University
LECTURE
Series

Volume 38

Lectures on Quasiconformal Mappings

Second Edition

Lars V. Ahlfors

with additional chapters by
C. J. Earle and I. Kra
M. Shishikura
J. H. Hubbard

American Mathematical Society

University LECTURE Series

Volume 38

Lectures on Quasiconformal Mappings

Second Edition

Lars V. Ahlfors

with additional chapters by
C. J. Earle and I. Kra
M. Shishikura
J. H. Hubbard

American Mathematical Society
Providence, Rhode Island

2000 *Mathematics Subject Classification.* Primary 30C62, 30Cxx.

For additional information and updates on this book, visit
www.ams.org/bookpages/ulect-38

Library of Congress Cataloging-in-Publication Data

Ahlfors, Lars Valerian, 1907–
Lectures on quasiconformal mappings / Lars V. Ahlfors ; with additional chapters by C. J. Earle... [et al.]. – 2nd. ed.
p. cm. — (University lecture series ; v. 38)
Originally published: Toronto ; New York : D. Van Nostrand Co., c1966.
ISBN 0-8218-3644-7 (alk. paper)
1. Quasiconformal mappings. I. Title. II. University lecture series (Providence, R. I.); 38.

QA360.A57 2006
515′.93—dc22 2006040650

10 9 8 7 6 5 4 3 2 1 11 10 09 08 07 06

Lectures on Quasiconformal Mappings

Second Edition

Preface

Lars Ahlfors's book *Lectures on Quasiconformal Mappings* was first published in 1966, and its special qualities were soon recognized. For example, a Russian translation was published in 1969, and, after seeing an early version of the notes that were the basis for Ahlfors's book, Lipman Bers, Fred Gardiner and Kra abandoned their plans to produce a book based on Bers's two-semester 1964 course at Columbia on quasiconformal mappings and Teichmüller spaces.

Ahlfors's classic continues to be widely read by graduate students and other mathematicians who are learning the foundations of the theories of quasiconformal mappings and Teichmüller spaces. It is particularly suitable for that purpose because of the elegance with which it presents the fundamentals of the theory of quasiconformal mappings. The early chapters provide precisely what is needed for the big results in Chapters V and VI. At the same time they give the reader an informative picture of how quasiconformal mappings work.

One reason for the economy of Ahlfors's presentation is that his book represents the contents of a one-semester course, given at Harvard University in the spring term of 1964. It was a remarkable achievement; in one semester he developed the theory of quasiconformal mappings from scratch, gave a self-contained treatment of Beltrami's equation (Chapter V of the book), and covered the basic properties of Teichmüller space, including the Bers embedding and the Teichmüller curve (see Chapter VI and §2 of our chapter in the appendix). Along the way, Ahlfors found time for some estimates in Chapter III B involving elliptic integrals and a treatment of an extremal problem of Teichmüller in Chapter III D that even now can be found in few other sources. The fact that quasiconformal mappings turned out to be important tools in 2 and 3-dimensional geometry, complex dynamics and value distribution theory created a new audience for a book that provides a uniquely efficient introduction to the subject. It illustrates Ahlfors's remarkable ability to get straight to the heart of the matter and present major results with a minimum set of prerequisites.

The notes on which the book is based were written by Ahlfors himself. It was his practice in advanced courses to write thorough lecture notes (in longhand, with a fountain pen), leaving them after class in a ring binder in the mathematics library reading room for the benefit of the people attending the course.

With this practice in mind, Fred Gehring invited Ahlfors to publish the spring 1964 lecture notes in the new paperback book series *Van Nostrand Mathematical Studies* that he and Paul Halmos were editing. Ahlfors, in turn, invited his recent student Earle, who had completed his graduate studies and left Harvard shortly before 1964, to edit the longhand notes and see to their typing. The published text hews close to the original notes, and of course Ahlfors checked and approved the few alterations that were suggested.

Unfortunately, *Lectures on Quasiconformal Mappings* has been out of print for many years. We are grateful to the American Mathematical Society and the Ahlfors family for making it available once again. In this new edition, the original text has been typeset in TeX but is otherwise unchanged except for correction of some misprints and slips of the pen.

A new feature of this edition is an appendix consisting of three chapters. The first is chiefly devoted to further developments in the theory of Teichmüller spaces. The second, by Shishikura, describes how quasiconformal mappings have revitalized the subject of complex dynamics. The third, by Hubbard, illustrates the role of quasiconformal mappings in Thurston's theory of hyperbolic structures on 3-manifolds. All three chapters demonstrate the continuing importance of quasiconformal mappings in many different areas.

The theory of quasiconformal mappings has itself grown dramatically since the first edition of this book appeared. These developments cannot be described in a book of modest size. Fortunately, they are reported in many sources that will be readily accessible to any reader of this book. He or she will find references to a number of these sources in the early pages of our chapter in the appendix.

We are certain that the appendix will be useful to the reader. But our deepest admiration is reserved for the 1966 Lars Ahlfors manuscript and his remarkably influential 1964 course. The fact that after 40 years the Ahlfors book is being reprinted once again is a loud and clear message to the current generation of researchers.

December 2005,
Clifford J. Earle, Ithaca, New York,
Irwin Kra, Stony Brook, New York.

Contents

The Ahlfors Lectures

Acknowledgments

The manuscript was prepared by Dr. Clifford Earle from rough longhand notes of the author. He has contributed many essential corrections, checked computations and supplied many of the bridges that connect one fragment of thought with the next. Without his devoted help the manuscript would never have attained readable form.

In keeping with the informal character of this little volume there is no index and the references are very spotty, to say the least. The experts will know that the history of the subject is one of slow evolution in which the authorship of ideas cannot always be pinpointed.

The typing was excellently done by Mrs. Caroline W. Browne in Princeton, and financed by Air Force Grant AFOSR-393-63.

CHAPTER I

Differentiable Quasiconformal Mappings

Introduction

There are several reasons why quasiconformal mappings have recently come to play a very active part in the theory of analytic functions of a single complex variable.

1. The most superficial reason is that q.c. mappings are a natural generalization of conformal mappings. If this were their only claim they would soon have been forgotten.

2. It was noticed at an early stage that many theorems on conformal mappings use only the quasiconformality. It is therefore of some interest to determine when conformality is essential and when it is not.

3. Q.c. mappings are less rigid than conformal mappings and are therefore much easier to use as a tool. This was typical of the utilitarian phase of the theory. For instance, it was used to prove theorems about the conformal type of simply connected Riemann surfaces (now mostly forgotten).

4. Q.c. mappings play an important role in the study of certain elliptic partial differential equations.

5. Extremal problems in q.c. mappings lead to analytic functions connected with regions or Riemann surfaces. This was a deep and unexpected discovery due to Teichmüller.

6. The problem of moduli was solved with the help of q.c. mappings. They also throw light on Fuchsian and Kleinian groups.

7. Conformal mappings degenerate when generalized to several variables, but q.c. mappings do not. This theory is still in its infancy.

A. The Problem and Definition of Grötzsch

The notion of a quasiconformal mapping, but not the name, was introduced by H. Grötzsch in 1928. If Q is a square and R is a rectangle, not a square, there is no conformal mapping of Q on R which maps vertices on vertices. Instead, Grötzsch asks for the most nearly conformal mapping of this kind. This calls for a measure of approximate conformality, and in supplying such a measure Grötzsch took the first step toward the creation of a theory of q.c. mappings.

All the work of Grötzsch was late to gain recognition, and this particular idea was regarded as a curiosity and allowed to remain dormant for several years. It reappears in 1935 in the work of Lavrentiev, but from the point of view of partial differential equations. In 1936 I included a reference to the q.c. case in my theory of covering surfaces. From then on the notion became generally known, and in 1937 Teichmüller began to prove important theorems by use of q.c. mappings, and later theorems about q.c. mappings.

We return to the definition of Grötzsch. Let $w = f(z)$ ($z = x + iy$, $w = u + iv$) be a C^1 homeomorphism from one region to another. At a point z_0 it induces a linear mapping of the differentials

$$\begin{aligned} du &= u_x\,dx + u_y\,dy \\ dv &= v_x\,dx + v_y\,dy \end{aligned} \tag{1}$$

which we can also write in the complex form

$$dw = f_z dz + f_{\bar z} d\bar z \tag{2}$$

with

$$f_z = \tfrac{1}{2}(f_x - i f_y), \quad f_{\bar z} = \tfrac{1}{2}(f_x + i f_y). \tag{3}$$

Geometrically, (1) represents an affine transformation from the (dx, dy) to the (du, dv) plane. It maps circles about the origin into similar ellipses. We wish to compute the ratio between the axes as well as their direction.

In classical notation one writes

$$du^2 + dv^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2 \tag{4}$$

with

$$E = u_x^2 + v_x^2, \quad F = u_x u_y + v_x v_y, \quad G = u_y^2 + v_y^2.$$

The eigenvalues are determined from

$$\begin{vmatrix} E - \lambda & F \\ F & G - \lambda \end{vmatrix} = 0 \tag{5}$$

and are

$$\lambda_1, \lambda_2 = \frac{E + G \pm [(E - G)^2 + 4F^2]^{1/2}}{2}. \tag{6}$$

The ratio $a : b$ of the axes is

$$\left(\frac{\lambda_1}{\lambda_2}\right)^{1/2} = \frac{E + G + [(E - G)^2 + 4F^2]^{1/2}}{2(EG - F^2)^{1/2}}. \tag{7}$$

The complex notation is much more convenient. Let us first note that

$$\begin{aligned} f_z &= \tfrac{1}{2}(u_x + v_y) + \tfrac{i}{2}(v_x - u_y) \\ f_{\bar z} &= \tfrac{1}{2}(u_x - v_y) + \tfrac{i}{2}(v_x + u_y). \end{aligned} \tag{8}$$

This gives

$$|f_z|^2 - |f_{\bar z}|^2 = u_x v_y - u_y v_x = J \tag{9}$$

which is the Jacobian. The Jacobian is positive for sense preserving and negative for sense reversing mappings. For the moment we shall consider only the sense preserving case. Then $|f_{\bar z}| < |f_z|$.

It now follows immediately from (2) that

$$(|f_z| - |f_{\bar z}|)|dz| \le |dw| \le (|f_z| + |f_{\bar z}|)|dz| \tag{10}$$

where both limits can be attained. We conclude that the ratio of the major to the minor axis is

$$D_f = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \ge 1. \tag{11}$$

This is called the *dilatation* at the point z. It is often more convenient to consider

$$d_f = \frac{|f_{\bar{z}}|}{|f_z|} < 1 \tag{12}$$

related to D_f by

$$D_f = \frac{1+d_f}{1-d_f}, \quad d_f = \frac{D_f - 1}{D_f + 1}. \tag{13}$$

The mapping is conformal at z if and only if $D_f = 1$, $d_f = 0$.

The maximum is attained when the ratio

$$\frac{f_{\bar{z}} d\bar{z}}{f_z dz}$$

is positive, the minimum when it is negative. We introduce now the *complex dilatation*

$$\mu_f = \frac{f_{\bar{z}}}{f_z} \tag{14}$$

with $|\mu_f| = d_f$. The maximum corresponds to the direction

$$\arg dz = \alpha = \tfrac{1}{2} \arg \mu, \tag{15}$$

the minimum to the direction $\alpha \pm \pi/2$. In the dw-plane the direction of the major axis is

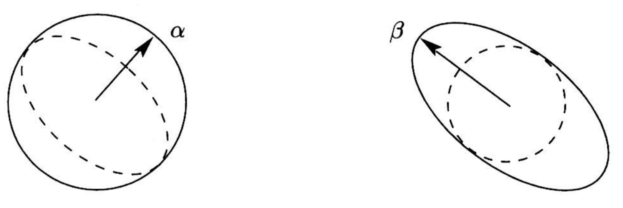
$$\arg dw = \beta = \tfrac{1}{2} \arg \nu \tag{16}$$

where we have set

$$\nu_f = \frac{f_{\bar{z}}}{\overline{f_{\bar{z}}}} = \left(\frac{f_z}{|f_z|}\right)^2 \mu_f. \tag{17}$$

The quantity ν_f may be called the *second complex dilatation.*

We will illustrate by the following self-explanatory figure:



Observe that $\beta - \alpha = \arg f_z$.

DEFINITION 1. The mapping f is said to be quasiconformal if D_f is bounded. It is K-quasiconformal if $D_f \le K$.

The condition $D_f \le K$ is equivalent to $d_f \le k = (K-1)/(K+1)$. A 1-quasiconformal mapping is conformal.

Let it be said at once that the restriction to C^1-mappings is most unnatural. One of our immediate aims is to get rid of this restriction. For the moment, however, we prefer to push this difficulty aside.

The usual rules are applicable and we find

$$(1)\qquad \begin{aligned}(g\circ f)_z &= (g_\zeta\circ f)f_z + (g_{\overline{\zeta}}\circ f)\overline{f}_z\\ (g\circ f)_{\overline{z}} &= (g_\zeta\circ f)f_{\overline{z}} + (g_{\overline{\zeta}}\circ f)\overline{f}_{\overline{z}}.\end{aligned}$$

When solved they give

$$(2)\qquad \begin{aligned}g_\zeta\circ f &= \frac{1}{J}[(g\circ f)_z\overline{f}_{\overline{z}} - (g\circ f)_{\overline{z}}\overline{f}_z]\\ g_{\overline{\zeta}}\circ f &= \frac{1}{J}[(g\circ f)_{\overline{z}}f_z - (g\circ f)_z f_{\overline{z}}]\end{aligned}$$

where $J = |f_z|^2 - |f_{\overline{z}}|^2$.

For $g = f^{-1}$ the formulas become

$$(3)\qquad (f^{-1})_\zeta\circ f = \overline{f}_{\overline{z}}/J, \quad (f^{-1})_{\overline{\zeta}}\circ f = -f_{\overline{z}}/J.$$

One derives, for instance,

$$(4)\qquad \mu_{f^{-1}} = -\nu_f\circ f^{-1}$$

and, on passing to the absolute values,

$$(5)\qquad d_{f^{-1}} = d_f\circ f^{-1}.$$

In other words, inverse mappings have the same dilatation at corresponding points.

From (2) we obtain

$$(6)\qquad \mu_g\circ f = \frac{f_z}{\overline{f}_{\overline{z}}}\frac{\mu_{g\circ f}-\mu_f}{1-\overline{\mu}_f\mu_{g\circ f}}.$$

If g is conformal, then $\mu_g = 0$ and we find

$$(7)\qquad \mu_{g\circ f} = \mu_f.$$

If f is conformal, $\mu_f = 0$ and

$$(8)\qquad \mu_g\circ f = \left(\frac{f'}{|f'|}\right)^2\mu_{g\circ f},$$

which can also be written as

$$(9)\qquad \nu_g\circ f = \nu_{g\circ f}.$$

In any case, the dilatation is invariant with respect to all conformal transformations.

If we set $g\circ f = h$ we find from (6)

$$(10)\qquad \mu_{h\circ f^{-1}}\circ f = \frac{f_z}{\overline{f}_{\overline{z}}}\frac{\mu_h-\mu_f}{1-\overline{\mu}_f\mu_h}.$$

For the dilatation

$$(11)\qquad d_{h\circ f^{-1}}\circ f = \left|\frac{\mu_h-\mu_f}{1-\mu_f\overline{\mu}_h}\right|$$

and

$$(12)\qquad \log D_{h\circ f^{-1}}\circ f = [\mu_h,\mu_f],$$

the non-euclidean distance (with respect to the metric $ds = \frac{2|dw|}{1-|w|^2}$ in $|w|<1$).

We can obviously use $\sup[\mu_h,\mu_f]$ as a distance between the mappings f and h (the Teichmüller distance). It is a metric provided one identifies mappings that differ by a conformal transformation.