CONTACT STRESS ANALYSIS

CONTACT STRESS ANALYSIS

Editors: A D Roberts and J E Mottershead

Proceedings of two one-day meetings of the
Stress Analysis and Tribology Groups
of The Institute of Physics

7 December 1988 and 16 November 1989
London

In collaboration with
The Institute of Metals
The Institution of Mechanical Engineers

IOP Short Meetings Series No 25
Institute of Physics

CODEN — IPS SE3

British Library Cataloguing in Publication Data

Contact stress analysis.
1. Stress analysis.
I. Roberts, A. D.
II. Mottershead, J. E.
III. Series 620.1′126

ISBN 0-85498-525-5
ISSN 0269-8986

Published by IOP Publishing Ltd
Techno House, Redcliffe Way, Bristol BS1 6NX, United Kingdom

Printed in the United Kingdom by Dotesios (Printers) Ltd

CONTACT STRESS ANALYSIS

Acknowledgment. The illustration on the front cover and figures 2, 3, 5, 7, and 10 in the paper by K L Johnson are reproduced with permission of Cambridge University Press from *Contact Mechanics* (1985) by K L Johnson.

Foreword

The aim of the two meetings was to emphasise how suitable analysis can lead to a better appreciation of the factors influencing tribological processes. Several areas were covered, so that an overview is gained of how contact mechanics can contribute to understanding and perhaps lead to the transfer of ideas across fields.

In the analysis of contact problems both analytical and numerical solutions are considered. How to carry out numerical contact stress calculations efficiently and reliably is one area of importance dealt with. Situations treated range from low pressure rubber contacts to the extreme pressures encountered in metal machining. Each illustrates particular problems of analysis.

The Editors wish to acknowledge the help of staff at IOP Publishing, Bristol and at the Malaysian Rubber Producers Research Association, Hertford in the preparation of manuscripts.

J E Mottershead, University of Liverpool
A D Roberts, MRPRA

Contents

CLASSICAL VERSUS NUMERICAL METHODS OF ELASTIC CONTACT STRESS ANALYSIS

K L Johnson

Department of Engineering, University of Cambridge

As the title suggests classical and numerical methods of analysis are frequently viewed as competitive and mutually exclusive. This is not as it should be ! Most progress is achieved if both approaches can be combined so that the strengths of one compensate the weaknesses of the other.

The finite-element method is not intrinsically well suited to the high stress gradients of contact stress problems. Alternative methods will be described and ways in which some of the disadvantages can be ameliorated will be discussed.

Finally, attention will be drawn to the 'elastic foundation' and 'wire brush' models of normal and tangential contact for obtaining quick, approximate solutions to contact problems.

Hertz theory

In general, a numerical method may be necessary when the Hertz conditions are violated. The restrictions of the Hertz theory may be listed as

(1) Small strains
(2) Smooth, continuous, second order surfaces
(3) Contact area small compared with size of bodies (elastic half-space idealisation) and the contact stress distribution can be treated separately from the general distribution of stress in the two bodies.
(4) Frictionless
(5) Non-conforming

The sort of situations may require numerical solution are

- (i) conforming surfaces; relax (5)
- (ii) contacts with friction; relax (4)
- (iii) discontinuous surfaces in contact region; relax (2)
- (iv) 'large' contact areas; relax (3)

Each of these cases will be considered. The difficulties of elastic contact stress theory arise because the displacement at any point in the contact surface depends upon the distribution of pressure throughout the whole contact. To find the pressure at any point in the contact generally requires the solution of an integral equation. This difficulty is avoided if the solids can be modelled by an elastic foundation (see final section).

CONTACT OF CONFORMING SURFACES

Many non-Hertzian contact problems do not permit analytical solutions, such as the case of conforming contacts where the initial separation cannot be described by a second-order polynomial. Thus the object is to determine the normal pressure distribution p (x,y) which satisfies the boundary conditions at the interface (fig.1) given by the

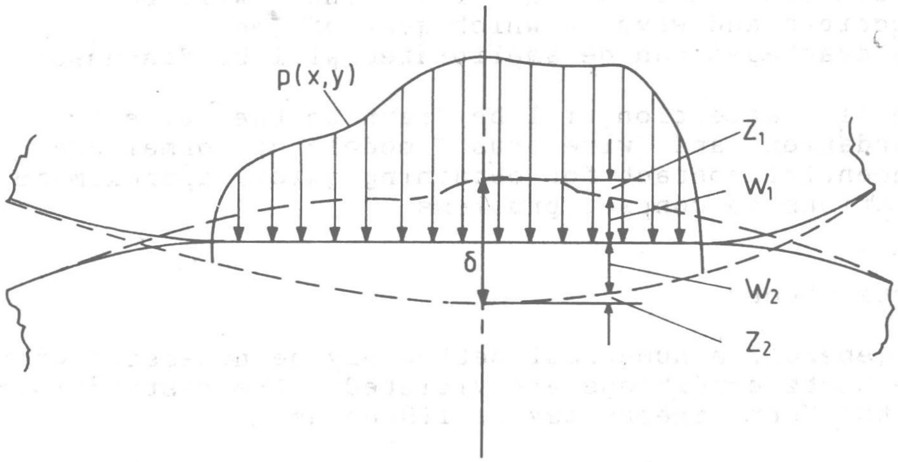


FIG.1 - Contact of frictionless elastic bodies

profiles $z_1(x,y)$ and $z_2(x,y)$. Here the normal pressure is found on the assumption that the surfaces are frictionless, and that the contacting bodies can be regarded as half-spaces.

In the contact area

$$\delta - \{ Z_1(x,y) + Z_2(x,y) \} = \omega_1(x,y) + \omega_2(x,y)$$

$$= \frac{1}{2\pi}\left[\frac{1-\nu_1}{G_1} + \frac{1-\nu_2}{G_2}\right] \iint_A \frac{p(\xi,\eta)}{\sqrt{(x-\xi)^2-(y-\eta)^2}}\, d\xi\, d\eta \quad (1)$$

where ν is Poisson's ratio, G the shear modulus and ξ and η are dummy variables in the integral. This is an integral equation for $p(x,y)$ acting on a contact area A whose shape and size are not known at the outset. Three methods of solution will be described.

Method (i) Analytical

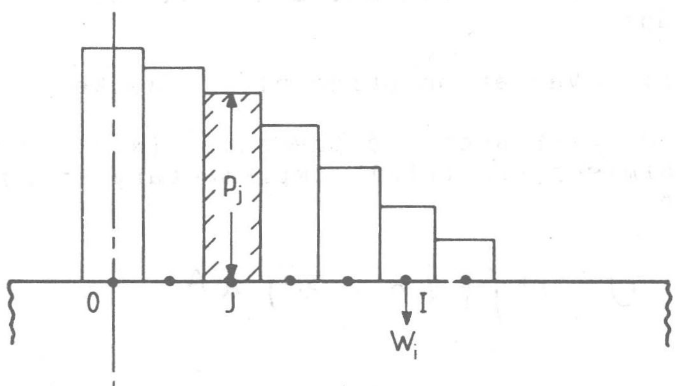
The case of plane strain or axi-symmetric problems requires the solution for $p(x)$ of the integral equation

$$\frac{\partial w_1(x)}{\partial x} + \frac{\partial w_2(x)}{\partial x} = \frac{1}{\pi}\left[\frac{1-\nu_1}{G_1} + \frac{1-\nu_2}{G_2}\right]\int_{-b}^{a} \frac{p(\xi)}{x-\xi}\,d\xi \quad (2)$$

The classical method which has been applied to line contact and axi-symmetric problems in which the shape of the contact area is known, is to represent the pressure distribution by an infinite series of known functions, most conveniently by Chebychev polynomials

Method (ii) Matrix inversion

A different approach afforded by modern computers is to deal with a discrete set of 'traction elements'. The boundary conditions are then satisfied at a discrete number of points - the 'matching points'. The traction can be represented either by adjacent columns of uniform pressure leading to a piecewise constant distribution (Fig.2a) or by overlapping triangular elements, giving rise to a piecewise linear distribution (Fig.2b). The surface displacements are finite everywhere, in the former, but the displacement gradients are infinite between adjacent elements. The latter produces surface



(a)

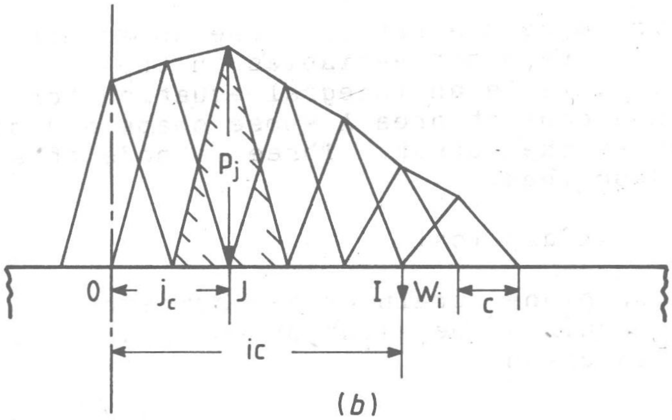


FIG.2 - Discretized traction elements in elastic contact : (a)piecewise constant; (b)piecewise linear

displacements which are smooth and continuous. One determines analytically the influence coefficients C_{ij} which express the displacement w_i due to unit pressure p_j. The total displacement is then given by

$$\{\omega\}_L = -\frac{(1-\nu)c}{2G} \sum C_{ij} P_j$$

It is first necessary to estimate the contact area. A suitable first guess is provided by the surface of interpenetration of the undeformed profiles.

Given $\omega_1(x) + \omega_2(x)$ from profiles, one inverts the above matrix to obtain $p(x)$.

For the second iteration, relax p at all points in which it has a negative (tensile) value and repeat the process. The procedure converges to the correct solution in which the pressure is positive throughout the true contact area and falls to zero at its edges.

Method (iii) Variation principles (Kalker)

The true contact area and pressure $p(x,y)$ is that which minimises the total complementary energy V^*, where

$$V^* = U^* + \int_A p(h - \delta)\, dA \qquad (3)$$

subject to $p > 0$ everywhere. For linear elastic materials

$$U^* = \frac{1}{2}\int_A p(w_1 + w_2)\, dA \qquad (4)$$

$h(x,y)$ = gap before deformation = $\bar{z}_1(x,y) + z_2(x,y)$

δ = compression at origin

For an elastic half-space the relationships between p_j and w_1 are linear so that the strain energy U^* is quadratic in p_i. Therefore V^* can be minimised by use of a quadratic programming routine.

An example is provided by the contact of an isotropic wavy surface with a flat (Johnson et al, Int.J.Mech.Sci.27, 383, 1985) as shown (fig.3).

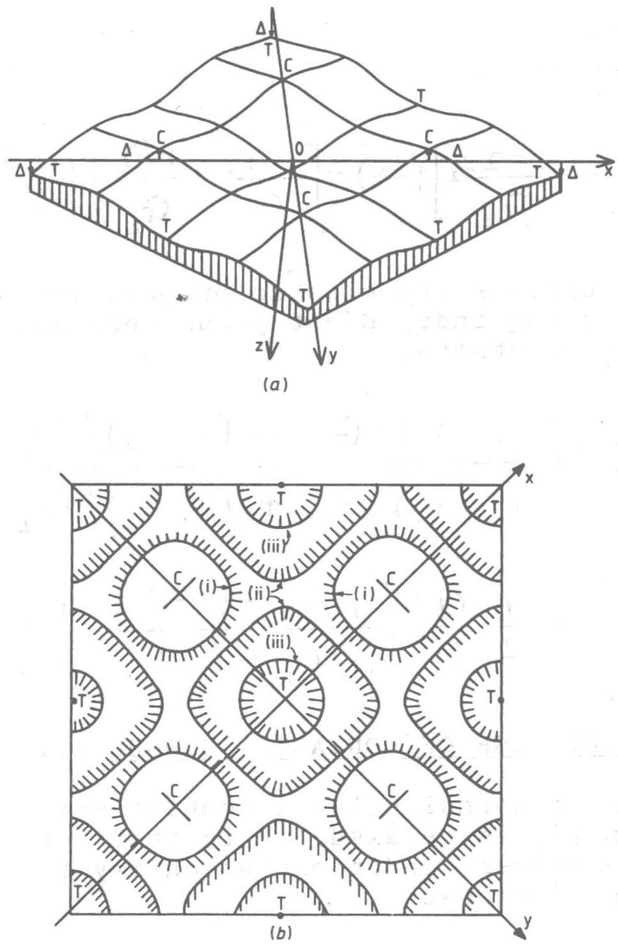


FIG.3 - Areas of contact and pressure distribution found by using equations (3) and (4).

Half-space problems with friction

There are no well-established general methods for problems involving friction. With friction the history of loading must be followed which requires incremental formulation. The interaction between the shear traction q and pressure p gives coupled integral equations: one for normal and the other for tangential deformation. In plane strain they may be written.

$$\frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial x} =$$

$$-\left[\frac{1-\nu_1}{G_1} + \frac{1-\nu_2}{G_2}\right]\int \frac{p(\xi)}{x-\xi}\,d\xi + \frac{1}{2}\left[\frac{1-2\nu_1}{G_1} - \frac{1-2\nu_2}{G_2}\right]q(x) \quad (5a)$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x} = \quad (5b)$$

$$-\frac{1}{2}\left[\frac{1-2\nu_1}{G_1} - \frac{1-2\nu_2}{G_2}\right]p(x) - \left[\frac{1-\nu_1}{G_1} - \frac{1-\nu_2}{G_2}\right]\int \frac{q(\xi)}{x-\xi}\,d\xi$$

In these equations the elastic properties can be expressed by two independent parameters known as Dundur's constants:

$$\alpha = \frac{(1-\nu_1)/G_1 - (1-\nu_2)/G_2}{(1-\nu_1)/G_1 + (1-\nu_2)/G_2}$$

$$\beta = \frac{1}{2}\,\frac{(1-2\nu_1)/G_1 - (1-2\nu_2/G_2}{(1-\nu_1)/G_1 + (1-\nu_2)/G_2}$$

CONDITIONS AT EDGE OF CONTACT - SINGULARITIES

In problems of normal elastic contact where Hertz restriction (2) is relaxed, it is instructive to examine the stress conditions which may arise close to the edge of contact.

(a) Continuous surfaces

When two non-conforming elastic bodies having continuous profiles are pressed into contact the pressure must fall continuously to zero at the edge of contact to avoid interference outside the contact area (see fig.4).

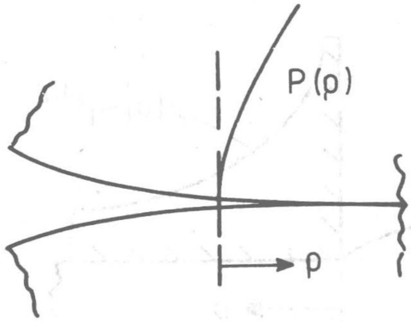


FIG.4 - Continuous surfaces : pressure falls to zero at edge.

For example, with contacting cylinders the resulting pressure distribution is semi-ellipsoidal, of the form $p(1 - x^2/a^2)^{1/2}$ falling to zero at $x = \pm a$.

(b) Non-continuous surfaces

If one or both bodies has a discontinuous profile at the edge of the contact the situtation is different, and usually a high stress concentration would be expected at the edge. In general there will be a singularity at O(fig.5) such that p varies as ρ^n where

$$n = f\,[\theta, \mu, \alpha, \beta]$$

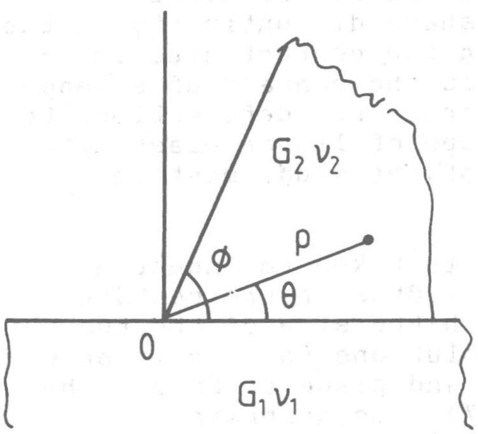


FIG.5 - Edge of contact : general conditions

An example would be the indentation of an elastic half-space by a rigid square punch, whence

$$\phi = \frac{\pi}{2}\ , \quad \mu = 0\ , \quad \alpha = 1$$

$$\nu_1 = 0.3 \longrightarrow \beta = \frac{1}{2}\left(\frac{1 - 2\nu_1}{1 - \nu_1}\right) = 0.286$$

If $\nu = 0$ or 0.5, $\beta = 0$ which gives $n = -1/2$ (Fig.6).

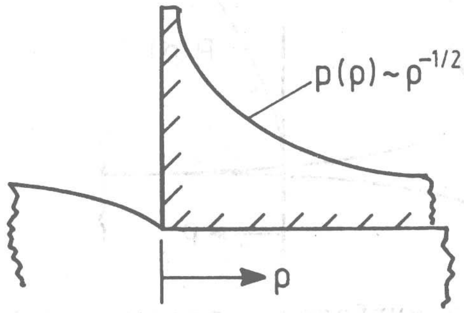


FIG.6 - Edge of contact : Rigid square punch gives square-root singularity.

It is recognised that, in reality, this infinite stress cannot exist. The elasticity theory is only valid for small strains, and real materials will yield plastically at a finite stress.

Even so, as developments in linear elastic fracture mechanics have shown, the strength of stress singularities calculated by linear elastic theory is capable of providing useful information about the intensities of stress concentrations and the probable extent of plastic flow.

Discontinuities within the contact

For smooth and continuous surface profiles the stresses are finite everywhere; a rigid punch introduces an infinite pressure at the contact edge. The influence of a sharp discontinuity in the slope of the profile within the contact area is now examined by reference to the contact of a wedge with a plane surface. In order for deformations to be small and within the scope of linear elasticity theory, the semi-angle α of the wedge must be close to 90°.

If a two-dimensional wedge is taken to indent a flat surface such that the widths of the contact strip is small compared with the size of the two solids, then the elastic solutions for a half-space can be used for both wedge and plane surface. The deformation is shown (fig.7). Logarithmic

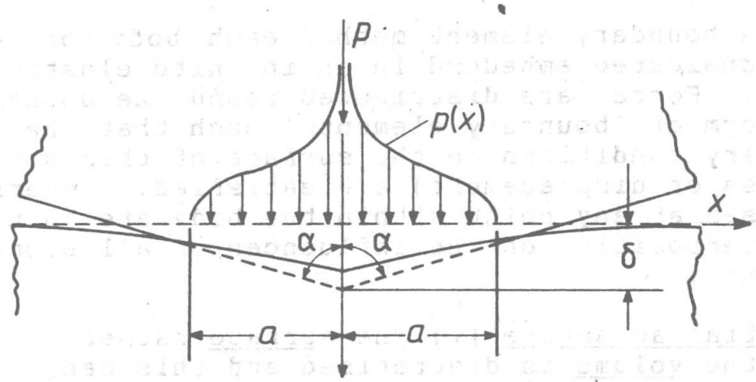


FIG.7 - Indentation by a blunt wedge: pressure distribution. Discontinuity of slope at apex gives a logarithmic singularity

singularities arise and the pressure distribution may be deduced as

$$p(x) = \frac{E^* \cot\alpha}{2\pi} \ln\left[\frac{a+\sqrt{a^2-x^2}}{a-\sqrt{a^2-x^2}}\right] \qquad (6)$$

such that the pressure $p(o) \rightarrow \infty$, that is, a logarithmic singularity at the apex of the wedge.

The information in this section is useful in numerical solutions, in suggesting suitable 'singularity elements' for incorporation in a finite element scheme.

FINITE SIZE BODIES

When the bulk stress field and the contact stress field cannot be separated, then one is usually obliged to resort to numerical methods. In gauging the importance of the proximity of the boundaries of the body, it should be kept in mind that :

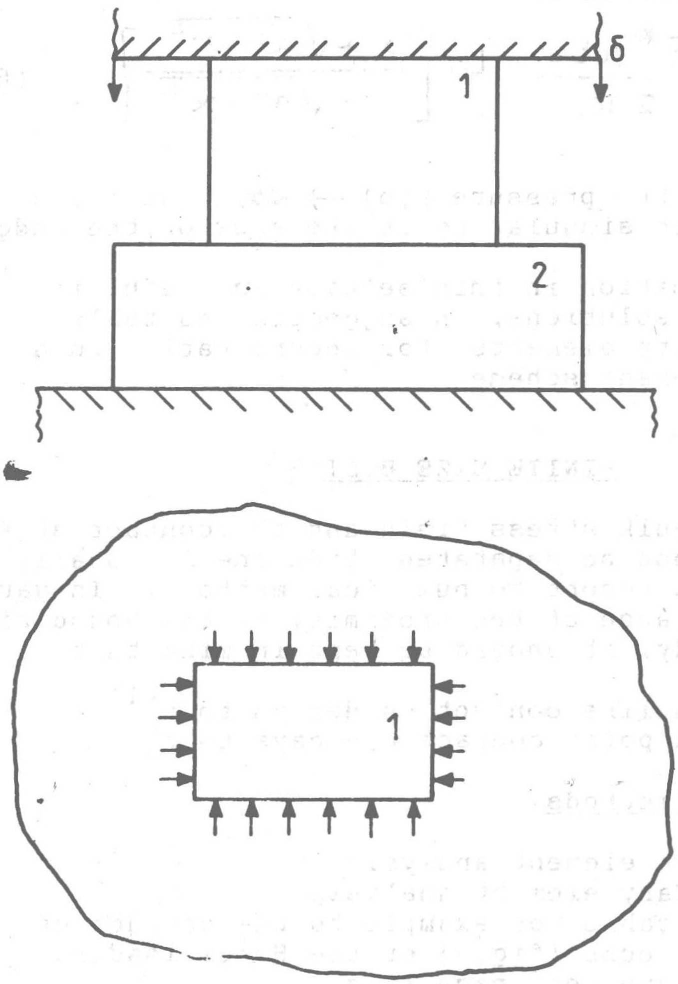
in line contact σ decays to r^{-1}
and in point contact σ decays to r^{-2}

Numerical methods

(i) Finite-element analysis
(ii) Boundary element analysis.
Applicable for example to the contact of two blocks (fig.8) or the Hertz loading of a quarter-space (Fig.9).

In the boundary element method each body (or both) are considered embedded in an infinite elastic solid. Forces are distributed round the boundary in the form of `boundary elements' such that the boundary conditions on the surface of this body (stress or displacement) are satisfied. Internal stresses at any point within the body are then found by superposition of the influences of all boundary elements.

<u>Potential advantage</u> (i) The <u>surface</u> rather than the <u>volume</u> is discretised and this can result in more efficient solutions than FE.
(ii) The internal stresses (away from the boundary) are calculated analytically and are thus negligibly influenced by the finite size of the boundary elements.



<u>FIG.8</u> - Boundary element method: each of the bodies is considered embedded in an infinite elastic solid.