P.C. Matthews

Vector Calculus

向量微积分

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Preface

Vector calculus is the fundamental language of mathematical physics. It provides a way to describe physical quantities in three-dimensional space and the way in which these quantities vary. Many topics in the physical sciences can be analysed mathematically using the techniques of vector calculus. These topics include fluid dynamics, solid mechanics and electromagnetism, all of which involve a description of vector and scalar quantities in three dimensions.

This book assumes no previous knowledge of vectors. However, it is assumed that the reader has a knowledge of basic calculus, including differentiation, integration and partial differentiation. Some knowledge of linear algebra is also required, particularly the concepts of matrices and determinants.

The book is designed to be self-contained, so that it is suitable for a programme of individual study. Each of the eight chapters introduces a new topic, and to facilitate understanding of the material, frequent reference is made to physical applications. The physical nature of the subject is clarified with over sixty diagrams, which provide an important aid to the comprehension of the new concepts. Following the introduction of each new topic, worked examples are provided. It is essential that these are studied carefully, so that a full understanding is developed before moving ahead. Like much of mathematics, each section of the book is built on the foundations laid in the earlier sections and chapters. In addition to the worked examples, a section of exercises is included at the middle and at the end of each chapter. Solutions to all the exercises are given at the back of the book, but the student is encouraged to attempt all of the exercises before looking up the answers! At the end of each chapter, a one-page summary is given, listing the most essential points of the chapter.

The first chapter covers the basic concepts of vectors and scalars, the ways in which vectors can be multiplied together and some of the applications of vectors to physics and geometry.

Chapter 2 defines the ways in which vector and scalar quantities can be integrated, covering line integrals, surface integrals and volume integrals. Again, these are illustrated with physical applications.

Techniques for differentiating vectors and scalars are given in Chapter 3, which forms the essential core of the subject of vector calculus. The key concepts of gradient, divergence and curl are defined, which provide the basis for the following chapters.

Chapter 4 introduces a new and powerful notation, suffix notation, for manipulating complicated vector expressions. Quantities that run to several lines using conventional vector notation can be written extremely compactly using suffix notation. One of the main reasons for writing this book is that there are very few other books that make full use of suffix notation, although it is commonly used in undergraduate mathematics courses.

Two important theorems, the divergence theorem and Stokes's theorem, are covered in Chapter 5. These help to tie the subject together, by providing links between the different forms of integrals from Chapter 2 and the derivatives of vectors from Chapter 3.

Chapter 6 covers the general theory of orthogonal curvilinear coordinate systems and describes the two most important examples, cylindrical polar coordinates and spherical polar coordinates.

Chapter 7 introduces a more rigorous, mathematical definition of vectors and scalars, which is based on the way in which they transform when the coordinate system is rotated. This definition is extended to a more general class of objects known as tensors. Some physical examples of tensors are given to aid the understanding of what can be a difficult concept to grasp.

The final chapter gives a brief overview of some of the applications of the subject, including the flow of heat within a body, the mechanics of solids and fluids and electromagnetism.

Table of Contents

1
Vector Algebra

1.1 Vectors and scalars

This book is concerned with the mathematical description of physical quantities. These physical quantities include vectors and scalars, which are defined below.

1.1.1 Definition of a vector and a scalar

A *vector* is a physical quantity which has both magnitude and direction. There are many examples of such quantities, including velocity, force and electric field. A *scalar* is a physical quantity which has magnitude only. Examples of scalars include mass, temperature and pressure.

In this book, vectors will be written in bold italic type (for example, $\boldsymbol{u}$ is a vector) while scalar quantities will be written in plain italic type (for example, a is a scalar). There are two other commonly used ways of denoting vectors which are more convenient when writing by hand: an arrow over the symbol ($\vec{u}$) or a line under the symbol ($\underline{u}$).

Vectors can be represented diagrammatically by a line with an arrow at the end, as shown in Figure 1.1. The length of the line shows the magnitude of the vector and the arrow indicates its direction. If the vector has magnitude one, it is said to be a *unit vector*. Two vectors are said to be equal if they have the same magnitude and the same direction.

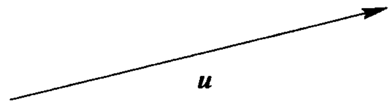

Fig. 1.1. Representation of a vector.

Example 1.1

Classify the following quantities according to whether they are vectors or scalars: energy, electric charge, electric current.

Energy and electric charge are scalars since there is no direction associated with them. Electric current is a vector because it flows in a particular direction.

1.1.2 Addition of vectors

Two vector quantities can be added together by the 'triangle rule' as shown in Figure 1.2. The vector $\boldsymbol{a} + \boldsymbol{b}$ is obtained by drawing the vector $\boldsymbol{a}$ and then drawing the vector $\boldsymbol{b}$ starting from the arrow at the end of $\boldsymbol{a}$.

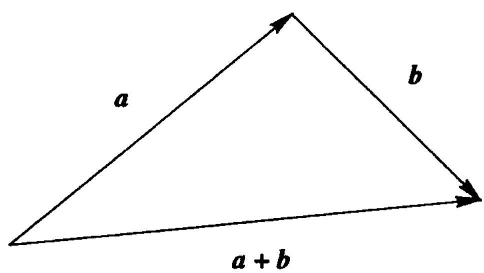

Fig. 1.2. Addition of vectors.

The vector $-\boldsymbol{a}$ is defined as the vector with magnitude equal to that of $\boldsymbol{a}$ but pointing in the opposite direction.

By adding $\boldsymbol{a}$ and $-\boldsymbol{a}$ we obtain the *zero vector*, $\mathbf{0}$. This has magnitude zero and so does not have a direction; nevertheless it is sensible to regard $\mathbf{0}$ as a vector.

1.1.3 Components of a vector

Vectors are often written using a Cartesian coordinate system with axes x, y, z. Such a system is usually assumed to be *right-handed*, which means that a screw rotated from the x-axis to the y-axis would move in the direction of the z-axis. Alternatively, if the thumb of the right hand points in the x direction and the first finger in the y direction, then the second finger points in the z direction.

Suppose that a vector $\boldsymbol{a}$ is drawn in a Cartesian coordinate system and extends from the point $(x_1,\ y_1,\ z_1)$ to the point $(x_2,\ y_2,\ z_2)$, as shown in Figure 1.3. Then the *components* of the vector are defined to be the three numbers $a_1 = x_2 - x_1$, $a_2 = y_2 - y_1$ and $a_3 = z_2 - z_1$. The vector can then be written in the form $\boldsymbol{a} = (a_1,\ a_2,\ a_3)$.

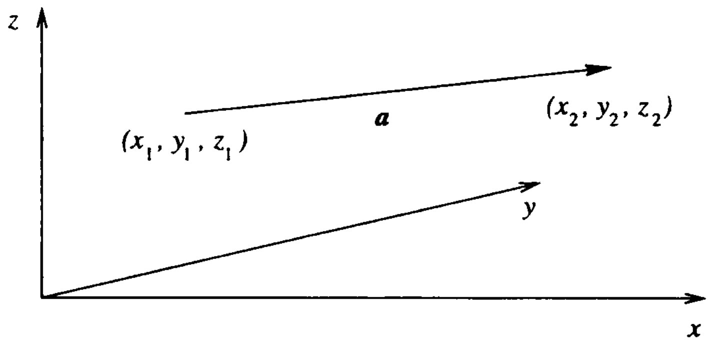

Fig. 1.3. The components of the vector $\boldsymbol{a}$ are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

By introducing three unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, which point along the coordinate axes x, y and z respectively, the vector can also be written in the form $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$. Using this form, the sum of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is $\boldsymbol{a}+\boldsymbol{b} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 + b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3 = (a_1+b_1)\boldsymbol{e}_1 + (a_2 + b_2)\boldsymbol{e}_2 + (a_3+b_3)\boldsymbol{e}_3$. It follows that vectors can be added simply by adding their components, so that the vector equation $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ is equivalent to the three equations $c_1 = a_1 + b_1$, $c_2 = a_2 + b_2$, $c_3 = a_3 + b_3$.

The magnitude of the vector is written $|\boldsymbol{a}|$. It can be deduced from Pythagoras's theorem that the magnitude of the vector can be written in terms of its components as $|\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

The position of a point in space (x, y, z) defines a vector which points from the origin of the coordinate system to the point (x, y, z). This vector is called the ***position vector*** of the point, and is usually denoted by the symbol $\boldsymbol{r}$, with components given by $\boldsymbol{r} = (x, y, z)$.

Example 1.2

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are defined by $\boldsymbol{a} = (1, 1, 1)$, $\boldsymbol{b} = (1, 2, 2)$. Find the magnitudes of $\boldsymbol{a}$ and $\boldsymbol{b}$, and find the vectors $\boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{a} - \boldsymbol{b}$.

The magnitude of the vector $\boldsymbol{a}$ is $|\boldsymbol{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. The magnitude of $\boldsymbol{b}$ is $|\boldsymbol{b}| = \sqrt{1^2 + 2^2 + 2^2} = 3$. The vector $\boldsymbol{a} + \boldsymbol{b}$ is $(1, 1, 1) + (1, 2, 2) = (2, 3, 3)$ and $\boldsymbol{a} - \boldsymbol{b} = (0, -1, -1)$.

1.2 Dot product

The ***dot product*** or ***scalar product*** of two vectors is a scalar quantity. It is written $\boldsymbol{a} \cdot \boldsymbol{b}$ and is defined as the product of the magnitudes of the two vectors and the cosine of the angle between them:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta. \tag{1.1}$$

A number of properties of the dot product follow from this definition:

- The dot product is commutative, i.e. $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$.
- If the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular (orthogonal) then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- Conversely, if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ then either the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular or one of the vectors is the zero vector.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Since the quantity $|\boldsymbol{b}| \cos\theta$ represents the component of the vector $\boldsymbol{b}$ in the direction of the vector $\boldsymbol{a}$, the scalar $\boldsymbol{a} \cdot \boldsymbol{b}$ can be thought of as the magnitude of $\boldsymbol{a}$ multiplied by the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ (see Figure 1.4).
- The dot product is distributive over addition, i.e. $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$. This follows geometrically from the fact that the component of $\boldsymbol{b} + \boldsymbol{c}$ in the direction of $\boldsymbol{a}$ is the same as the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ plus the component of $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$ (see Figure 1.5).

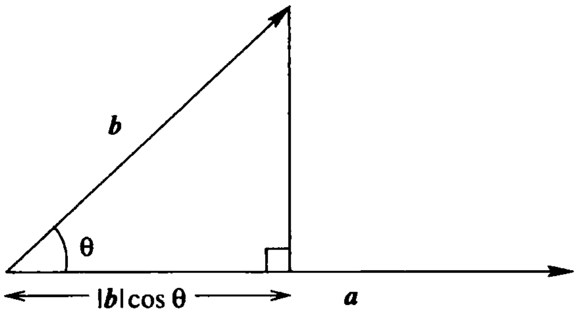

Fig. 1.4. The component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is $|\boldsymbol{b}|\cos\theta$.

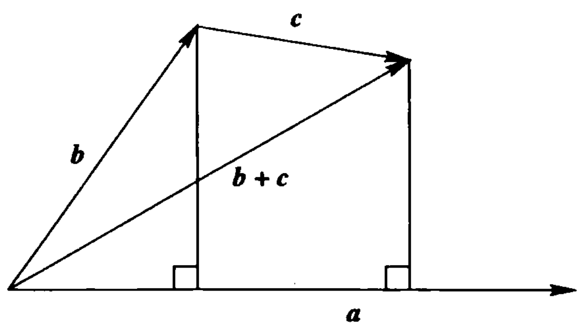

Fig. 1.5. Geometrical demonstration that the dot product is distributive over addition.

A formula for the dot product $\boldsymbol{a} \cdot \boldsymbol{b}$ in terms of the components of the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be derived from the above properties. Considering first the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$, it follows from the fact that these vectors have magnitude 1 and are orthogonal to each other that

$$\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 = 1,\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_2 = 1,\ \boldsymbol{e}_3 \cdot \boldsymbol{e}_3 = 1,\ \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0,\ \boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = 0,\ \boldsymbol{e}_3 \cdot \boldsymbol{e}_1 = 0.$$

The dot product of $\boldsymbol{a}$ and $\boldsymbol{b}$ is therefore

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= (a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3) \cdot (b_1\boldsymbol{e}_1 + b_2\boldsymbol{e}_2 + b_3\boldsymbol{e}_3) \\ &= a_1b_1\boldsymbol{e}_1 \cdot \boldsymbol{e}_1 + a_2b_2\boldsymbol{e}_2 \cdot \boldsymbol{e}_2 + a_3b_3\boldsymbol{e}_3 \cdot \boldsymbol{e}_3 \\ &= a_1b_1 + a_2b_2 + a_3b_3. \end{aligned} \tag{1.2}$$

Example 1.3

Find the dot product of the vectors $(1, 1, 2)$ and $(2, 3, 2)$.

$(1, 1, 2) \cdot (2, 3, 2) = 1 \times 2 + 1 \times 3 + 2 \times 2 = 9.$

Example 1.4

For what value of c are the vectors $(c, 1, 1)$ and $(-1, 2, 0)$ perpendicular?

They are perpendicular when their dot product is zero. The dot product is $-c + 2 + 0$ so the vectors are perpendicular if $c = 2$.

Example 1.5

Show that a triangle inscribed in a circle is right-angled if one of the sides of the triangle is a diameter of the circle.

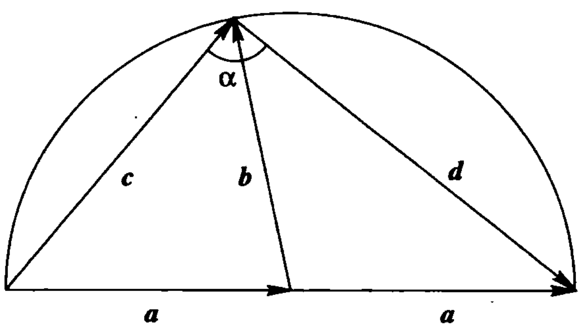

Fig. 1.6. Geometrical construction to show that α is a right angle.

Introduce two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as shown in Figure 1.6. Since these two vectors are both along radii of the circle they are of equal magnitude. The two sides $\boldsymbol{c}$ and $\boldsymbol{d}$ of the triangle are then given by $\boldsymbol{c} = \boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{d} = \boldsymbol{a}-\boldsymbol{b}$. The dot product of these two vectors is $\boldsymbol{c}\cdot\boldsymbol{d} = (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = |\boldsymbol{a}|^2 - \boldsymbol{a}\cdot\boldsymbol{b}+\boldsymbol{b}\cdot\boldsymbol{a}-|\boldsymbol{b}|^2 = 0$. Since the dot product is zero the vectors are perpendicular, so the angle α is a right angle. This is just one of many geometrical results that can be obtained using vector methods.

1.2.1 Applications of the dot product

Work done against a force

Suppose that a constant force $\boldsymbol{F}$ acts on a body and that the body is moved a distance $\boldsymbol{d}$. Then the work done against the force is given by the magnitude of the force times the distance moved in the direction opposite to the force; this is simply $-\boldsymbol{F}\cdot\boldsymbol{d}$ (Figure 1.7).

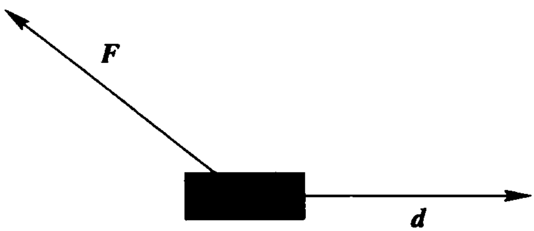

Fig. 1.7. The work done against a force $\boldsymbol{F}$ when an object is moved a distance $\boldsymbol{d}$ is $-\boldsymbol{F}\cdot\boldsymbol{d}$.

Equation of a plane

Consider a two-dimensional plane in three-dimensional space (Figure 1.8). Let $\boldsymbol{r}$ be the position vector of any point in the plane, and let $\boldsymbol{a}$ be a vector perpendicular to the plane. The condition for a point with position vector $\boldsymbol{r}$ to lie in the plane is that the component of $\boldsymbol{r}$ in the direction of $\boldsymbol{a}$ is equal to the perpendicular distance p from the origin to the plane. The general form of the equation of a plane is therefore

$$\boldsymbol{r}\cdot\boldsymbol{a} = \text{constant}.$$

An alternative way to write this is in terms of components. Writing $\boldsymbol{r} = (x, y, z)$ and $\boldsymbol{a} = (a_1, a_2, a_3)$, the equation of a plane becomes

$$a_1x + a_2y + a_3z = \text{constant}. \tag{1.3}$$

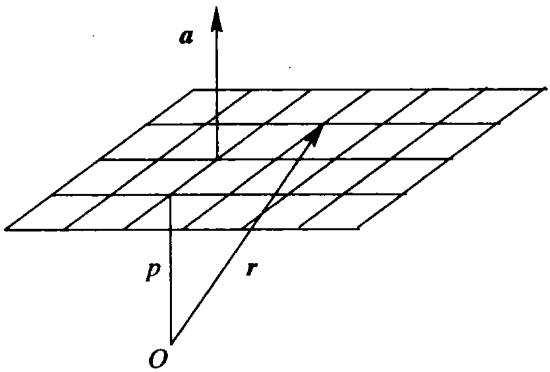

Fig. 1.8. The equation of a plane is $\boldsymbol{r} \cdot \boldsymbol{a} =$ constant.

EXERCISES

1.1 Classify the following quantities according to whether they are vectors or scalars: density, magnetic field strength, power, momentum, angular momentum, acceleration.

1.2 If $\boldsymbol{a} = (2, 0, 3)$ and $\boldsymbol{b} = (1, 0, -1)$, find $|\boldsymbol{a}|$, $|\boldsymbol{b}|$, $\boldsymbol{a} + \boldsymbol{b}$, $\boldsymbol{a} - \boldsymbol{b}$ and $\boldsymbol{a} \cdot \boldsymbol{b}$. What is the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$?

1.3 If $\boldsymbol{u} = (1, 2, 2)$ and $\boldsymbol{v} = (-6, 2, 3)$, find the component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ and the component of $\boldsymbol{v}$ in the direction of $\boldsymbol{u}$.

1.4 Find the equation of the plane that is perpendicular to the vector $(1, 1, -1)$ and passes through the point $x = 1$, $y = 2$, $z = 1$.

1.5 Use vector methods to show that the diagonals of a rhombus are perpendicular.

1.6 What is the angle between any two diagonals of a cube?

1.7 Use vectors to show that for any triangle, the three lines drawn from each vertex to the midpoint of the opposite side all pass through the same point.