Learning and Expectations in Macroeconomics

George W. Evans and Seppo Honkapohja

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To Pauline and Sirkku
and to our parents

"Good judgment comes from experience. Experience comes from bad judgment."

(Higdon's Law)

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Preface

This book provides a systematic treatment of the learning approach to modeling expectations formation in macroeconomics. This approach goes beyond rational expectations, the current standard hypothesis about expectations in macroeconomic theory. We focus on adaptive learning in which, at each moment of time, agents make forecasts using forecast functions formulated on the basis of available data, and these forecast functions are revised over time as new data become available. The body of this book is devoted to the statistical or econometric approach to learning which futher postulates that econometric techniques are used to estimate the parameters of the forecast functions.

Most of the research on adaptive learning within macroeconomics has been in this area, though other approaches have also been studied. While a number of surveys (including two by us) are available, a full treatise has been missing and this book aims to fill the gap.

Models of adaptive learning introduce a specific form of bounded rationality, as economic agents are assumed to maximize utility or profit given their forecasts at each moment of time, and the method used to estimate the forecast parameters is based on standard econometric techniques such as least squares. Rational expectations then becomes an equilibrium or fixed point of the learning dynamics, and in fact some early contributions have informally justified rational expectations as the outcome of a trial-and-error process. One can view the study of adaptive learning as making this justification explicit. In contrast, rational learning retains the rational expectations assumption continuously over time.

The study of adaptive learning offers much more than just a rationale for rational expectations. It provides a check on the robustness of equilibria with respect to expectational errors. It offers a way of selecting among multiple equilibria, which is a major conundrum for many rational expectations models. Another use of learning dynamics is computational, since the recursive algorithms provide a method for numerically computing equilibria. Learning also offers new possibilities for modeling dynamic macroeconomic phenomena. Some of these possible uses of adaptive learning in applied modeling have only recently been explored by researchers.

We have tried to provide a systematic treatment of the econometric approach to adaptive learning in a way that should be accessible and useful to both graduate students and professional economists. After an introductory part, the presentation covers both general techniques and their application to widely used standard frameworks. In several places we extend the state of knowledge in directions that should prove useful in both applied and theoretical macroeconomics. Chapter 10 contains a systematic treatment of adaptive learning for linear multivariate models. Until now this has not been systematically developed in a way that is useful for applications. In Chapter 13 and 14 we provide a discussion of the implications of misspecification in learning, including dynamics of learning with constant-gain algorithms. A further notable feature is Part II, in which the general mathematical techniques are explained in some detail.

Who Should Read This Book and How?

We have intended this book for two audiences. First, the book should be useful for professional economists interested in dynamic macroeconomic theory and applied macroeconomic modeling. This includes practitioners with some previous knowledge but who need a systematic treatment of the subject. It also includes graduate students who encounter the subject for the first time. As explained below, the book is structured so that it contains both an introduction and a systematic treatise of the subject.

The second audience consists of researchers and graduate students in other areas of economics and related fields in which the modeling of expectations is an integral part of their analytical frameworks. A case in point is financial economics in which expectations play a key role. We have not ventured far in this field (or others), though we do discuss the standard present-value model of asset princing.

Part I of the book provides an introduction to adaptive learning. The level of exposition is geared towards first-year graduate students who have some familiarity with modern dynamic macroeconomic theory. It starts with a historical overview and general discussion in Chapter 1, and this is followed by a leisurely introduction of the basic approach and technique in Chapter 2. Chapters 3 and 4 offer, respectively, some variations on the basic approach and an exposition on how to formulate and analyze learning in several standard models.

Part II is devoted to an exposition of the general mathematical technique on which adaptive learning is formally based. After a background chapter on basic

mathematical concepts, two main chapters provide a detailed discussion of the methods of stochastic approximation. Chapter 6 is a treatment of the basic stability and instability results for stochastic recursive algorithms. Chapter 7 contains a number of further developments including nonstochastic algorithms, a speed-of-convergence result for the usual algorithms, and some results for algorithms with constant gain. These last results have quite recently found application in economics. It should be noted that this part is technically much more demanding than other parts of the book. However, its detailed reading is not necessary since the formal theorems can be consulted when necessary.

Part III is a systematic discussion of linear models. Chapter 8 contains the most central results for standard univariate frameworks arising from many macroeconomic models. The level of discussion is relatively elementary, and this chapter is accessible to anyone who has read only through Part I. Chapter 9 takes up several further specialized topics for univariate linear models. Chapter 10 is devoted to multivariate linear models. This chapter develops the analysis of learning for general frameworks covering many macroeconomic models that appear frequently in the literature. Two appendices treat the linearization of multivariate models and the Blanchard–Kahn solution technique for both regular and irregular models.

In Part IV attention is directed at nonlinear models with an emphasis on stochastic frameworks that are appropriate when, for example, technology or preference shocks are present. Chapter 11 contains the basic stability and instability results for steady-state equilibria, and these are applied to a number of specific economic models. Chapter 12 takes up nonlinear models of endogenous fluctuations. Both periodic cycles and sunspot equilibria are considered, and these types of equilibria in several models are analyzed for stability under learning. Most of the material in this part should be accessible if the reader is familiar with only Part I.

Part V, the last part of the book, looks at extensions and recent developments. Chapters 13 and 14 consider a number of cases in which learning does not converge to a rational expectations equilibrium. This happens if agents are not using all relevant information or if they use a learning rule that does not fully converge because it allows for recurring structural shifts. Many of these issues have only quite recently been analyzed, and our treatment both introduces and contributes to the literature. Chapter 15 provides an overview of alternative approaches and some further issues. Finally, Chapter 16 offers some perspectives and conclusions on the subject.

Parts of the book are designed so that they can be used during a first-year graduate course in macroeconomics. Familiarity with standard rational expectations modeling is a prerequisite for Part I. This part, by itself or together with

Chapters 8, 11, and 15, would form an attractive introduction to the field for first- or second-year graduate students. Material in the other parts can be added for specialist courses. A web site with problem sets and other supplementary material is available at the address www.valt.helsinki.fi/geshbook/.

Acknowledgments

The book has grown out of our joint research since the late 1980s. The concrete impetus for writing the book came during the Nordic Research Course that we taught in Helsinki in the summer of 1995. Over the years we have given, both together and separately, courses and series of lectures on the subject at the London School of Economics, University of California at Los Angeles, University of Århus, Stockholm School of Economics, Norwegian School of Economics, Hitotsubashi University in Tokyo, Centre for Economic Studies in University of Munich, Bank of Finland, University of Helsinki, and University of Oregon. We have also given a large number of research seminars on our papers in the field. The comments and criticisms from the audiences on these occasions have helped us a great deal. The individuals are too numerous to list.

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Learning and Expectations in Macroeconomics

Part I

View of the Landscape

Chapter 1

Expectations and the Learning Approach

1.1 Expectations in Macroeconomics

Modern economic theory recognizes that the central difference between economics and natural sciences lies in the forward-looking decisions made by economic agents. In every segment of macroeconomics expectations play a key role. In consumption theory the paradigm life-cycle and permanent income approaches stress the role of expected future incomes. In investment decisions present-value calculations are conditional on expected future prices and sales. Asset prices (equity prices, interest rates, and exchange rates) clearly depend on expected future prices. Many other examples can be given.

Contemporary macroeconomics gives due weight to the role of expectations. A central aspect is that expectations influence the time path of the economy, and one might reasonably hypothesize that the time path of the economy influences expectations. The current standard methodology for modeling expectations is to assume *rational expectations* (RE), which is in fact an equilibrium in this two-sided relationship. Formally, in dynamic stochastic models, RE is usually defined as the mathematical conditional expectation of the relevant variables. The expectations are conditioned on all of the information available to the decision makers. For reasons that are well known, and which we will later explain, RE implicitly makes some rather strong assumptions.

Rational expectations modeling has been the latest step in a very long line of dynamic theories which have emphasized the role of expectations. The earliest references to economic expectations or forecasts date to the ancient Greek philosophers. In Politics (1259a), Aristotle recounts an anecdote concerning the pre-Socratic philosopher Thales of Miletus (c. 636–c. 546 B.C.). Forecasting one winter that there would be a great olive harvest in the coming year, Thales placed deposits for the use of all the olive presses in Chios and Miletus. He then made a large amount of money letting out the presses at high rates when the harvest time arrived.[1] Stories illustrating the importance of expectations in economic decision making can also be found in the Old Testament. In Genesis 41–47 we are told that Joseph (on behalf of the Pharaoh) took actions to store grain from years of good harvest in advance of years in which he forecasted famine. He was then able to sell the stored grains back during the famine years, eventually trading for livestock when the farmers' money ran out.[2]

Systematic economic theories or analyses in which expectations play a major role began as early as Henry Thornton's treatment of paper credit, published in 1802, and Émile Cheysson's 1887 formulation of a framework which had features of the "cobweb" cycle.[3] There is also some discussion of the role of expectations by the Classical Economists, but while they were interested in dynamic issues such as capital accumulation and growth, their method of analysis was essentially static. The economy was thought to be in a stationary state which can be seen as a sequence of static equilibria. A part of this interpretation was the notion of perfect foresight, so that expectations were equated with actual outcomes. This downplayed the significance of expectations.

Alfred Marshall extended the classical approach to incorporate the distinction between the short and the long run. He did not have a full dynamic theory, but he is credited with the notion of "static expectations" of prices. The first explicit analysis of stability in the cobweb model was made by Ezekiel (1938). Hicks (1939) is considered to be the key systematic exposition of the temporary equilibrium approach, initiated by the Stockholm school, in which expectations

[1] In giving this story, as well as another about a Sicilian who bought up all the iron from the iron mines, Aristotle also emphasized the advantage of creating a monopoly.

[2] The forecasting methods used in these stories provide an interesting contrast with those analyzed in this book. Thales is said to have relied on his skill in the stars, and Joseph's forecasts were based on the divine interpretation of dreams.

[3] This is pointed out in Schumpeter (1954, pp. 720 and 842, respectively). Hebert (1973) discusses Cheysson's formulation. The bibliographical references are Cheysson (1887) and Thornton (1939).

of future prices influence demands and supplies in a general equilibrium context.[4] Finally, Muth (1961) was the first to formulate explicitly the notion of rational expectations and did so in the context of the cobweb model.[5]

In macroeconomic contexts the importance of the state of long-term expectations of prospective yields for investment and asset prices was emphasized by Keynes in his General Theory.[6] Keynes emphasized the central role of expectations for the determination of investment, output, and employment. However, he often stressed the subjective basis for the state of confidence and did not provide an explicit model of how expectations are formed.[7] In the 1950s and 1960s expectations were introduced into almost every area of macroeconomics, including consumption, investment, money demand, and inflation. Typically, expectations were mechanically incorporated in macroeconomic modeling using adaptive expectations or related lag schemes. Rational expectations then made the decisive appearance in macroeconomics in the papers of Lucas (1972) and Sargent (1973).[8]

We will now illustrate some of these ways of modeling expectations with the aid of two well-known models. The first one is the cobweb model, though it may be noted that a version of the Lucas (1973) macroeconomic model is formally identical to it. The second is the well-known Cagan model of inflation (see Cagan, 1956). Some other models can be put in the same form, in particular versions of the present-value model of asset pricing.

These two examples are chosen for their familiarity and simplicity. This book will analyze a large number of macroeconomic models, including linear as well as nonlinear expectations models and univariate as well as multivariate models. Recent developments in modeling expectations have gone beyond rational expectations in specifying learning mechanisms which describe the evolution of expectation rules over time. The aim of this book is to develop systematically this new view of expectations formation and its implications for macroeconomic theory.

[4] Lindahl (1939) is perhaps the clearest discussion of the approach of the Stockholm school. Hicks (1965) has a discussion of the methods of dynamic analysis in the context of capital accumulation and growth. However, Hicks does not consider rational expectations.

[5] Sargent (1993) cites Hurwicz (1946) for the first use of the term "rational expectations."

[6] See Keynes (1936, Chapter 12).

[7] Some passages, particularly in Keynes (1937), suggest that attempting to forecast very distant future events can almost overwhelm rational calculation. For a forceful presentation of this view, see Loasby (1976, Chapter 9).

[8] Most of the early literature on rational expectations is collected in the volumes Lucas and Sargent (1981) and Lucas (1981).

1.2 Two Examples

1.2.1 The Cobweb Model

Consider a single competitive market in which there is a time lag in production. Demand is assumed to depend negatively on the prevailing market price

$$d_t = m_I - m_p p_t + v_{1t},$$

while supply depends positively on the expected price

$$s_t = r_I + r_p p_t^e + v_{2t},$$

where m_p, $r_p > 0$ and m_I and r_I denote the intercepts. We have introduced shocks to both demand and supply. v_{1t} and v_{2t} are unobserved white noise random variables. The interpretation of the supply function is that there is a one-period production lag, so that supply decisions for period t must be based on information available at time $t-1$. We will typically make the representative agent assumption that all agents have the same expectation, but at some points of the book we explicitly take up the issue of heterogeneous expectations. In the preceding equation p_t^e can be interpreted as the average expectation across firms.

We assume that markets clear, so that $s_t = d_t$. The reduced form for this model is

$$p_t = \mu + \alpha p_t^e + \eta_t, \qquad (1.1)$$

where $\mu = (m_I - r_I)/m_p$ and $\alpha = -r_p/m_p$. Note that $\alpha < 0$. $\eta_t = (v_{1t} - v_{2t})/m_p$ so that we can write $\eta_t \sim \text{iid}(0, \sigma_\eta^2)$. Equation (1.1) is an example of a temporary equilibrium relationship in which the current price depends on price expectations.

The well-known Lucas (1973) aggregate supply model can be put in the same form. Suppose that aggregate output is given by

$$q_t = \bar{q} + \pi(p_t - p_t^e) + \zeta_t,$$

where $\pi > 0$, while aggregate demand is given by the quantity theory equation

$$m_t + v_t = p_t + q_t,$$

where v_t is a velocity shock. Here all variables are in logarithmic form. Finally, assume that money supply is random around a constant mean

$$m_t = \bar{m} + u_t.$$

Here u_t, v_t, and ζ_t are white noise shocks. The reduced form for this model is

$$p_t = (1+\pi)^{-1}(\bar{m} - \bar{q}) + \pi(1+\pi)^{-1} p_t^e + (1+\pi)^{-1}(u_t + v_t - \zeta_t).$$

This equation is precisely of the same form as equation (1.1) with $\alpha = \pi(1+\pi)^{-1}$ and $\eta_t = (1+\pi)^{-1}(u_t + v_t - \zeta_t)$. Note that in this example $0 < \alpha < 1$.

Our formulation of the cobweb model has been made very simple for illustrative purposes. It can be readily generalized, e.g., to incorporate observable exogenous variables. This will be done in later chapters.

1.2.2 The Cagan Model

In a simple version of the Cagan model of inflation, the demand for money depends linearly on expected inflation,

$$m_t - p_t = -\psi(p_{t+1}^e - p_t), \qquad \psi > 0,$$

where m_t is the log of the money supply at time t, p_t is the log of the price level at time t, and p_{t+1}^e denotes the expectation of p_{t+1} formed in time t. We assume that m_t is iid with a constant mean. Solving for p_t, we get

$$p_t = \alpha p_{t+1}^e + \beta m_t, \qquad (1.2)$$

where $\alpha = \psi(1+\psi)^{-1}$ and $\beta = (1+\psi)^{-1}$.

The basic model of asset pricing under risk neutrality takes the same form. Under suitable assumptions all assets earn the expected rate of return $1 + r$, where $r > 0$ is the real net interest rate, assumed constant. If an asset pays dividend d_t at the beginning of period t, then its price p_t at t is given by $p_t = (1+r)^{-1} p_{t+1}^e + d_t$.[9] This is clearly of the same form as equation (1.2).

1.3 Classical Models of Expectation Formation

The reduced forms (1.1) and (1.2) of the preceding examples clearly illustrate the central role of expectations. Indeed, both of them show how the current

[9] See, e.g., Blanchard and Fischer (1989, pp. 215–216).

market-clearing price depends on expected prices. These reduced forms thus describe a temporary equilibrium which is conditioned by the expectations. Developments since the Stockholm School, Keynes, and Hicks can be seen as different theories of expectations formation, i.e., how to close the model so that it constitutes a fully specified dynamic theory. We now briefly describe some of the most widely used schemes with the aid of the examples.

Naive or static expectations were widely used in the early literature. In the context of the cobweb model they take the form of

$$p_t^e = p_{t-1}.$$

Once this is substituted into equation (1.1), one obtains $p_t = \mu + \alpha p_{t-1} + \eta_t$, which is a stochastic process known as an AR(1) process. In the early literature there were no random shocks, yielding a simple difference equation $p_t = \mu + \alpha p_{t-1}$. This immediately led to the question whether the generated sequence of prices converged to the stationary state over time. The convergence condition is, of course, $|\alpha| < 1$. Whether this is satisfied depends on the relative slopes of the demand and supply curves.[10] In the stochastic case this condition determines whether the price converges to a stationary stochastic process.

The origins of the *adaptive expectations* hypothesis can be traced back to Irving Fisher (see Fisher, 1930). It was formally introduced in the 1950s by Cagan (1956), Friedman (1957), and Nerlove (1958). In terms of the price level the hypothesis takes the form

$$p_t^e = p_{t-1}^e + \lambda(p_{t-1} - p_{t-1}^e),$$

and in the context of the cobweb model one obtains the system

$$p_t^e = (1 - \lambda(1-\alpha))p_{t-1}^e + \lambda\mu + \lambda\eta_{t-1}.$$

This is again an AR(1) process, now in the expectations p_t^e, which can be analyzed for stability or stationarity in the usual way.

Note that adaptive expectations can also be written in the form

$$p_t^e = \lambda \sum_{i=0}^{\infty} (1-\lambda)^i p_{t-1-i},$$

which is a distributed lag with exponentially declining weights. Besides adaptive expectations, other distributed lag formulations were used in the literature to allow for extrapolative or regressive elements. Adaptive expectations played a prominent role in macroeconomics in the 1960s and 1970s. For example, inflation expectations were often modeled adaptively in the analysis of the expectations-augmented Phillips curve.

[10] In the Lucas model the condition is automatically satisfied.

The *rational expectations* revolution begins with the observations that adaptive expectations, or any other fixed-weight distributed lag formula, may provide poor forecasts in certain contexts and that better forecast rules may be readily available. The optimal forecast method will in fact depend on the stochastic process which is followed by the variable being forecast, and as can be seen from our examples this implies an interdependency between the forecasting method and the economic model which must be solved explicitly. On this approach we write

$$p_t^e = E_{t-1} p_t \quad \text{and} \quad p_{t+1}^e = E_t p_{t+1}$$

for the cobweb example and for the Cagan model, respectively. Here $E_{t-1} p_t$ denotes the mathematical (statistical) expectation of p_t conditional on variables observable at time $t-1$ (including past data) and similarly $E_t p_{t+1}$ denotes the expectation of p_{t+1} conditional on information at time t.

We emphasize that rational expectations is in fact an equilibrium concept. The actual stochastic process followed by prices depends on the forecast rules used by agents, so that the optimal choice of the forecast rule by any agent is conditional on the choices of others. An RE equilibrium imposes the consistency condition that each agent's choice is a best response to the choices by others. In the simplest models we have representative agents and these choices are identical.

For the cobweb model we now have $p_t = \mu + \alpha E_{t-1} p_t + \eta_t$. Taking conditional expectations E_{t-1} of both sides yields $E_{t-1} p_t = \mu + \alpha E_{t-1} p_t$, so that expectations are given by $E_{t-1} p_t = (1-\alpha)^{-1}\mu$ and we have

$$p_t = (1-\alpha)^{-1}\mu + \eta_t.$$

(This step implicitly imposes the consistency condition described in the previous paragraph.) This is the unique way to form expectations which are "rational" in the model (1.1).

Similarly, in the Cagan model we have $p_t = \alpha E_t p_{t+1} + \beta m_t$ and if m_t is iid with mean $\bar{m}$, a rational expectations solution is $E_t p_{t+1} = (1-\alpha)^{-1}\beta\bar{m}$ and

$$p_t = (1-\alpha)^{-1}\alpha\beta\bar{m} + \beta m_t.$$

For this model there are in fact other rational expectations solutions, a point we will temporarily put aside but which we will discuss at length later in the book.

Two related observations should be made. First, under rational expectations the appropriate way to form expectations depends on the stochastic process followed by the exogenous variables η_t or m_t. If these are not iid processes, then the rational expectations will themselves be random variables, and they often form a complicated stochastic process. Second, it is apparent from our examples that neither static nor adaptive expectations are in general rational. Static or adaptive expectations will be "rational" only in certain special cases.

The rational expectations hypothesis became widely used in the 1970s and 1980s and it is now the benchmark paradigm in macroeconomics. In the 1990s, approaches incorporating learning behavior in expectation formation have been increasingly studied.

Paralleling the rational expectations modeling, there was further work refining the temporary equilibrium approach in general equilibrium theory. Much of this work focused on the existence of a temporary equilibrium for given expectation functions. However, the dynamics of sequences of temporary equilibria were also studied and this latter work is conceptually connected to the learning approach analyzed in this book.[11] The temporary equilibrium modeling was primarily developed using nonstochastic models, whereas the approach taken in this book emphasizes that economies are subject to random shocks.

1.4 Learning: The New View of Expectations

The rational expectations approach presupposes that economic agents have a great deal of knowledge about the economy. Even in our simple examples, in which expectations are constant, computing these constants requires the full knowledge of the structure of the model, the values of the parameters, and that the random shock is iid.[12] In empirical work economists, who postulate rational expectations, do not themselves know the parameter values and must estimate them econometrically. It appears more natural to assume that the agents in the economy face the same limitations on knowledge about the economy. This suggests that a more plausible view of rationality is that the agents act like statisticians or econometricians when doing the forecasting about the future state of the economy. This insight is the starting point of the adaptive learning approach to modeling expectations formation. This viewpoint introduces a specific form of "bounded rationality" to macroeconomics as discussed in Sargent (1993, Chapter 2).

[11] Many of the key papers on temporary equilibrium are collected in Grandmont (1988). A recent paper in this tradition, focusing on learning in a nonstochastic context, is Grandmont (1998).

[12] The strong assumptions required in the rational expectations hypothesis were widely discussed in the late 1970s and early 1980s; see, e.g., Blume, Bray, and Easley (1982), Frydman and Phelps (1983), and the references therein. Arrow (1986) has a good discussion of these issues.

More precisely, this viewpoint is called adaptive learning, since agents adjust their forecast rule as new data becomes available over time. There are alternative approaches to modeling learning, and we will explain their main features in Chapter 15. However, adaptive learning is the central focus of the book.

Taking this approach immediately raises the question of its relationship to rational expectations. It turns out that in many cases learning can provide at least an asymptotic justification for the RE hypothesis. For example, in the cobweb model with unobserved iid shocks, if agents estimate a constant expected value by computing the sample mean from past prices, one can show that expectations will converge over time to the RE value. This property turns out to be quite general for the cobweb-type models, provided agents use the appropriate econometric functional form. If the model includes exogenous observable variables or lagged endogenous variables, the agents will need to run regressions in the same way that an econometrician would.[13]

Another major advantage of the learning approach arises in connection with the issue of multiple equilibria. We have briefly alluded to the possibility that under the RE hypothesis the solution will not always be unique. To see this we consider a variation of the Cagan model $p_t = \alpha E_t p_{t+1} + \beta m_t$, where now money supply is assumed to follow a feedback rule $m_t = \bar{m} + \xi p_{t-1} + u_t$. This leads to the equation

$$p_t = \beta\bar{m} + \alpha E_t p_{t+1} + \beta\xi p_{t-1} + \beta u_t.$$

It can be shown that for many parameter values this equation yields two RE solutions of the form

$$p_t = k_1 + k_2 p_{t-1} + k_3 u_t, \tag{1.3}$$

where the k_i depend on the original parameters $\alpha, \beta, \xi, \bar{m}$. In some cases both of these solutions are even stochastically stationary.

[13] Bray (1982) was the first to provide a result showing convergence to rational expectations in a model in which expectations influence the economy and agents use an econometric procedure to update their expectations over time. Friedman (1979) and Taylor (1975) considered expectations which are formed using econometric procedures, but in contexts where expectations do not influence the economy. The final section in Chapter 2 provides a guide to the literature on learning.