CLASSICS IN MATHEMATICS

Saunders Mac Lane

Homology

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Springer

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www.wpcbj.com.cn

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Reprint of the 1975 Edition

Springer

Saunders Mac Lane
Department of Mathematics, University of Chicago
Chicago, IL 60637-1514
USA

Originally published as Vol. 114 of the
Grundlehren der mathematischen Wissenschaften

Mathematics Subject Classification (1991): 18-02, 18AXX, 18CXX, 18GXX

ISBN 3-540-58662-8 Springer-Verlag Berlin Heidelberg New York

CIP data applied for

图书在版编目（CIP）数据

同调＝Homology：英文/（美）莱恩著．—北京：世界图书出版公司北京公司，2009.8

ISBN 978-7-5100-0501-5

Ⅰ．同… Ⅱ．莱… Ⅲ．同调—研究生—教材—英文 Ⅳ．O189.22

中国版本图书馆 CIP 数据核字（2009）第 100931 号

书　　名：	Homology
作　　者：	Saunders Mac Lane
中 译 名：	同调
责任编辑：	高蓉
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号　100010）
联系电话：	010-64021602，010-64015659
电子信箱：	kjb@ wpcbj. com. cn
开　　本：	24 开
印　　张：	18.5
版　　次：	2009 年 08 月
版权登记：	图字：01-2009-1086
书　　号：	978-7-5100-0501-5/O·717　　**定　　价：** 50.00 元

Saunders Mac Lane was born on August 4, 1909 in Connecticut. He studied at Yale University and then at the University of Chicago and at Göttingen, where he received the D. Phil. in 1934. He has taught at Harvard, Cornell and the University of Chicago.

Mac Lane's initial research was in logic and in algebraic number theory (valuation theory). With Samuel Eilenberg he published fifteen papers on algebraic topology. A number of them involved the initial steps in the cohomology of groups and in other aspects of homological algebra – as well as the discovery of category theory. His famous undergraduate textbook *Survey of modern algebra,* written jointly with G. Birkhoff, has remained in print for over 50 years. Mac Lane is also the author of several other highly successful books.

To Dorothy

Preface

In presenting this treatment of homological algebra, it is a pleasure to acknowledge the help and encouragement which I have had from all sides. Homological algebra arose from many sources in algebra and topology. Decisive examples came from the study of group extensions and their factor sets, a subject I learned in joint work with OTTO SCHILLING. A further development of homological ideas, with a view to their topological applications, came in my long collaboration with SAMUEL EILENBERG; to both collaborators, especial thanks. For many years the Air Force Office of Scientific Research supported my research projects on various subjects now summarized here; it is a pleasure to acknowledge their lively understanding of basic science.

Both REINHOLD BAER and JOSEF SCHMID read and commented on my entire manuscript; their advice has led to many improvements. ANDERS KOCK and JACQUES RIGUET have read the entire galley proof and caught many slips and obscurities. Among the others whose suggestions have served me well, I note FRANK ADAMS, LOUIS AUSLANDER, WILFRED COCKCROFT, ALBRECHT DOLD, GEOFFREY HORROCKS, FRIEDRICH KASCH, JOHANN LEICHT, ARUNAS LIULEVICIUS, JOHN MOORE, DIETER PUPPE, JOSEPH YAO, and a number of my current students at the University of Chicago – not to mention the auditors of my lectures at Chicago, Heidelberg, Bonn, Frankfurt, and Aarhus. My wife, DOROTHY, has cheerfully typed more versions of more chapters than she would like to count. Messrs. SPRINGER have been unfailingly courteous in the preparation of the book; in particular, I am grateful to F. K. SCHMIDT, the Editor of this series, for his support. To all these and others who have helped me, I express my best thanks.

Chicago, 17. February 1963

SAUNDERS MAC LANE

Added Preface

In the third printing, several errors have been corrected. In particular, the previous erroneous construction of the splitting in the Homology classification theorem (Theorem III, 4.3) has been replaced by a correct proof, due essentially to DOLD (A. DOLD, *Lectures on Algebraic Topology,* Grundlehren der mathematischen Wissenschaften, vol. 200, Springer 1972). Also, the axioms on page 260 for allowable short exact sequences have been modified, so that they will actually apply where they are used on page 376.

SAUNDERS MAC LANE

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Introduction

Our subject starts with homology, homomorphisms, and tensors.

Homology provides an algebraic "picture" of topological spaces, assigning to each space X a family of abelian groups $H_0(X), \ldots, H_n(X), \ldots$, to each continuous map $f: X \to Y$ a family of group homomorphisms $f_n: H_n(X) \to H_n(Y)$. Properties of the space or the map can often be effectively found from properties of the groups H_n or the homomorphisms f_n. A similar process associates homology groups to other Mathematical objects; for example, to a group Π or to an associative algebra Λ. Homology in all such cases is our concern.

Complexes provide a means of calculating homology. Each n-dimensional "singular" simplex T in a topological space X has a boundary consisting of singular simplices of dimension $n-1$. If K_n is the free abelian group generated by all these n-simplices, the function ∂ assigning to each T the alternating sum ∂T of its boundary simplices determines a homomorphism $\partial: K_n \to K_{n-1}$. This yields (Chap. II) a "complex" which consists of abelian groups K_n and boundary homomorphisms ∂, in the form

$$0 \leftarrow K_0 \xleftarrow{\partial} K_1 \xleftarrow{\partial} K_2 \xleftarrow{\partial} K_3 \xleftarrow{\partial} \cdots.$$

Moreover, $\partial\partial = 0$, so the kernel C_n of $\partial: K_n \to K_{n-1}$ contains the image ∂K_{n+1}. The factor group $H_n(K) = C_n / \partial K_{n+1}$ is the n-th homology group of the complex K or of the underlying space X. Often a smaller or simpler complex will suffice to compute the same homology groups for X. Given a group Π, there is a corresponding complex whose homology is that appropriate to the group. For example, the one dimensional homology of Π is its factor commutator group $\Pi/[\Pi, \Pi]$.

Homomorphisms of appropriate type are associated with each type of algebraic system; under composition of homomorphisms the systems and their homomorphisms constitute a "category" (Chap. I). If C and A are abelian groups, the set $\operatorname{Hom}(C, A)$ of all group homomorphisms $f: C \to A$ is also an abelian group. For C fixed, it is a covariant "functor" on the category of all abelian groups A; each homomorphism $\alpha: A \to A'$ induces the map $\alpha_*: \operatorname{Hom}(C, A) \to \operatorname{Hom}(C, A')$ which carries each f into its composite αf with α. For A fixed, Hom is contravariant: Each $\gamma: C' \to C$ induces the map γ^* in the opposite direction, $\operatorname{Hom}(C, A) \to \operatorname{Hom}(C', A)$, sending f to the composite $f\gamma$. Thus $\operatorname{Hom}(?, A)$ applied to a complex $K=?$ turns the arrows around to give a complex

$$\mathrm{Hom}(K_0, A) \xrightarrow{\partial^*} \mathrm{Hom}(K_1, A) \xrightarrow{\partial^*} \mathrm{Hom}(K_2, A) \to \cdots.$$

Here the factor group (Kernel ∂^*)/(Image ∂^*) is the cohomology $H^n(K,A)$ of K with coefficients A. According to the provenance of K, it yields the cohomology of a space X or of a group Π.

An extension of a group A by a group C is a group $B \supset A$ with $B/A \cong C$; in diagramatic language, an extension is just a sequence

$$E: 0 \to A \to B \to C \to 0$$

of abelian groups and homomorphisms which is exact in the sense that the kernel of each homomorphism is exactly the image of the preceding one. The set $\mathrm{Ext}^1(C,A)$ of all extensions of A by C turns out to be an abelian group and a functor of A and C, covariant in A and contravariant in C.

Question: Does the homology of a complex K determine its cohomology? The answer is almost yes, provided each K_n is a free abelian group. In this case $H^n(K,A)$ is determined "up to a group extension" by $H_n(K)$, $H_{n-1}(K)$, and A; specifically, the "universal coefficient theorem" (Chap. III) gives an exact sequence

$$0 \to \mathrm{Ext}^1\big(H_{n-1}(K), A\big) \to H^n(K,A) \to \mathrm{Hom}\big(H_n(K), A\big) \to 0$$

involving the functor Ext^1 just introduced. If the K_n are not free groups, there is a more complex answer, involving the spectral sequences to be described in Chap. XI.

Tensors arise from vector spaces U, V, and W and bilinear functions $B(u,v)$ on $U \times V$ to W. Manufacture the vector space $U \otimes V$ generated by symbols $u \otimes v$ which are bilinear in $u \in U$ and $v \in V$ and nothing more. Then $u \otimes v$ is a universal bilinear function; to any bilinear B there is a unique linear transformation $T: U \otimes V \to W$ with $B(u,v) = T(u \otimes v)$. The elements of $V \otimes V$ turn out to be just the classical tensors (in two indices) associated with the vector space V. Two abelian groups A and G have a tensor product $A \otimes G$ generated by bilinear symbols $a \otimes g$; it is an abelian group, and a functor covariant in A and G. In particular, if K is a complex, so is $A \otimes K: A \otimes K_0 \leftarrow A \otimes K_1 \leftarrow \cdots$.

Question: Does the homology of K determine that of $A \otimes K$? Answer: Almost yes; if each K is free, there is an exact sequence

$$0 \to A \otimes H_n(K) \to H_n(A \otimes K) \to \mathrm{Tor}_1\big(A, H_{n-1}(K)\big) \to 0.$$

Here $\mathrm{Tor}_1(A,G)$ is a new covariant functor of the abelian groups A and G, called the "torsion product"; it depends (Chap. V) on the elements of finite order in A and G and is generated, subject to suitable relations, by pairs of elements $a \in A$ and $g \in G$ for which there is an integer m with $ma = 0 = mg$.

Take the cartesian product $X \times Y$ of two spaces. Can we calculate its homology from that of X and Y? A study of complexes constructed from simplices (Chap. VIII) reduces this question to the calculation of the homology of a tensor product $K \otimes L$ of two complexes. This calculation again involves the torsion product, via an exact sequence (the Künneth Thm, Chap. V)

$$0 \to \sum_{p+q=n} H_p(K) \otimes H_q(L) \to H_n(K \otimes L) \to \sum_{p+q=n-1} \operatorname{Tor}_1(H_p(K), H_q(L)) \to 0.$$

But woe, if A is a subgroup of B, $A \otimes G$ is not usually a subgroup of $B \otimes G$; in other words, if $E: 0 \to A \to B \to C \to 0$ is exact, the sequence of tensor products

$$0 \to A \otimes G \to B \otimes G \to C \otimes G \to 0,$$

is exact, *except* possibly at $A \otimes G$. Happily, the torsion product repairs the trouble; the given sequence E defines a homomorphism $E_*: \operatorname{Tor}_1(C,G) \to A \otimes G$ with image exactly the kernel of $A \otimes G \to B \otimes G$, and the sequence

$$0 \to \operatorname{Tor}_1(A, G) \to \operatorname{Tor}_1(B, G) \to \operatorname{Tor}_1(C, G) \xrightarrow{E_*} A \otimes G \to B \otimes G$$

is exact. Call E_* the connecting homomorphism for Tor_1 and $\otimes$.

But again woe, if A is a subgroup of B, a homomorphism $f: A \to G$ may not be extendable to a homomorphism $B \to G$; in other words, the exact sequence $0 \to A \to B \to C \to 0$ induces a sequence (opposite direction by contravariance!)

$$0 \to \operatorname{Hom}(C, G) \to \operatorname{Hom}(B, G) \to \operatorname{Hom}(A, G) \to 0$$

which may not be exact at $\operatorname{Hom}(A,G)$. Ext^1 to the rescue: There is a "connecting" homomorphism E^* which produces a longer exact sequence

$$0 \to \operatorname{Hom}(C, G) \to \operatorname{Hom}(B, G) \to \operatorname{Hom}(A, G) \xrightarrow{E^*}$$
$$\xrightarrow{E^*} \operatorname{Ext}^1(C, G) \to \operatorname{Ext}^1(B, G) \to \operatorname{Ext}^1(A, G) \to 0.$$

Now generalize; replace abelian groups by modules over any commutative ring R. Then $\operatorname{Ext}^1(A,G)$ ist still defined as an R-module, but the longer sequence may now fail of exactness at $\operatorname{Ext}^1(A,G)$. There is a new functor $\operatorname{Ext}^2(A,G)$, a new connecting homomorphism $E^*: \operatorname{Ext}^1(A,G) \to \operatorname{Ext}^2(C,G)$, and an exact sequence extending indefinitely to the right as

$$\cdots \to \operatorname{Ext}^n(C, G) \to \operatorname{Ext}^n(B, G) \to \operatorname{Ext}^n(A, G) \xrightarrow{E^*} \operatorname{Ext}^{n+1}(C, G) \to \cdots.$$

The elements of $\operatorname{Ext}^n(C, G)$ are suitable equivalence classes of long exact sequences

$$0 \to G \to B_{n-1} \to \cdots \to B_0 \to C \to 0$$

running from G to C through n intermediate modules. Similarly for the tensor product; there are functors $\operatorname{Tor}_n(A, G)$, described via suitable generators and relations, which enter into a long exact sequence

$$\cdots \to \operatorname{Tor}_{n+1}(C, G) \xrightarrow{E_*} \operatorname{Tor}_n(A, G) \to \operatorname{Tor}_n(B, G) \to \operatorname{Tor}_n(C, G) \to \cdots$$

induced by each $E: 0 \to A \to B \to C \to 0$. They apply also if the ring is not commutative — and A, B, and C are right R-modules, G a left R-module.

These functors Tor_n and Ext^n are the subject of homological algebra. They give the cohomology of various algebraic systems. If Π is a group, take R to be the group ring generated by Π over the integers. Then the group Z of integers is (trivially) an R-module; if A is any other R-module, the groups $\operatorname{Ext}_R^n(Z, A)$ are the cohomology groups $H^n(\Pi, A)$ of the group Π with coefficients in A. If $n=2$, $H^2(\Pi, A)$ turns out, as it should, to be the group of all extensions B of the abelian group A by the (non-abelian) group Π, where the structure of A as a Π-module specifies how A is a normal subgroup of B. If $n=3$, $H^3(\Pi, A)$ is a group whose elements are "obstructions" to an extension problem. Similarly, $\operatorname{Tor}_n(Z, A)$ gives the homology groups of Π. Again, if Λ is an algebra over the field F, construct Ext^n by long exact sequences of two-sided Λ-modules A. The algebra Λ is itself such a module, and $\operatorname{Ext}^n(\Lambda, A)$ is the cohomology of Λ with coefficients A; again Ext^2 and Ext^3 correspond to extension problems for algebras.

A module P is projective if every homomorphism $P \to B/A$ lifts to a homomorphism $P \to B$. Any free module is projective; write any module in terms of generators; this expresses it as a quotient of a free module, and hence of a projective module.

How can Tor_n and Ext^n be calculated? Write A as a quotient of a projective module P_0; that is, write an exact sequence $0 \leftarrow A \leftarrow P_0$. The kernel of $P_0 \to A$ is again a quotient of a projective P_1. This process continues to give an exact sequence $0 \leftarrow A \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$. The complex P is called a "projective resolution" of A. It is by no means unique; compare two such

$$\begin{array}{ccccccccc}
0 \leftarrow & A & \leftarrow & P_0 & \xleftarrow{\partial} & P_1 & \leftarrow & P_2 & \leftarrow \cdots \\
 & \| & & \downarrow f_0 & & \downarrow f_1 & & \downarrow & \\
0 \leftarrow & A & \leftarrow & P_0' & \xleftarrow{\partial} & P_1' & \leftarrow & P_2' & \leftarrow \cdots .
\end{array}$$

Since P_0 is projective, the map $P_0 \to A$ lifts to $f_0: P_0 \to P_0'$. The composite map $P_1 \to P_0'$ lifts in turn to an $f_1: P_1 \to P_1'$ with $\partial f_1 = f_0 \partial$, and so on by recursion. The resulting comparison $f_n: P_n \to P'_n$ of complexes induces a homomorphism $H_n(P \otimes G) \to H_n(P' \otimes G)$. Reversing the roles of P and P' and deforming $P \to P' \to P$ to the identity (deformations are called homotopies) shows this an isomorphism $H_n(P \otimes G) \cong H_n(P' \otimes G)$. Therefore the homology groups $H_n(P \otimes G)$ do not depend on the choice of the projective resolution P, but only on A and G. They turn out to be the groups $\mathrm{Tor}_n(A, G)$. Similarly, the cohomology groups $H^n(P, G)$ are the groups $\mathrm{Ext}^n(A, G)$, while the requisite connecting homomorphisms E^* may be obtained from a basic exact homology sequence for complexes (Chap. II). Thus Tor and Ext may be calculated from projective resolutions. For example, if Π is a group, the module Z has a standard "bar resolution" (Chap. IX) whose cohomology is that of Π. For particular groups, particular resolutions are more efficient.

Qualitative considerations ask for the minimum length of a projective resolution of an R-module A. If there is a projective resolution of A stopping with $P_{n+1} = 0$, A is said to have homological dimension at most n. These dimensions enter into the arithmetic structure of the ring R; for example, if R is the ring Z of integers, every module has dimension at most 1; again for example, the Hilbert Syzygy Theorem (Chap. VII) deals with dimensions of graded modules over a polynomial ring.

Two exact sequences $0 \to A \to B \to C \to 0$ and $0 \to C \to D \to F \to 0$ may be "spliced" at C to give a longer exact sequence

$$0 \to A \to B \dashrightarrow D \to F \to 0; \qquad B \searrow C \nearrow D$$

in other words, an element of $\mathrm{Ext}^1(C, A)$ and an element of $\mathrm{Ext}^1(F, C)$ determine a two-fold extension which is an element of $\mathrm{Ext}^2(F, A)$, called their product (Chap. III). These and similar products for Tor can be computed from resolutions (Chap. VIII).

Every R-module is also an abelian group; that is, a module over the ring Z of integers. Call an extension $E: A \to B \to C$ of R-modules Z-split if the middle module B, regarded just as an abelian group, is the direct sum of A and C. Construct the group $\mathrm{Ext}^1_{(R,Z)}(C, A)$ using only such Z-split extensions. This functor has connecting homomorphisms E^* for those E which are Z-split. With the corresponding torsion functors and their connecting homomorphisms, it is the subject of relative homological algebra (Chap. IX). The cohomology of a group is such a relative functor. Again, if Λ is an algebra over the commutative ring K, all appropriate concepts are relative to K; in particular, the cohomology of Λ arises from exact sequences of Λ-bimodules which are split as sequences of K-modules.

Modules appear to be the essential object of study. But the exactness of a resolution and the definition of a projective are properties of homomorphisms; all the arguments work if the modules and the homomorphisms are replaced by any objects $A, B, \ldots$ with "morphisms" $\alpha: A \to B$ which can be added, compounded, and have suitable kernels, cokernels $(B/\alpha A)$, and images. Technically, this amounts to developing homological algebra in an abelian category (Chap. IX). From the functor $T_0(A) = A \otimes G$ we constructed a sequence of functors $T_n(A) = \mathrm{Tor}_n(A, G)$. More generally, let T_0 be any covariant functor which is additive $[T_0(\alpha_1 + \alpha_2) = T_0\alpha_1 + T_0\alpha_2]$ and which carries each exact sequence $0 \to A \to B \to C \to 0$ into a right exact sequence $T_0(A) \to T_0(B) \to T_0(C) \to 0$. We again investigate the kernel of $T_0(A) \to T_0(B)$ and construct new functors to describe it. If the category has "enough" projectives, each A has a projective resolution P, and $H_n(T_0(P))$ is independent of the choice of P and defines a functor $T_n(A)$ which enters into a long exact sequence

$$\cdots \to T_n(A) \to T_n(B) \to T_n(C) \xrightarrow{E_*} T_{n-1}(A) \to \cdots.$$

Thus T_0 determines a whole sequence of derived functors T_n and of connecting homomorphisms $E_*: T_n(C) \to T_{n-1}(A)$. These "derived" functors can be characterized conceptually by three basic properties (Chap. XII):

(i) The long sequence above is exact,

(ii) If P is projective and $n > 0$, $T_n(P) = 0$,

(iii) If $E \to E'$ is a homomorphism of exact sequences, the diagram of connecting homomorphisms commutes (naturality!):

$$\begin{array}{ccc} T_n(C) & \to & T_{n-1}(A) \\ \downarrow & & \downarrow \\ T_n(C') & \to & T_{n-1}(A'). \end{array}$$

In particular, given $T_0(A) = A \otimes G$, these axioms characterize $\mathrm{Tor}_n(A, G)$ as functors of A. There is a similar characterization of the functors $\mathrm{Ext}^n(C, A)$ (Chap. III). Alternatively, each derived functor T_n can be characterized just in terms of the preceding T_{n-1}: If $E: S_n(C) \to S_{n-1}(A)$ is another natural connecting homomorphism between additive functors, each "natural" map of S_{n-1} into T_{n-1} extends to a unique natural map of S_n into T_n. This "universal" property of T_n describes it as the left satellite of T_{n-1}; it may be used to construct products.

Successive and interlocking layers of generalizations appear throughout homological algebra. We go from abelian groups to modules to bimodules to objects in an abelian category; from rings to groups to algebras to Hopf algebras (Chap. VI); from exact sequences to Z-split