hh}}{(s_h)^2} - \frac{B'}{\gamma B S_h}\right)K' + \left(-\frac{k^2(s_h)^{\gamma-1}}{\gamma B}\right)K(K')^{\gamma+1} = 0. \tag{2.34}
$$

To solve equation (2.34), we assume

$$
\begin{aligned}
\frac{s_{hh}}{(s_h)^2} - \frac{B'}{\gamma B S_h} &= 0, && (2.35)\\
\text{Case(i)}: \quad \frac{k^2(s_h)^{\gamma-1}}{\gamma B} &= lh^n, && (2.36)\\
\text{Case(ii)}: \quad \frac{k^2(s_h)^{\gamma-1}}{\gamma B} &= l_0 e^{n_1 h}. && (2.37)
\end{aligned}
$$

Integrating equation (2.35) twice leads to

$$
s = \int B^{\frac{1}{\gamma}}\,dh. \tag{2.38}
$$

Case (i):

Now substituting s in equation (2.35) gives

$$B = \left(\frac{l\gamma}{k^2}\right)^{-\gamma} h^{-n\gamma}. \tag{2.39}$$

Equations (2.38) and (2.39) lead to

$$s = \frac{k^2 h^{1-n}}{l\gamma(1-n)}. \tag{2.40}$$

Substituting equation (2.35) and (2.36) together in equation (2.34), it takes the form

$$K'' + lh^n K(K')^{\gamma+1} = 0. \tag{2.41}$$

To solve equation (2.41), assume $K = \sigma s^\theta$. Substituting K in (2.41), we obtain

$$\theta = \frac{n\gamma - \gamma + 1}{(n-1)(\gamma+1)}, \tag{2.42}$$

$$\text{and } \sigma = (-1)^{\frac{n}{(n-1)(1+\gamma)}} \left[\frac{n-2}{(1+\gamma)^{1-\gamma}(n\gamma-\gamma+1)^\gamma}\right]^{\frac{1}{1+\gamma}} \left[\frac{l\gamma^n}{(n-1)^{(\gamma-n\gamma-1)}k^{2n}}\right]^{\frac{1}{(n-1)(1+\gamma)}} \tag{2.43}$$

Therefore

$$K = (-1)^{\frac{n}{(n-1)(1+\gamma)}} \left[\frac{n-2}{(1+\gamma)^{1-\gamma}(n\gamma-\gamma+1)^\gamma}\right]^{\frac{1}{1+\gamma}} \left[\frac{l\gamma^n}{(n-1)^{(\gamma-n\gamma-1)}k^{2n}}\right]^{\frac{1}{(n-1)(1+\gamma)}} s^{\frac{n\gamma-\gamma+1}{(n-1)(\gamma+1)}}. \tag{2.44}$$

Since $H(h) = K(s(h))$, we get

$$H(h) = (-1)^{\frac{n}{(n-1)(1+\gamma)}} \left[\frac{n-2}{(1+\gamma)^{1-\gamma}(n\gamma-\gamma+1)^\gamma}\right]^{\frac{1}{1+\gamma}} \left[\frac{l\gamma^n}{(n-1)^{(\gamma-n\gamma-1)}k^{2n}}\right]^{\frac{1}{(n-1)(1+\gamma)}} \left[\frac{k^2}{l\gamma(1-n)}\right]^{\frac{n\gamma-\gamma+1}{(n-1)(1+\gamma)}} h^{\frac{\gamma-n\gamma-1}{1+\gamma}} \tag{2.45}$$

Hence the solutions of $E(h,t)$ are
(i) $c_* = 0$ and $\lambda = -k^2$.

$$E(h,t) = (-1)^{\frac{n}{(n-1)(1+\gamma)}} (\gamma+1) \left[\frac{k^{2\gamma}(n-2)\gamma^{\frac{n-n\gamma+\gamma-1}{n-1}}}{2(\gamma-1)l^\gamma(n\gamma-\gamma+1)^\gamma}\right]^{\frac{1}{\gamma+1}} h^{\frac{\gamma-n\gamma-1}{1+\gamma}} t^{\frac{2}{\gamma+1}}, \tag{2.46}$$

(ii) $c_* \neq 0$, $\lambda = -k^2$.

$$E(h,t) = (-1)^{\frac{n}{(n-1)(1+\gamma)}} \left[\frac{n-2}{(1+\gamma)^{1-\gamma}(n\gamma-\gamma+1)^\gamma}\right]^{\frac{1}{1+\gamma}} \left[\frac{l\gamma^n}{(n-1)^{(\gamma-n\gamma-1)}k^{2n}}\right]^{\frac{1}{(n-1)(1+\gamma)}}$$

$$\left[\frac{k^2}{l\gamma(1-n)}\right]^{\frac{n\gamma-\gamma+1}{(n-1)(1+\gamma)}} h^{\frac{\gamma-n\gamma-1}{1+\gamma}} \left[\frac{1}{6}\left[-108\left(-\frac{32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2 - 8a^3\right)\right.\right.$$
$$+12\left(\frac{-32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)^{\frac{1}{2}}\left(81\left(\frac{-32k^6}{(1-\gamma)^3c_*^3}\right.\right.$$
$$\left.\left.\left.-\frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right) + 12a^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \frac{12a^2}{3}$$
$$\left[-108\left(\frac{-32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right) - 8a^3\right.$$
$$+12\left(\frac{-32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)^{\frac{1}{2}}$$
$$\left(81\left(-\frac{32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)\right.$$
$$\left.\left.+12a^3\right)^{\frac{1}{2}}\right]^{\frac{-1}{3}} - \left.\frac{a}{3}\right]^{\frac{1}{\gamma-1}}, \tag{2.47}$$

<u>**Case (ii):**</u>

In this case substituting s into equation (2.35) gives

$$B = \left(\frac{l_0\gamma}{k^2}\right)^{-\gamma} e^{-\gamma n_1 h}. \tag{2.48}$$

Hence from equation (2.38)

$$s = \frac{-k^2}{n_1 l_0 \gamma} e^{-n_1 h}. \tag{2.49}$$

Substituting equation(2.35) and(2.37) together in equation (2.34), it takes the form

$$K'' + l_0 e^{n_1 h} K (K')^{\gamma+1} = 0. \tag{2.50}$$

To solve equation (2.50), assume $K = \sigma_1 s^{\theta_1}$. Substituting K in (2.50), we obtain

$$\theta_1 = \frac{\gamma}{\gamma+1}, \tag{2.51}$$

$$\text{and} \quad \sigma_1 = \left[\frac{-n_1(\gamma)^{1-\gamma}}{k^2(\gamma+1)^{1-\gamma}}\right]^{\frac{1}{\gamma+1}}. \tag{2.52}$$

Therefore

$$K = \left[\frac{-n_1(\gamma)^{1-\gamma}}{k^2(\gamma+1)^{1-\gamma}}\right]^{\frac{1}{\gamma+1}} s^{\frac{\gamma}{\gamma+1}}. \tag{2.53}$$

Since $H(h) = K(s(h))$, we get

$$H(h) = \left[\frac{-n_1(\gamma)^{1-\gamma}}{k^2(\gamma+1)^{1-\gamma}}\right]^{\frac{1}{\gamma+1}} \left[\frac{-k^2}{n_1 l_0 \gamma}\right]^{\frac{\gamma}{\gamma+1}} e^{\left(\frac{-n_1\gamma}{\gamma+1}\right)h} \tag{2.54}$$

Hence the solutions of $E(h,t)$ are
i) $c_* = 0$ and $\lambda = -k^2$.

$$E(h,t) = -(\gamma+1)\left[\frac{(n_1)^{1-\gamma}(\gamma)^{1-2\gamma}k^{2\gamma}}{2(\gamma-1){l_0}^{\gamma}}\right]^{\frac{1}{\gamma+1}} t^{\frac{2}{\gamma+1}} e^{\left(\frac{-n_1\gamma}{\gamma+1}\right)h} \tag{2.55}$$

ii) $c_* \neq 0$, $\lambda = -k^2$.

$$\begin{aligned} E(h,t) = & \left[\frac{-n_1(\gamma)^{1-\gamma}}{k^2(\gamma+1)^{1-\gamma}}\right]^{\frac{1}{\gamma+1}}\left[\frac{-k^2}{n_1 l_0 \gamma}\right]^{\frac{\gamma}{\gamma+1}} e^{\left(\frac{-n_1\gamma}{\gamma+1}\right)h} \\ & \left[\frac{1}{6}\left[-108\left(\frac{-32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2 - 8a^3\right)\right.\right. \\ & +12\left(\frac{-32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)^{\frac{1}{2}}\left(81\left(\frac{-32k^6}{(1-\gamma)^3c_*^3}\right.\right. \\ & \left.\left.\left. -\frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right) + 12a^3\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \frac{12a^2}{3} \\ & \left[-108\left(\frac{-32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right) - 8a^3\right. \\ & +12\left(\frac{-32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)^{\frac{1}{2}} \\ & \left(81\left(\frac{-32k^6}{(1-\gamma)^3c_*^3} - \frac{9c_*}{4}(\gamma-1)^2(t+m)^2\right)\right. \\ & \left.\left. +12a^3\right)^{\frac{1}{2}}\right]^{\frac{-1}{3}} - \left.\frac{a}{3}\right]^{\frac{1}{\gamma-1}}. \end{aligned} \tag{2.56}$$

3. Direct Method

We seek the solution of (1.7) in the form

$$E(h,t) = \beta(t)f(z), \quad z = z(h,t). \tag{3.1}$$

We substitute (3.1) in (1.7) and require the resulting equation to be an ordinary differential equation governing the function $f(z)$ to get

$$f'' + \Lambda_1(f')^{\gamma+2} + \Lambda_2(f')^{\gamma+1}f + \Lambda_3 f\prime + \Lambda_4 f'^{(\gamma+1)}f'' = 0. \tag{3.2}$$

The functions $\Lambda_n = \Lambda_n(z)$, $n = 1,2,3,4$ are introduced according to

$$2\beta^{\gamma+1}\beta' z_h^{\gamma+1} z_t + \beta z_{tt} = -\gamma B(h)\beta z_h^2 \Lambda_1(z), \tag{3.3}$$

$$\beta^{\gamma+1}\beta'' z_h^{\gamma+1} = -\gamma B(h)\beta z_h^2 \Lambda_2(z), \tag{3.4}$$

$$B'(h)\beta z_h - \gamma B(h)\beta z_{hh} = -\gamma B(h)\beta z_h^2 \Lambda_3(z), \tag{3.5}$$

$$\beta^{\gamma+2} z_h^{\gamma+1} z_t^2 = -\gamma B(h)\beta z_h^2 \Lambda_4(z). \tag{3.6}$$

Now with the help of only one remark as against the 3 remarks given in the direct method,we solve equations (3.3)- (3.6) for the functions $\beta(t)$, $z(h,t)$ and $\Gamma_n(z) = 1, 2, 3, 4$.
Remark: If $z(h,t)$ is determined from an equation of the form $f(z) = \hat{z}(h,t)$, where $f(z)$ is any invertible function then we may take $f(z) = z$, without loss of generality. Setting

$$\Lambda_3(z) = -\frac{\Gamma_3''(z)}{\Gamma_3'(z)}, \tag{3.7}$$

in (3.5) and integrating with respect to z twice, we get

$$\Gamma_3(z) = \int [B(h)]^{\frac{1}{\gamma}}\, dh \tag{3.8}$$

In view of the above remark, we choose $\Gamma_3(z) = z$ so that $\Lambda_3(z) = 0$ and

$$z = \int [B(h)]^{\frac{1}{\gamma}}\, dh. \tag{3.9}$$

It is clear from equation (3.9) that $z_t = 0$. Since $z_t = 0$ we deduce from equations (3.3)-(3.6)that $\Lambda_1(z) = \Lambda_4(z) = 0$. At this point equation (3.4) takes the form

$$\beta^\gamma \beta'' = -\gamma \left(B^{\frac{-1}{\gamma}}\right) \Lambda_2(z). \tag{3.10}$$

We now consider the following two cases:

Case 3.1: $B(h)$ is algebraic.

Assuming $B(h) = b_2 h^{m_2}$ in (3.9) leads to

$$z = (b_2)^{1/\gamma} \left(\frac{\gamma}{m_2+\gamma}\right) h^{(m_2+\gamma)/\gamma}. \tag{3.11}$$

To solve (3.10) we write

$$\beta^{\gamma+1}\beta'' = \mathrm{A}, \tag{3.12}$$

and

$$\Lambda_2(z) = a_2 z^{a_3}, \tag{3.13}$$

where A, a_2 and a_3 are constants. Substituting (3.11)- (3.13) and replacing $B(h)$ by $B(h) = b_2 h^{m_2}$ in (3.9), we obtain

$$\Lambda_2(z) = \frac{-A\,(b_2)^{\frac{-1}{m_2+\gamma}}}{\gamma} \left(\frac{m_2+\gamma}{\gamma}\right)^{\frac{-m_2}{m_2+\gamma}} z^{\frac{-m_2}{m_2+\gamma}}. \tag{3.14}$$

To solve (3.12) assume

$$\beta = \beta_0 t^{\beta_1}. \tag{3.15}$$

Substituting (3.15) into(3.12) we obtain

$$\beta(t) = \left[\frac{\mathrm{A}(1+\gamma)^2}{2(1-\gamma)}\right]^{1/(1+\gamma)} t^{2/(1+\gamma)}. \tag{3.16}$$