Volume 20

# CRM PROCEEDINGS & LECTURE NOTES

Centre de Recherches Mathématiques
Université de Montréal

# Semi-Analytic Methods for the Navier-Stokes Equations

Katie Coughlin
*Editor*

American Mathematical Society

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# Selected Titles in This Series

20 **Katie Coughlin, Editor,** Semi-analytic methods for the Navier-Stokes equations, 1999
19 **Rajiv Gupta and Kenneth S. Williams, Editors,** Number theory, 1999
18 **Serge Dubuc and Gilles Deslauriers, Editors,** Spline functions and the theory of wavelets, 1999
17 **Olga Kharlampovich, Editor,** Summer school in group theory in Banff, 1996, 1998
16 **Alain Vincent, Editor,** Numerical methods in fluid mechanics, 1998
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# Semi-Analytic Methods for the Navier-Stokes Equations

# List of Participants

**Majid Ahmadi**
Concordia University
Montréal, Qc H3G 1M8 (CANADA)
ahmadi@CERCA.UMontreal.CA

**Paul Arminjon**
Université de Montréal
Montréal, Qc H3C 3J7 (CANADA)
arminjon@DMS.UMontreal.CA

**Nadine Aubry**
Cornell University
Ithaca, NY 14853 (USA)
na23@cornell.edu

**Sivaramakrishnan Balachandar**
Univ. of Illinois at Urbana-Champaign
Urbana, IL 61801 (USA)
bala1s@h.tam.uiuc.edu

**Peter Bartello**
Environnement Canada
Dorval, Qc H9P 1J3 (CANADA)
pbartello@rpn.aes.doe.CA

**Stephen Bergeron**
Université de Montréal
Montréal, Qc H3C 3J7, (CANADA)
bergeron@CERCA.UMontreal.CA

**Anne Bourlioux**
Université de Montréal
Montréal, Qc H3C 3J7 (CANADA)
bourliou@CERCA.UMontreal.CA

**Pierre Carrier**
CERCA, Université de Montréal
Montréal, Qc H3X 2H9 (CANADA)
pierre@CERCA.UMontreal.CA

**Emily Ching**
Department of Physics
The Chinese University of Hong Kong
Shatin, New Territories (HONG KONG)
ching@phy.cuhk.edu.hk

**Katie Coughlin**
CERCA, Université de Montréal
Montréal, Qc H3X 2H9 (CANADA)
coughlin@CERCA.UMontreal.CA

**Henry S. Greenside**
Duke University
Durham, NC 27708 (USA)
hsg@phy.duke.edu

**John G. Heywood**
University of British Columbia
Vancouver, BC V6T 1Y4 (CANADA)
heywood@math.ubc.ca

**Doug Hundley**
Colorado State University
Fort Collins, CO 80523 (USA)
hundley@math.colostate.edu

**Mohamed Jardak**
CERCA, Université de Montréal
Montréal, Qc H3X 2H9 (CANADA)
mohamed@CERCA.UMontreal.CA

**Roger Khayat**
Indust. Materials Inst. NRC
Boucherville, QC J4B 6Y4 (CANADA)
roger.khayat@anrc.ca

**Michael J. Kirby**
Colorado State University
Fort Collins, CO 80523 (USA)
kirby@math.colostate.edu

**Changhoon Lee**
University of Seoul
Dongdaemun-gu
Seoul 130–743 (KOREA)
chlee@seoul.uos.ac.kr

**Alexandre Lepage**
Université de Montréal
Montréal, Qc H3C 3J7 (CANADA)
lepage@CERCA.UMontreal.CA

**Thomas S. Lund**
University of Texas at Arlington
Arlington, TX 76019 (USA)

**Hung Ly**
University of California, Irvine
Irvine, CA 92717 (USA)
hly@math.uci.edu

**Andrei Malevsky**
CERCA, Université de Montréal
Montréal, Qc H3G 1C2 (CANADA)
malevsky@CERCA.UMontreal.CA

**Philip S. Marcus**
University of California, Berkeley
Berkeley, CA 94720 (USA)
pmarcus@me.berkeley.edu

**Martine Marion**
École Centrale de Lyon
69131 Ecully (FRANCE)
marion@cc.ec-lyon.fr

**Sherwin A. Maslowe**
McGill University
Montréal, Qc H3A 2K6 (CANADA)
maslowe@math.McGill.CA

**Maurice Meneguzzi**
Université Paris Sud
91405 Orsay Cedex (FRANCE)
mngz@CERCA.UMontreal.CA

**Charles Meneveau**
The Johns Hopkins University
Baltimore, MD 21212 (USA)
meneveau@polaris.me.jhu.edu

**Georges Michaud**
CERCA, Université de Montréal
Montréal, Qc H3X 2H9 (CANADA)
michaudg@CERCA.UMontreal.CA

**R. Miranda**
Colorado State University
Fort Collins, CO 80523 (USA)
miranda@math.colostate.edu

**Julie Plante**
Université de Montréal
Montréal, Qc H3C 3J7 (CANADA)
Julie@CERCA.UMontreal.CA

**Joel D. Shore**
McGill University
Montréal, Qc H3A 2T8 (CANADA)
shore@physics.mcgill.ca

**David Straub**
McGill University
Montréal, Qc H3A 2K6 (CANADA)
straub@CERCA.UMontreal.CA

**Edriss Titi**
University of California, Irvine
Irvine, CA 92717 (USA)
etiti@math.uci.edu

**Slavko Vasic**
CERCA, Université de Montréal
Montréal, Qc H3X 2H9 (CANADA)
slavko@CERCA.UMontreal.CA

**Alain Vincent**
CERCA, Université de Montréal
Montréal, Qc H3X 2H9 (CANADA)
vincent@CERCA.UMontreal.CA

**Tom Warn**
McGill University
Montréal, Qc H3A 2K6 (CANADA)
my26@musica.mcgill.ca

**Jeffrey B. Weiss**
University of Colorado
Boulder, CO 80309 (USA)
jweiss@colorado.edu

**Shannon Wynne**
University of California, Irvine
Irvine, CA 92717 (USA)
swynne@math.uci.edu

# List of Speakers

**N. Aubry**
City University of New York

**S. Balachandar**
University of Illinois

**E. Ching**
The Chinese University of Hong Kong

**H. S. Greenside**
Duke University

**J. G. Heywood**
University of British Columbia

**M. J. Kirby**
Colorado State University

**P. S. Marcus**
University of California, Berkeley

**M. Marion**
École Centrale de Lyon

**C. Meneveau**
The Johns Hopkins University

**J. D. Shore**
McGill University

**E. Titi**
University of California, Irvine

**T. Warn**
McGill University

**J. B. Weiss**
University of Colorado

# Preface

The lectures collected in this book were given during a week-long workshop entitled "Semi-analytic methods for the Navier-Stokes equations", hosted by the Centre de Recherche Mathématique in Montréal, as part of their 1995 thematic year on numerical analysis.

The title of the workshop was chosen to reflect a current reality in fluid dynamics: while a single set of equations (the Navier-Stokes equations or NSE) describe fluid behavior in a wide range of physical situations, the solutions of these equations are sufficiently varied and complex that another level of analysis is clearly needed. The fundamental problem is not just to solve the NSE, but also to understand what the solutions mean. This requires creating models of the observed phenomena which are simpler than the NSE, which make use of all the information available from numerical and laboratory experiments, and which will lead to a better understanding of the essential physical processes involved.

This kind of work is being done in a variety of disciplines, and one of the goals of the workshop was to bring together people working in different fields, but sharing a common perspective on the nature of the problem to be solved. The participants included mathematicians, physicists, and engineers, all using a 'semi-analytic' approach to the study of the NSE, based on judicious use of numerical simulation in creating and testing new theoretical ideas. The actual technical methods used are quite diverse.

One of the fundamental, still problematic phenomena encountered in fluid flow is the existence of turbulence, or the coexistence of turbulence with ordered flow features (coherent structures). Strategies for treating turbulence theoretically and numerically are discussed in the papers of Ching, Meneveau & Lund, and Weiss. Ching focuses on the construction of model equations describing some statistical properties of turbulent flows. The goal is to bypass the NSE and come up with equations describing directly the evolution of probability density functions. Meneveau & Lund address the problem of modeling small scales in large eddy simulation of turbulent flow. In a widely-used class of models, small scale fluctuations are modeled by an eddy-viscosity—a term formally resembling the viscous term in the NSE with a variable coefficient. This coefficient needs to be calculated dynamically, but, as the notion of eddy-viscosity is only valid in an average sense, it cannot be computed instantaneously. They propose a Lagrangian time-history averaging, where the relevant averages are taken over the path of a given fluid parcel.

Weiss gives an overview of a particular approach to modeling two-dimensional turbulence, based on extensive numerical observations that the flow is dominated by a system of long-lived, coherent vortices. The conservative evolution of the vortex system is 'punctuated' by occasional dissipative events such as mergers.

This gives a different definition of a weak perturbation to a Hamiltonian system—here there is a strong perturbation, but only for short and relatively rare times. The paper of Marcus & Lee also arises from the study two-dimensional turbulent flows, complicated by the presence of a spatially varying Coriolis force. Here the phenomenon of interest is the formation of east-west jets (as observed for example on Jupiter) and their subsequent dynamics; a great deal can be learned from relatively simple physical models of the jets.

Another phenomenon observed in fluids and other spatially extended systems is spatiotemporal chaos, characterized by the existence of slowly varying but disordered patterns. Greenside presents a lucid review of the problem and the theoretical and numerical questions it poses. One of the fundamental issues is whether, and how, the existence of spatial order and slow dynamics can be used to simplify the equations, in the sense of reducing the number of degrees of freedom that need to be explicitly calculated. This requires an understanding of how certain physical space characteristics are expressed in the phase space structure of the attractor for the system. This question is also addressed in the articles of Kirby *et al.*, who are interested in the problem of finding the coordinates in phase space that allow a given solution to be represented with maximal efficiency. They describe an approach based on the Karhunen-Loeve decomposition, which is implemented using neural networks.

While the technical details involved in numerical and theoretical work on the NSE can quickly become overwhelming, one of the pleasures of this workshop was in seeing that there is a common ground, and that the conceptual issues involved in different problems can be communicated, without too much trouble, across disciplinary boundaries. I would like to thank all of the participants, and in particular the authors of this volume, for their contributions to a very stimulating and enjoyable conference.

Katie Coughlin
Montréal, January 1999

# Contents

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# Probabilities and Conditional Averages in Turbulence

Emily S. C. Ching

ABSTRACT. In the study of turbulent fluid flows, one of the key issues is to understand the statistics of the fluctuations of the physical quantities of interest. Specifically, one would like to obtain results for the probability density function (PDF) which describes the statistics. One might attempt to derive the PDF's directly from the equations of motion. However, this is known to be a very difficult problem and very few explicit results have been obtained so far. In this paper, we describe a different approach to the problem. We show that the probability density function can be expressed implicitly as an exact formula in terms of certain conditional averages. Results for the PDF's can then be obtained via the study and understanding of these conditional averages.

## 1. Introduction

When fluid flows become turbulent, the physical quantities of interest, such as the velocity and the temperature fields, exhibit highly irregular fluctuations both in time and space. It is thus natural to use a statistical approach to study turbulence. The statistics of any fluctuating quantity are described by its probability density function (PDF). As a result, a key problem in turbulence research is to derive results for the PDF's of the physical quantities of interest. For example, if the PDF's of the velocity differences across different separating distances are known, the scaling of the velocity structure functions, and especially the question of whether there is any correction to that predicted by Kolmogorov some fifty years ago [**11**], would be solved.

When studying the PDF's of fluctuating turbulent quantities, another interesting aspect is what their shapes are. The statistics of velocity and temperature derivatives, which are believed to be small-scale characteristics, have been known to deviate significantly from Gaussian and this is related to the problem of intermittency[1], which is a fundamental problem in turbulence. More recent interest

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This is the final version of the paper.

[1]See, for example, [**14**] and references therein.

stems from the discovery that the PDF of temperature fluctuations in turbulent Rayleigh-Bénard convection changes from Gaussian in the lower Rayleigh-number regime (known as soft turbulence) to exponential-like in the higher Rayleigh-number regime (known as hard turbulence) [**2, 22**]. This discovery has prompted various studies which try to understand the non-Gaussianity of scalar fluctuations in turbulent flows [**1, 3, 4, 8–10, 17–21**].

One might attempt to calculate the PDF's of the turbulent quantities of interest directly from the equations of motion, i.e., for example, to calculate the PDF of velocity fluctuations from the Navier-Stokes equation. However, this task is well-known to be highly nontrivial. As of today, it has not yet been accomplished even for the relatively simpler problem of temperature being a passive scalar (not driving the motion) advected by a random velocity field. Because of the difficulty in obtaining explicit results, it would be useful to derive implicit results that relate the PDF's to other physical quantities especially when the latter could be studied and understood more easily.

In this paper, we review and discuss the work on obtaining implicit formulas for the PDF in terms of certain conditional averages. In Section 2, the result for decaying homogeneous temperature fluctuations, derived by Sinai and Yakhot [**19**], is reviewed. Building upon Sinai and Yakhot's work, results for stationary turbulent fluctuations have been obtained [**4**] and are presented in Section 3. Exact results for fluctuations in general stationary or statistically homogeneous processes have been derived [**5, 16**] and are reviewed in Section 4. In Section 5, we discuss some properties of the conditional averages. The paper ends with a summary in Section 6.

## 2. Decaying Homogeneous Temperature Fluctuations

An implicit formula for the PDF can be derived directly from the equation of motion for the case of decaying turbulent temperature fluctuations in homogeneous and incompressible fluid flows. Such a derivation was done by Sinai and Yakhot [**19**] and is outlined below. The governing equation is

$$\partial_t T + \vec{u} \cdot \vec{\nabla} T = \kappa \nabla^2 T, \tag{2.1}$$

where $T(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{x}, t)$ represent respectively the temperature and the velocity field. Since the flow is incompressible, we have

$$\vec{\nabla} \cdot \vec{u} = 0. \tag{2.2}$$

For temperature field that is passive, the velocity field is prescribed by the Navier-Stokes equation. On the other hand, when the temperature field drives the motion (i.e., being active), for example in Rayleigh Bénard convection, it is coupled with the velocity field via

$$\partial_t \vec{u} + \vec{u} \cdot \vec{\nabla} \vec{u} = -\vec{\nabla} p + \nu \nabla^2 \vec{u} + g\alpha T \hat{z}, \tag{2.3}$$

where $p$ is the pressure divided by density, $\hat{z}$ is the unit vector in the vertical direction, $g$ is the acceleration due to gravity; and $\alpha$, $\nu$ and $\kappa$ are properties of the fluid, respectively its volume expansion coefficient, kinematic viscosity and thermal diffusivity. In (2.3), the Boussinesq approximation is used which assumes that any variation in density with temperature is retained only as a buoyancy term.

Multiplying (2.1) by $2nT^{2n-1}$ and taking ensemble average (which is equivalent to spatial average for homogeneous flows) over the whole system, we have

$$\partial_t\langle T^{2n}\rangle = \kappa\big\langle\nabla^2(T^{2n})\big\rangle - 2n(2n-1)\kappa\big\langle T^{2n-2}|\nabla T|^2\big\rangle. \tag{2.4}$$

The term with the velocity field vanishes because of the no slip velocity boundary condition. Since the temperature field is homogeneous, there cannot be any temperature difference maintained across the system, i.e., there is no forcing on the boundary and $\big\langle\nabla^2(T^{2n})\big\rangle = 0$. As a result, the temperature fluctuations decay in time:

$$\partial_t\langle T^{2n}\rangle = -2n(2n-1)\kappa\big\langle T^{2n-2}|\nabla T|^2\big\rangle. \tag{2.5}$$

Based on information from numerical experiments, Sinai and Yakhot [**19**] assumed that the normalized field, $X \equiv T/\sqrt{\langle T^2\rangle}$, reaches a stationary limit which implies that

$$\partial_t\langle X^{2n}\rangle = 0. \tag{2.6}$$

With (2.6), Sinai and Yakhot [**19**] obtained

$$\langle X^{2n}\rangle\big\langle|\nabla X|^2\big\rangle = (2n-1)\big\langle X^{2n-2}|\nabla X|^2\big\rangle. \tag{2.7}$$

Using this statistical relation, they derived an implicit formula for the PDF of the normalized temperature fluctuation, namely,

$$P(X=x) = \frac{C_N}{\langle|\nabla X|^2 \mid X=x\rangle}\exp\left[\int_0^x \frac{-\langle|\nabla X|^2\rangle x'}{\langle|\nabla X|^2 \mid X=x'\rangle}\,dx'\right], \tag{2.8}$$

where $C_N$ is a constant fixed by the normalization: $\int P(x)\,dx = 1$.

Equation (2.8) is an implicit result relating the PDF of the normalized fluctuation $X$ to the conditional average $\big\langle|\nabla X|^2 \mid X=x\big\rangle$. The conditional average is an average of $|\nabla X|^2$ taken only when $X$ is at a given value $x$ and is, in general, a function of $x$.

## 3. Stationary Temperature Fluctuations

As can be seen clearly, the derivation of (2.8) by Sinai and Yakhot is not directly applicable to the case of Rayleigh Bénard convection when a temperature difference is maintained, i.e., the system is forced. Less obviously, their analyses are also inapplicable for stationary passive temperature fluctuations where external heat is supplied to maintain the fluctuations for measurements.

The present author has assumed [**4**] that a statistical relation similar to (2.7) holds for general stationary turbulent temperature fluctuations, namely

$$\langle X^{2n}\rangle\langle\dot X^2\rangle = (2n-1)\langle X^{2n-2}\dot X^2\rangle, \tag{3.1}$$

where $X = \big(T-\langle T\rangle\big)/\sqrt{\langle(T-\langle T\rangle)^2\rangle}$ is the normalized temperature fluctuation and an overdot indicates derivative with respect to time. Then following Sinai and Yakhot, an expression has been obtained [**4**] for the PDF of $X$:

$$P(X=x) = \frac{C_N}{\langle\dot X^2 \mid X=x\rangle}\exp\left[\int_0^x \frac{-\langle\dot X^2\rangle x'}{\langle\dot X^2 \mid X=x'\rangle}\,dx'\right]. \tag{3.2}$$

This result has been tested [**4**] against convective turbulence data [**2**] and some passive temperature data measured in the wake of a heated cylinder[2]. Good agreement has been found.

This method has been further extended [**4**] to study the statistics of temperature differences, $T_\tau(x,t) \equiv T(x,t+\tau) - T(x,t)$, between two different times separated by a time interval $\tau$. Similar results to (3.2), with $T$ replaced by $T_\tau$, were obtained for their PDF's, which were again checked to hold very well for both the convective and passive temperature data, except for very short time separations.

The good agreement between the experimental data and the implicit formula (3.2) establishes that the statistical relation (3.1) holds reasonably well for stationary turbulent temperature fluctuations including both passive and active data. This unexpected result will be further explored in Section 5.

Moreover, (3.2) implies that the shape of the PDF, especially its tails, is governed by the functional dependence of $\langle \dot{X}^2 \mid X = x\rangle$ on $x$. Specifically, the PDF is Gaussian if $\langle \dot{X}^2 \mid X = x\rangle$ is independent of $x$. On the other hand, if $\langle \dot{X}^2 \mid X = x\rangle \sim |x|$ then the PDF is nearly exponential. Such a difference in behavior of the conditional average $\langle \dot{X}^2 \mid X = x\rangle$ has indeed been found [**4, 5**] which accounts for the observed change of the PDF from Gaussian to exponential-like when going from soft turbulence to hard turbulence [**2**].

## 4. General Stationary or Statistically Homogeneous Fluctuations

In order to understand the results (2.8) and (3.2) better, we have turned to study the statistics of general fluctuations [**5, 16**]. General exact results for the PDF have been obtained and their derivations are shown below.

First consider a stationary process. Let $X(t)$ be a physical variable measured as a function of time $t$ at a certain fixed spatial location. For example, $X(t)$ can be the temperature or a component of the velocity measured in a stationary turbulent flow. Then the PDF of $X$, $P(X = x)$, is given formally by[3].

$$P(x) = \big\langle p(x,t)\big\rangle \text{ where } p(x,t) \equiv \delta\big(X(t) - x\big), \tag{4.1}$$

and ensemble average is equivalent to time average in stationary processes.

Differentiating $p(x,t)$ with respect to time, multiplying a general function $f(t)$ to the resulting equation, and taking ensemble average gives

$$\langle \dot{f} \mid x\rangle P(x) = \frac{d}{dx}\big[\langle f\dot{X} \mid x\rangle P(x)\big]. \tag{4.2}$$

We have used the fact that $\big\langle \partial(fp)/\partial t\big\rangle$ vanishes for stationary processes and that $\big\langle p(x,t)F(t)\big\rangle = \big\langle F(t) \mid x\big\rangle P(x)$ for any function $F(t)$. Equation (4.2), which can also be derived following the method used in Ref. 19, is valid for any differentiable function $f(t)$. Solving (4.2) gives a formula for $P(x)$:

$$P(x) = \frac{C_N}{|\langle f\dot{X} \mid x\rangle|} \exp\bigg(\int_0^x \frac{\langle \dot{f} \mid x'\rangle}{\langle f\dot{X} \mid x'\rangle}\, dx'\bigg). \tag{4.3}$$

Equation (4.3) expresses the PDF of $X$ in terms of two conditional averages, $\langle f\dot{X} \mid x\rangle$ and $\langle \dot{f} \mid x\rangle$, where $f(t)$ is any general differentiable function such that $\langle f\dot{X} \mid x\rangle \neq 0$ for all values of $x$. Since the same $P(x)$ can be expressed in terms of conditional

[2]The temperature data were taken at a fixed point downstream of the cylinder on the wake centerline at a Reynolds number $= 5.2 \times 10^3$.

[3]See, for example, [**15**]

averages involving different $f$'s, the conditional averages for different choices of $f$ are related to each other [**5**].

An analogous formula for the PDF in statistically homogeneous processes can be similarly derived [**5**]. Suppose $X(\mathbf{r})$ is now a physical variable measured as a function of position $\mathbf{r}$ at a certain time in a statistically homogeneous fluid flow. The PDF of $X$ is found to be

$$P(X = x) = \frac{C_N}{|\langle \mathbf{g} \cdot \boldsymbol{\nabla} X \mid x \rangle|} \exp\biggl( \int_0^x \frac{\langle \boldsymbol{\nabla} \cdot \mathbf{g} \mid x' \rangle}{\langle \mathbf{g} \cdot \boldsymbol{\nabla} X \mid x' \rangle} \, dx' \biggr), \tag{4.4}$$

for any differentiable function $\mathbf{g}(\mathbf{r})$.

We emphasize that the two results (4.3) and (4.4) are exact and that their derivations assume only stationarity or statistical homogeneity and differentiability with respect to time or space. It is interesting to consider some special cases. Taking $f = \dot{X}$ and $\mathbf{g} = \boldsymbol{\nabla} Y$ respectively gives

$$P(x) = \frac{C_N}{\langle \dot{X}^2 \mid x \rangle} \exp\biggl( \int_0^x \frac{\langle \ddot{X} \mid x' \rangle}{\langle \dot{X}^2 \mid x' \rangle} \, dx' \biggr), \tag{4.5}$$

which was first derived in Ref. 16 and

$$P(x) = \frac{C_N}{\langle |\boldsymbol{\nabla} X|^2 \mid x \rangle} \exp\biggl( \int_0^x \frac{\langle \nabla^2 X \mid x' \rangle}{\langle |\boldsymbol{\nabla} X|^2 \mid x' \rangle} \, dx' \biggr), \tag{4.6}$$

which is the exact analog [**5, 21**] to (4.5) and has been obtained independently by several others [**12**].

## 5. Conditional Averages

In Sections 2 to 4, we have shown various results for the PDF of fluctuations in different circumstances. All of these implicit formulas relate the PDF of the quantity of interest to some conditional averages of either its time or spatial derivatives. Hence, understanding the PDF amounts to understanding these conditional averages.

It turns out that these conditional averages could be studied or understood more easily. For instance, for the decaying homogeneous temperature fluctuations, the conditional average $\langle \nabla^2 X \mid X = x \rangle$ is a simple function of $x$:

$$\langle \nabla^2 X \mid X = x \rangle = -\bigl\langle |\nabla X|^2 \bigr\rangle x. \tag{5.1}$$

This can be seen by comparing (4.6) with (2.8). We note again that in the derivation of (2.8), stationarity of the normalized temperature fluctuation is assumed. Thus, it would be worthwhile to directly test (5.1) using data from numerical simulations. Moreover, under other circumstances, the conditional average $\bigl\langle |\boldsymbol{\nabla} X|^2 \mid X = x \bigr\rangle$ can be evaluated directly from the equations of motion [**6**].

On the other hand, comparing the exact result (4.5) for a general stationary fluctuation with the result (3.2) obtained for stationary turbulent temperature fluctuations and temperature differences, we find the following interesting feature:

$$r(x) \equiv \frac{\langle \ddot{X} \mid X = x \rangle}{\langle \dot{X}^2 \rangle} = -x, \tag{5.2}$$

for the stationary turbulent temperature data. Equation (5.2) has been verified directly using both the convective and the wake temperature data. This linearity of $r(x)$ has also been found to hold approximately for spanwise vorticity data taken in several different turbulent shear flows [**13**]. The existence of such simple and general statistical feature in turbulence, a complicated phenomenon, is quite surprising[4]. It is exciting at the same time since a general feature is likely to be understood more easily than a complicated one.

We now show that (5.2) follows directly from (3.1). Because of stationarity, the ensemble average of any total time derivative vanishes. As a result, (3.1) can be written as

$$\langle X^{2n}\rangle\langle \dot{X}^2\rangle = -\langle X^{2n-1}\ddot{X}\rangle. \tag{5.3}$$

Expressing (5.3) in terms of integrals involving $P(x)$, we have

$$\langle \dot{X}^2\rangle \int x^{2n} P(X=x)\,dx = -\int \langle \ddot{X} \mid X=x\rangle x^{2n-1} P(X=x)\,dx. \tag{5.4}$$

The fact that (5.4) holds for all $n$ implies

$$\langle \dot{X}^2\rangle x = -\langle \ddot{X} \mid X=x\rangle, \tag{5.5}$$

which is exactly (5.2). Hence, to understand (5.2), we have to understand (5.3). We are in the process of understanding (5.3) for temperature differences between two different spatial locations from fundamental physical principles [**7**].

## 6. Summary

Implicit formulas in terms of conditional averages have been obtained for the PDF of fluctuating physical variables in both stationary and statistically homogeneous turbulent fluid flows. These conditional averages could be found exactly from the equations of motion in certain circumstances. Moreover, one of the conditional averages, $\langle \ddot{X} \mid X=x\rangle$, is found to have simple features that hold generally for various fluctuations $X$ in different turbulent flows. Hence, it is expected that physical insights and understanding of the statistics of the fluctuations would be gained through the study of the conditional averages. Work along these lines is in progress and will be reported elsewhere.

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[4]It is noted in Ref. 13 that linearity is one possible form of $r(x)$ such that the statistical constraints $\langle r(x)\rangle = 0$ and $\langle xr(x)\rangle = -1$ are satisfied, regardless of what the PDF of $X$ is. Nevertheless, a full explanation remains to be worked out.