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Morris W. Hirsch

Differential Topology

微分拓扑学

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Preface

This book presents some of the basic topological ideas used in studying differentiable manifolds and maps. Mathematical prerequisites have been kept to a minimum; the standard course in analysis and general topology is adequate preparation. An appendix briefly summarizes some of the background material.

In order to emphasize the geometrical and intuitive aspects of differential topology, I have avoided the use of algebraic topology, except in a few isolated places that can easily be skipped. For the same reason I make no use of differential forms or tensors.

In my view, advanced algebraic techniques like homology theory are better understood after one has seen several examples of how the raw material of geometry and analysis is distilled down to numerical invariants, such as those developed in this book: the degree of a map, the Euler number of a vector bundle, the genus of a surface, the cobordism class of a manifold, and so forth. With these as motivating examples, the use of homology and homotopy theory in topology should seem quite natural.

There are hundreds of exercises, ranging in difficulty from the routine to the unsolved. While these provide examples and further developments of the theory, they are only rarely relied on in the proofs of theorems.

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Introduction

> Any problem which is non-linear in character, which involves more than one coordinate system or more than one variable, or where structure is initially defined in the large, is likely to require considerations of topology and group theory for its solution. In the solution of such problems classical analysis will frequently appear as an instrument in the small, integrated over the whole problem with the aid of topology or group theory.
>
> —M. Morse, *Calculus of Variations in the Large*, 1934

> La possibilité d'utiliser le modèle differential est, à mes yeux, la justification ultime de l'emploi des modèles quantitifs dans les sciences.
>
> —R. Thom, *Stabilité Structurelle et Morphogénèse*, 1972

In many branches of mathematics one finds spaces that can be described locally by n-tuples of real numbers. Such objects are called manifolds: *a manifold is a topological space which is locally homeomorphic to Euclidean n-space* $\mathbb{R}^n$. We can think of a manifold as being made of pieces of $\mathbb{R}^n$ glued together by homeomorphisms. If these homeomorphisms are chosen to be differentiable, we obtain a *differentiable manifold.* This book is concerned mainly with differentiable manifolds.

The Development of Differentiable Topology

The concept of manifold emerged gradually from the geometry and function theory of the nineteenth century. Differential geometers studied curves and surfaces in "ordinary space"; they were mainly interested in local concepts such as curvature. Function theorists took a more global point of view; they realized that invariants of a function F of several real or complex variables could be obtained from topological invariants of the sets $F^{-1}(c)$; for "most" values of c, these are manifolds.

Riemann broke new ground with the construction of what we call Riemann surfaces. These were perhaps the first abstract manifolds; that is, they were not defined as subsets of Euclidean space.

Riemann surfaces furnish a good example of how manifolds can be used to investigate global questions. The idea of a convergent power series (in one complex variable) is not difficult. This simple local concept becomes a complex global one, however, when the process of analytic continuation is introduced. The collection of all possible analytic continuations of a convergent power series has a global nature which is quite elusive. The global aspect suddenly becomes clear as soon as Riemann surfaces are introduced: the continuations fit together to form a (single valued) function on a surface. *The surface expresses the global nature of the analytic continuation process.* The problem has become geometrized.

Riemann introduced the global invariant of the *connectivity* of a surface: this meant maximal number of curves whose union does not disconnect the surface, plus one. It was known and "proved" in the 1860's that compact corientable surfaces were classified topologically by their connectivity. Strangely enough, no one in the nineteenth century saw the necessity for proving the subtle and difficult theorem that the connectivity of a compact surface is actually *finite*.

Poincaré began the topological analysis of 3-dimensional manifolds. In a series of papers on "Analysis Situs," remarkable for their originality and power, he invented many of the basic tools of algebraic topology. He also bequeathed to us the most important unsolved problem in differential topology, known as *Poincaré's conjecture*: is every simply connected compact 3-manifold, without boundary, homeomorphic to the 3-sphere?

It is interesting to note that Poincaré used purely differentiable methods at the beginning of his series of papers, but by the end he relied heavily on combinatorial techniques. For the next thirty years topologists concentrated almost exclusively on combinatorial and algebraic methods.

Although Herman Weyl had defined abstract differentiable manifolds in 1912 in his book on Riemann surfaces, it was not until Whitney's papers of 1936 and later that the concept of differentiable manifold was firmly established as an important mathematical object, having its own problems and methods.

Since Whitney's papers appeared, differential topology has undergone a rapid development. Many fruitful connections with algebraic and piecewise linear topology were found; good progress was made on such questions as embedding, immersions, and classification by homotopy equivalence or diffeomorphism. Poincaré's conjecture is still unsolved, however. In recent years techniques and results from differential topology have become important in many other fields.

The Nature of Differential Topology

In today's mathematical sciences manifolds are found in many different fields. In algebra they occur as Lie groups; in relativity as space-time; in economics as indifference surfaces; in mechanics as phase-spaces and energy surfaces. Wherever dynamical processes are studied, (hydrodynamics, population genetics, electrical circuits, etc.) manifolds are used for the "state-space," the setting for a model of the process by a differential equation or a mapping.

In most of these examples the historical development follows the *local-to-global* pattern. Lie groups, for example, were originally "local groups" having a single parametrization as a neighborhood of the origin in $\mathbb{R}^n$. Only later did global questions arise, such as the classification of compact groups. In each case the global nature of the subject became geometrized (at least partially) by the introduction of manifolds. In mechanics, for example, the differences in the possible long-term behavior of two physical systems become clear if it is known that one energy surface is a sphere and the other is a torus.

When manifolds occur "naturally" in a branch of mathematics, there is always present some extra structure: a Riemannian metric, a binary operation, a dynamical system, a conformal structure, etc. It is often this structure which is the main object of interest; the manifold is merely the setting. But the differential topologist studies the *manifold itself*; the extra structures are used only as tools.

The extra structure often presents fascinating local questions. In a Riemannian manifold, for instance, the curvature may vary from point to point. *But in differential topology there are no local questions.* (More precisely, they belong to calculus.) A manifold looks exactly the same at all points because it is locally Euclidean. In fact, a manifold (connected, without boundary) is homogeneous in a more exact sense: its diffeomorphism group acts transitively.

The questions which differential topology tries to answer are global; they involve the whole manifold. Some typical questions are: Can a given manifold be embedded in another one? If two manifolds are homeomorphic, are they necessarily diffeomorphic? Which manifolds are boundaries of compact manifolds? Do the topological invariants of a manifold have any special properties? Does every manifold admit a non-trivial action of some cyclic group?

Each of these questions is, of course, a shorthand request for a *theory.* The embedding question, for example, really means: define and compute diffeomorphism invariants that enable us to decide whether M embeds in N, and in how many essentially distinct ways.

If we know how to construct all possible manifolds and how to tell from "computable" invariants when two are diffeomorphic, we would be a long way toward answering any given question about manifolds. Unfortunately, such a classification theorem seems unattainable at present, except for very special classes of manifolds (such as surfaces). Therefore we must resort to more direct attacks on specific questions, devising different theories for different questions. Some of these theories, or parts of them, are presented in this book.

The Contents of This Book

The first difficulty that confronts us in analyzing manifolds is their homogeneity. A manifold has no distinguished "parts"; every point looks like every other point. How can we break it down into simpler objects?

The solution is to artificially impose on a manifold a nonhomogeneous structure of some kind which can be analyzed. The major task then is to derive intrinsic properties of the original manifold from properties of the artificial structure.

This procedure is common in many parts of mathematics. In studying vector spaces, for example, one imposes coordinates by means of a basis; the cardinality of the basis is then proved to depend only on the vector space. In algebraic topology one defines the homology groups of a polyhedron in terms of a particular triangulation, and then proves the groups to be independent of the triangulation.

Manifolds are, in fact, often studied by means of triangulations. A more natural kind of decomposition, however, consists of the level sets $f^{-1}(y)$ of a smooth map $f:M \to \mathbb{R}$, having the simplest kinds of critical points (where Df vanish). This method of analysis goes back to Poincaré and even to Möbius (1866); it received extensive development by Marston Morse and today is called *Morse theory*. Chapter 6 is devoted to the elementary aspects of Morse theory. In Chapter 9 Morse theory is used to classify compact surfaces.

A basic idea in differential topology is that of *general position* or *transversality*; this is studied in Chapter 3. Two submanifolds A, B of a manifold N are in general position if at every point of $A \cap B$ the tangent spaces of A and B span that of N. If A and B are not in general position, arbitrarily small perturbations of one of them will put them in general position. If they are in general position, they remain in it under all sufficiently small perturbations; and $A \cap B$ is then a submanifold of the "right" dimension. A map $f:M \to N$ is *transverse* to A if the graph of f and $M \times A$ are in general position in $M \times N$. This makes $f^{-1}(A)$ a submanifold of M, and the topology of $f^{-1}(A)$ reflects many properties of f. In this way an important connection between manifolds and maps is established.

Transversality is a great unifying idea in differential topology; many results, including most of those in this book, are ultimately based on transversality in one form or another.

The theory of degrees of maps, developed in Chapter 5, is based on transversality in the following way. Let $f:M \to N$ be a map between compact oriented manifolds of the same dimension, without boundary. Suppose f is transverse to a point $y \in N$; such a point is called a regular value of f. The degree of f is the "algebraic" number of points in $f^{-1}(y)$, that is, the number of such points where f preserves orientation minus the number where f reverses orientation. It turns out that this degree is independent of y and, in fact, depends only on the homotopy class of f. If $N = S^n$ then the degree is the *only* homotopy invariant. In this way we develop a bit of classical algebraic topology: the set of homotopy classes $[M,S^n]$ is naturally isomorphic to the group of integers.

The theory of fibre bundles, especially vector bundles, is one of the strongest links between algebraic and differential topology. Patterned on the tangent and normal bundles of a manifold, vector bundles are analogous to manifolds in form, but considerably simpler to analyze. Most of the deeper diffeomorphism invariants are invariants of the tangent bundle. In Chapter 4 we develop the elementary theory of vector bundles, including the classification theorem: isomorphism classes of vector bundles over M correspond naturally to homotopy classes of maps from M into a certain Grassmann manifold. This result relates homotopy theory to differential topology in a new and important way.

Further importance of vector bundles comes from the tubular neighborhood theorem: a submanifold $B \subset M$ has an essentially unique neighborhood looking like a vector bundle over B.

In 1954 René Thom proposed the equivalence relation of *cobordism*: two manifolds are cobordant if together they form the boundary of a compact manifold. The resulting set of equivalence classes in each dimension has a natural abelian group structure. In a *tour de force* of differential and algebraic topology, Thom showed that these groups coincide with certain homotopy groups, and he carried out a good deal of their calculation. The elementary aspects of Thom's theory, which is a beautiful mixture of transversality, tubular neighborhoods, and the classification of vector bundles, is presented in Chapter 7.

Of the remaining chapters, Chapter 1 introduces the basic definitions and, proves the "easy" Whitney embedding theorem: any map of a compact n-manifold into a $(2n + 1)$-manifold can be approximated by embeddings. Chapter 2 topologizes the set of maps from one manifold to another and develops approximation theorems. A key result is that for most purposes it can be assumed that every manifold is C^∞. Much of this chapter can be skipped by a reader interested chiefly in compact C^∞ manifolds. Chapter 8 is a technical chapter on isotopy, containing some frequently used methods of deforming embeddings; these results are needed for the final chapter on the classification of surfaces.

The first three chapters are fundamental to everything else in the book. Most of Chapter 6 (Morse Theory) can be read immediately after Chapter 3; while Chapter 7 (Cobordism) can be read directly after Chapter 4. The classification of surfaces, Chapter 9, uses material from all the other chapters except Chapter 7.

The more challenging exercises are starred, as are those requiring algebraic topology or other advanced topics. The few that have two stars are really too difficult to be considered exercises, but are included for the sake of the results they contain. Three-star "exercises" are problems to which I do not know the answer.

A reference to Theorem 1 of Section 2 in Chapter 3 is written 3.2.1, or as 2.1 if it appears in Chapter 3. The section is called Section 3.2. Numbers in brackets refer to the bibliography.

Acknowledgments

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Chapter 1

Manifolds and Maps

Il faut d'abord examiner la question de la définition des variétés.

—P. Heegard, *Dissertation*, 1892

The assemblage of points on a surface is a twofold manifoldness; the assemblage of points in tri-dimensional space is a threefold manifoldness; the values of a continuous function of n arguments an n-fold manifoldness.

—G. Chrystal, *Encyclopedia Brittanica*, 1892

The introduction of numbers as coordinates ... is an act of violence ...

—H. Weyl, *Philosophy of Mathematics and Natural Science*, 1949

Differential topology is the study of differentiable manifolds and maps. A manifold is a topological space which locally looks like Cartesian n-space $\mathbb{R}^n$; it is built up of pieces of $\mathbb{R}^n$ glued together by homeomorphisms. If these homeomorphisms are differentiable we obtain a differentiable manifold.

The task of differential topology is the discovery and analysis of global properties of manifolds. These properties are often quite subtle. In order to study them, or even to express them, a wide variety of topological, analytic and algebraic tools have been developed. Some of these will be examined in this book.

In this chapter the basic concepts of differential topology are introduced: differentiable manifolds, submanifolds and maps, and the tangent functor. This functor assigns to each differentiable manifold M another manifold TM called its tangent bundle, and to every differentiable map $f:M \to N$ it assigns a map $Tf:TM \to TN$. In local coordinates Tf is essentially the derivative of f. Although its definition is necessarily rather complicated, the tangent functor is the key to many problems in differential topology; it reveals much of the deeper structure of manifolds.

In Section 1.3 we prove some basic theorems about submanifolds, maps and embeddings. The key ideas of regular value and transversality are introduced. The regular value theorem, which is just a global version of the implicit function theorem, is proved. It states that if $f:M \to N$ is a map then under certain conditions $f^{-1}(y)$ will be a submanifold of M. The submanifolds

$f^{-1}(y)$ and of the map f are intimately related; in this way a powerful positive feedback loop is created:

Theorems about manifolds	$\longrightarrow$ $\longleftarrow$	Theorems about maps

This interplay between manifolds and maps will be exploited in later chapters.

Also proved in Section 1.3 is the pleasant fact that every compact manifold embeds in some $\mathbb{R}^q$. Borrowing an analytic lemma from a later chapter, we then prove a version of the deeper embedding theorem of Whitney: every map of a compact n-manifold into $\mathbb{R}^{2n+1}$ can be approximated by embeddings.

Manifolds with boundary, or ∂-manifolds, are introduced in Section 1.4. These form a natural and indeed indispensable extension of the manifolds defined in Section 1.1; their presence, however, tends to complicate the mathematics. The special arguments needed to handle ∂-manifolds are usually obvious; in order to present the main ideas without interruption we shall frequently postpone or omit entirely proofs of theorems about ∂-manifolds.

At the end of the chapter a convention is stated which is designed to exclude the pathology of non-Hausdorff and nonparacompact manifolds.

Running through the chapter is an idea that pervades all of differential topology: the passage from local to global. This theme is expressed in the very definition of manifold; every statement about manifolds necessarily repeats it, explicitly or implicitly. The proof of the regular value theorem, for example, consists in pointing out the local nature of the hypothesis and conclusion, and then applying the implicit function theorem (which is itself a passage from infinitesimal to local). The compact embedding theorem pieces together local embeddings to get a global one. Whitney's embedding theorem builds on this, using, in addition, a lemma on the existence of regular values. This proof of this lemma, as will be seen in Chapter 3, is a simple globalization of a rather subtle local property of differentiable maps.

Every concept in differential topology can be analyzed in terms of this local-global polarity. Often a definition, theorem or proof becomes clearer if its various local and global aspects are kept in mind.

0. Submanifolds of $\mathbb{R}^{n+k}$

Before giving formal definitions we first discuss informally the familiar space S^n and then more general submanifolds of Euclidean space.

The *unit n-sphere* is

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\},$$

where $|x| = \left(\sum_{i=1}^{n+1} x_i^2\right)^{1/2}$. We introduce local coordinates in S^n as follows.

For $j = 1, \ldots, n + 1$ define open hemispheres

$$U_{2j-1} = \{x \in S^n : x_j > 0\},$$
$$U_{2j} = \{x \in S^n : x_j < 0\}.$$

For $i = 1, \ldots, 2n + 2$ define maps

$$\varphi_i : U_i \to \mathbb{R}^n,$$
$$\varphi_i(x) = (x_1, \ldots, \hat{x}_j, \ldots, x_{n+1}) \qquad \text{if } i = 2j - 1 \text{ or } 2j;$$

this means the n-tuple obtained from x by deleting the jth coordinate. Clearly φ_i maps U_i homeomorphically onto the open n-disk

$$B = \{y \in \mathbb{R}^n : |y| < 1\}.$$

It is easy to see that $\varphi_i^{-1} : B \to \mathbb{R}^{n+1}$ is analytic.

Each (φ_i, U_i) is called a "chart" for S^n; the set of all (φ_i, U_i) is an "atlas". In terms of this atlas we say a map $f : S^n \to \mathbb{R}^k$ is "differentiable of class C^r" in case each composite map

$$f \circ \varphi_i^{-1} : B \to \mathbb{R}^k$$

is C^r. If it happens that $g : S^n \to \mathbb{R}^{m+1}$ is C^r in this sense, and $g(S^n) \subset S^m$, it is natural to call $g : S^n \to S^m$ a C^r map. This definition is equivalent to the following. Let $\{(\psi_j, V_j)\}$ be an atlas for S^n, $i = 1, \ldots, q$. Then $g : S^n \to S^m$ is C^r provided each map

$$\psi_j g \varphi_i^{-1} : \varphi_i g^{-1}(V_j) \to \mathbb{R}^m$$

is C^r; this makes sense because $\varphi_i g^{-1}(V_j)$ is an open subset of $\mathbb{R}^n$.

Thus *we have extended the notion of C^r map to the unit spheres S^n, $n = 1, 2, \ldots$.* It is easy to verify that the composition of C^r maps (in this extended sense) is again C^r.

A broader class of manifolds is obtained as follows. Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^k$ be a C^r map, $r \geqslant 1$, and put $M = f^{-1}(0)$. Suppose that f has rank k at every point of $f^{-1}(0)$; we call M a "regular level surface". An example is $M = S^n \subset \mathbb{R}^{n+1}$ where $f(x) = 1 - \sum_{i=1}^{n+1} x_i^2$.

Local coordinates are introduced into M as follows. Fix $p \in M$. By a linear coordinate change we can assume that the $k \times k$ matrix $\partial f_i / \partial x_j$, $1 \leqslant i, j \leqslant k$, has rank k at p. Now identify $\mathbb{R}^{n+k}$ with $\mathbb{R}^n \times \mathbb{R}^k$ and put $p = (a,b)$. According to the implicit function theorem there exist a neighborhood $U \times V$ of (a,b) in $\mathbb{R}^n \times \mathbb{R}^k$ and a C^r map $g : U \to V$, such that $g(x) = y$ if and only if $f(x,y) = 0$. Thus

$$\begin{aligned} M \cap (U \times V) &= \{(x, g(x)) : x \in U\} \\ &= \text{graph of } g. \end{aligned}$$

Define

$$W = M \cap (U \times V),$$
$$\varphi: W \to \mathbb{R}^n,$$
$$(x, g(x)) \mapsto x \qquad (x \in U).$$

Then (φ, W) is taken as a local coordinate system on M. In terms of such coordinates we can further extend the notion of C^r map to maps between regular level surfaces.

Exactly the same constructions are made when the domain of f is taken to be an open subset of $\mathbb{R}^{n+k}$, rather than all of $\mathbb{R}^{n+k}$.

A significantly broader class of manifolds comprises those subsets M of $\mathbb{R}^{n+k}$ which *locally* are regular level surfaces of C^r maps. That is, each point of M has a neighborhood $W \subset \mathbb{R}^{n+k}$ such that

$$W \cap M = f^{-1}(0)$$

for some C^r map $f: W \to \mathbb{R}^k$ having rank k at each point $W \cap M$. Local coordinates are introduced and C^r maps are defined as before. A manifold of this type is called an "n-dimensional submanifold of $\mathbb{R}^{n+k}$".

In each of these examples it is easy to see that *the coordinate changes are* C^r. These coordinate changes are the maps

$$\varphi_j \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

where (φ_i, U_i) and (φ_j, U_j) vary over an atlas for the manifold in question. (The domain and range of $\varphi_j \varphi_i^{-1}$ are open subsets of $\mathbb{R}^m$, so that it makes sense to say that $\varphi_j \varphi_i^{-1}$ is C^r.)

This has an important implication: to verify that a map $f: M \to N$ is C^r, it suffices to check that for each point $x \in M$ there is at least one pair of charts, (φ, U) for M and (ψ, V) for N, with $x \in U$ and $f(U) \subset V$, such that the map

$$\mathbb{R}^m \supset \varphi(U) \xrightarrow{\psi f \varphi^{-1}} \psi(V) \subset \mathbb{R}^n$$

is C^r. For suppose this is true, and let $(\bar{\varphi}, \bar{U}), (\bar{\psi}, \bar{V})$ be any charts for M, N; we must show that $\bar{\psi} f \bar{\varphi}^{-1}$ is C^r. An arbitrary point in the domain of $\bar{\psi} f \bar{\varphi}^{-1}$ is of the form $\bar{\varphi}(x)$ where $x \in \bar{U} \cap f^{-1}(\bar{V})$. Let $(\varphi, U), (\psi, V)$ be charts for M, N such that $x \in U, f(U) \subset V$ and $\psi f \varphi^{-1}$ is C^r. Then in a neighborhood of $\bar{\varphi}(x)$ we have

$$\bar{\psi} f \bar{\varphi}^{-1} = (\bar{\psi}\psi^{-1})(\psi f \varphi^{-1})(\varphi \bar{\varphi}^{-1}).$$

Thus $\bar{\psi} f \bar{\varphi}^{-1}$ is locally the composition of three C^r maps, so it is C^r.

Next we discuss the tangent bundle of an n-dimensional submanifold $M \subset \mathbb{R}^{n+k}$. Let $x \in M$ and let (φ, U) be a chart at x (that is, $x \in U$). Put $a = \varphi(x) \in \mathbb{R}^n$. Let $E_x \subset \mathbb{R}^{n+k}$ be the vector subspace which is the range of the linear map

$$D\varphi_a^{-1}: \mathbb{R}^n \to \mathbb{R}^{n+k}.$$

Because of the chain rule, E_x depends only on x, not on the choice of (φ, U).