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CONFORMAL INVARIANTS
Topics in Geometric Function Theory

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Preface

This is a textbook primarily intended for students with approximately a year's background in complex variable theory. The material has been collected from lecture courses given over a long period of years, mostly at Harvard University. The book emphasizes classic and semiclassic results which the author feels every student of complex analysis should know before embarking on independent research. The selection of topics is rather arbitrary, but reflects the author's preference for the geometric approach. There is no attempt to cover recent advances in more specialized directions.

Most conformal invariants can be described in terms of extremal properties. Conformal invariants and extremal problems are therefore intimately linked and form together the central theme of this book. An obvious reason for publishing these lectures is the fact that much of the material has never appeared in textbook form. In particular this is true of the theory of extremal length, instigated by Arne Beurling, which should really be the subject of a monograph of its own, preferably by Beurling himself. Another topic that has received only scant attention in the textbook literature is Schiffer's variational method, which I have tried to cover as carefully and as thoroughly as I know how. I hope very much that this account will prove readable. I have also included a new proof of $|a_4| \leq 4$ which appeared earlier in a *Festschrift* for M. A. Lavrentiev (in Russian).

The last two chapters, on Riemann surfaces, stand somewhat apart from the rest of the book. They are motivated by the need for a quicker approach to the uniformization theorem than can be obtained from Leo Sario's and my book "Riemann Surfaces."

Some early lectures of mine at Oklahoma A. and M. College had been transcribed by R. Osserman and M. Gerstenhaber, as was a lecture at Harvard University on extremal methods by E. Schlesinger. These writeups were of great help in assembling the present version. I also express my gratitude to F. Gehring without whose encouragement I would not have gone ahead with publication.

There is some overlap with Makoto Ohtsuka's book "Dirichlet Problem, Extremal Length and Prime Ends" (Van Nostrand, 1970) which is partly based on my lectures at Harvard University and in Japan.

Lars V. Ahlfors

Contents

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APPLICATIONS OF SCHWARZ'S LEMMA

1-1 THE NONEUCLIDEAN METRIC

The fractional linear transformation

$$S(z) = \frac{az + b}{\bar{b}z + \bar{a}} \tag{1-1}$$

with $|a|^2 - |b|^2 = 1$ maps the unit disk $\Delta = \{z; |z| < 1\}$ conformally onto itself. It is also customary to write (1-1) in the form

$$S(z) = e^{i\alpha} \frac{z - z_0}{1 - \bar{z}_0 z} \tag{1-2}$$

which has the advantage of exhibiting $z_0 = S^{-1}(0)$ and $\alpha = \arg S'(0)$.

Consider $z_1, z_2 \in \Delta$ and set $w_1 = S(z_1)$, $w_2 = S(z_2)$. From (1-1) we obtain

$$w_1 - w_2 = \frac{z_1 - z_2}{(\bar{b}z_1 + \bar{a})(\bar{b}z_2 + \bar{a})}$$

$$1 - \bar{w}_1 w_2 = \frac{1 - \bar{z}_1 z_2}{(b\bar{z}_1 + a)(\bar{b}z_2 + \bar{a})},$$

and hence
$$\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| = \left|\frac{w_1 - w_2}{1 - \bar{w}_1 w_2}\right|. \tag{1-3}$$

We say that
$$\delta(z_1,z_2) = \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| \tag{1-4}$$

is a *conformal invariant*. Comparison of (1-2) and (1-4) shows that $\delta(z_1,z_2) < 1$, a fact that can also be read off from the useful identity

$$1 - \delta(z_1,z_2)^2 = \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|1 - \bar{z}_1 z_2|^2}.$$

If z_1 approaches z_2, (1-3) becomes

$$\frac{|dz|}{1 - |z|^2} = \frac{|dw|}{1 - |w|^2}.$$

This shows that the Riemannian metric whose element of length is

$$ds = \frac{2|dz|}{1 - |z|^2} \tag{1-5}$$

is invariant under conformal self-mappings of the disk (the reason for the factor 2 will become apparent later). In this metric every rectifiable arc γ has an invariant length

$$\int_\gamma \frac{2|dz|}{1 - |z|^2},$$

and every measurable set E has an invariant area

$$\iint_E \frac{4dx\,dy}{(1 - |z|^2)^2}.$$

The shortest arc from 0 to any other point is along a radius. Hence the geodesics are circles orthogonal to $|z| = 1$. They can be considered straight lines in a geometry, the *hyperbolic* or *noneuclidean* geometry of the disk.

The noneuclidean distance from 0 to $r > 0$ is

$$\int_0^r \frac{2dr}{1 - r^2} = \log\frac{1 + r}{1 - r}.$$

Since $\delta(0,r) = r$, it follows that the noneuclidean distance $d(z_1,z_2)$ is connected with $\delta(z_1,z_2)$ through $\delta = \tanh(d/2)$.

The noneuclidean geometry can also be carried over to the half plane

$H = \{z = x + iy; y > 0\}$. The element of length that corresponds to the choice (1-5) is

$$ds = \frac{|dz|}{y}, \tag{1-6}$$

and the straight lines are circles and lines orthogonal to the real axis.

1-2 THE SCHWARZ–PICK THEOREM

The classic Schwarz lemma asserts the following: If f is analytic and $|f(z)| < 1$ for $|z| < 1$, and if $f(0) = 0$, then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality $|f(z)| = |z|$ with $z \neq 0$ or $|f'(0)| = 1$ can occur only for $f(z) = e^{i\alpha}z$, α a real constant.

There is no need to reproduce the well-known proof. It was noted by Pick that the result can be expressed in invariant form.

Theorem 1-1 An analytic mapping of the unit disk into itself decreases the noneuclidean distance between two points, the noneuclidean length of an arc, and the noneuclidean area of a set.

The explicit inequalities are

$$\frac{|f(z_1) - f(z_2)|}{|1 - \overline{f(z_1)}f(z_2)|} \leq \frac{|z_1 - z_2|}{|1 - \bar{z}_1 z_2|}$$

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

Nontrivial equality holds only when f is a fractional linear transformation of the form (1-1).

Pick does not stop with this observation. He also proves the following more general version which deserves to be better known.

Theorem 1-2 Let f: $\Delta \to \Delta$ be analytic and set $w_k = f(z_k)$, $k = 1, \ldots, n$. Then the Hermitian form

$$Q_n(t) = \sum_{h,k=1}^{n} \frac{1 - w_h \bar{w}_k}{1 - z_h \bar{z}_k} t_h \bar{t}_k$$

is positive definite (or semidefinite).

PROOF We assume first that f is analytic on the closed disk. The function $F = (1 + f)/(1 - f)$ has a positive real part, and if $F = U + iV$

we have the representation

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} U(e^{i\theta})\, d\theta + iV(0).$$

This gives

$$F(z_h) + \overline{F(z_k)} = \frac{1}{\pi} \int_0^{2\pi} \frac{1 - z_h \bar{z}_k}{(e^{i\theta} - z_h)(e^{-i\theta} - \bar{z}_k)} U\, d\theta,$$

and hence

$$\sum_{h,k=1}^{n} \frac{F_h + \bar{F}_k}{1 - z_h \bar{z}_k} t_h \bar{t}_k = \frac{1}{\pi} \int_0^{2\pi} \left| \sum_1^n \frac{t_k}{e^{i\theta} - z_k} \right|^2 U\, d\theta \geq 0.$$

Here $F_h + \bar{F}_k = 2(1 - w_h \bar{w}_k)/(1 - w_h)(1 - \bar{w}_k)$. The factors in the denominator can be incorporated in t_h, $\bar{t}_k$, and we conclude that $Q_n(t) \geq 0$. For arbitrary f we apply the theorem to $f(rz)$, $0 < r < 1$, and pass to the limit.

Explicitly, the condition means that all the determinants

$$D_k = \begin{vmatrix} \dfrac{1 - |w_1|^2}{1 - |z_1|^2} & \cdots & \dfrac{1 - w_1 \bar{w}_k}{1 - z_1 \bar{z}_k} \\ \cdots & \cdots & \cdots \\ \dfrac{1 - w_k \bar{w}_1}{1 - z_k \bar{z}_1} & \cdots & \dfrac{1 - |w_k|^2}{1 - |z_k|^2} \end{vmatrix}$$

are ≥ 0. It can be shown that these conditions are also sufficient for the interpolation problem to have a solution. If $w_1, \ldots, w_{n-1}$ are given and $D_1, \ldots, D_{n-1} \geq 0$, the condition on w_n will be of the form $|w_n|^2 + 2 \operatorname{Re}(a w_n) + b \leq 0$. This means that w_n is restricted to a certain closed disk. It turns out that the disk reduces to a point if and only if $D_{n-1} = 0$.

The proof of the sufficiency is somewhat complicated and would lead too far from our central theme. We shall be content to show, by a method due to R. Nevanlinna, that the possible values of w_n fill a closed disk. We do not prove that this disk is determined by $D_n \geq 0$.

Nevanlinna's reasoning is recursive. For $n = 1$ there is very little to prove. Indeed, there is no solution if $|w_1| > 1$. If $|w_1| = 1$ there is a unique solution, namely, the constant w_1. If $|w_1| < 1$ and f_1 is a solution, then

$$f_2(z) = \frac{f_1(z) - w_1}{1 - \bar{w}_1 f_1(z)} : \frac{z - z_1}{1 - \bar{z}_1 z} \tag{1-7}$$

is regular in Δ, and we have proved that $|f_2(z)| \leq 1$. Conversely, for any such function f_2 formula (1-7) yields a solution f_1.

For $n = 2$ the solutions, if any, are among the functions f_1 already determined, and $f_2(z_2)$ must be equal to a prescribed value $w_2^{(2)}$. There are the same alternatives as before, and it is clear how the process continues. We are trying to construct a sequence of functions f_k of modulus ≤ 1 with certain prescribed values $f_k(z_k) = w_k^{(k)}$ which can be calculated from $w_1, \ldots, w_k$. If $|w_k^{(k)}| > 1$ for some k, the process comes to a halt and there is no solution. If $|w_k^{(k)}| = 1$, there is a unique f_k, and hence a unique solution of the interpolation problem restricted to $z_1, \ldots, z_k$. In case all $|w_k^{(k)}| < 1$, the recursive relations

$$f_{k+1}(z) = \frac{f_k(z) - w_k^{(k)}}{1 - \bar{w}_k^{(k)} f_k(z)} : \frac{z - z_k}{1 - \bar{z}_k z} \qquad k = 1, \ldots, n$$

lead to all solutions f_1 of the original problem when f_{n+1} ranges over all analytic functions with $|f_{n+1}(z)| \leq 1$ in Δ.

Because the connection between f_k and f_{k+1} is given as a fractional linear transformation, the general solution is of the form

$$f_1(z) = \frac{A_n(z)f_{n+1}(z) + B_n(z)}{C_n(z)f_{n+1}(z) + D_n(z)},$$

where A_n, B_n, C_n, D_n are polynomials of degree n determined by the data of the problem. We recognize now that the possible values of $f(z)$ at a fixed point do indeed range over a closed disk.

This solution was given in R. Nevanlinna [42]. The corresponding problem for infinitely many z_k, w_k was studied by Denjoy [17], R. Nevanlinna [43], and more recently Carleson [13].

1-3 CONVEX REGIONS

A set is convex if it contains the line segment between any two of its points. We wish to characterize the analytic functions f that define a one-to-one conformal map of the unit disk on a convex region. For simplicity such functions will be called convex univalent (Hayman [27]).

Theorem 1-3 An analytic function f in Δ is convex univalent if and only if

$$\operatorname{Re} \frac{zf''(z)}{f'(z)} \geq -1 \tag{1-8}$$

for all $z \in \Delta$. When this is true the stronger inequality

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2|z|^2}{1 - |z|^2} \right| \leq \frac{2|z|}{1 - |z|^2} \tag{1-9}$$

is also in force.

Suppose for a moment that f is not only convex univalent but also analytic on the closed disk. It is intuitively clear that the image of the unit circle has a tangent which turns in the positive direction when $\theta = \arg z$ increases. This condition is expressed through $\partial/\partial\theta \arg df \geq 0$. But $\arg df = \arg f' + \arg dz = \arg f' + \theta + \pi/2$, and the condition becomes $\partial/\partial\theta\,(\arg f' + \theta) = \operatorname{Re}\,(zf''/f' + 1) \geq 0$ for $|z| = 1$. By the maximum principle the same holds for $|z| < 1$.

Although this could be made into a rigorous proof, we much prefer an idea due to Hayman. We may assume that $f(0) = 0$. If f is convex univalent, the function

$$g(z) = f^{-1}\left[\frac{f(\sqrt{z}) + f(-\sqrt{z})}{2}\right]$$

is well defined, analytic, and of absolute value <1 in Δ. Hence $|g'(0)| \leq 1$. But if $f(z) = a_1z + a_2z^2 + \cdots$, then $g(z) = (a_2/a_1)z + \cdots$, and we obtain $|a_2/a_1| \leq 1$, $|f''(0)/f'(0)| \leq 2$. This is (1-9) for $z = 0$.

We apply this result to $F(z) = f[(z + c)/(1 + \bar{c}z)]$, $|c| < 1$, which maps Δ on the same region. Simple calculations give

$$\frac{F''(0)}{F'(0)} = \frac{f''(c)}{f'(c)}\,(1 - |c|^2) - 2\bar{c},$$

and we obtain (1-9) and its consequence (1-8).

The proof of the converse is less elegant. It is evidently sufficient to prove that the image of $\Delta_r = \{z;\,|z| < r\}$ is convex for every $r < 1$. The assumption (1-8) implies that $\arg df$ increases with θ on $|z| = r$. Since f' is never zero, the change of $\arg df$ is 2π. Therefore, we can find θ_1 and θ_2 such that $\arg df$ increases from 0 to π on $[\theta_1,\theta_2]$ and from π to 2π on $[\theta_2,\theta_1 + 2\pi]$. If $f(re^{i\theta}) = u(\theta) + iv(\theta)$, it follows that v increases on the first interval and decreases on the second. Let v_0 be a real number between the minimum $v(\theta_1)$ and the maximum $v(\theta_2)$. Then $v(\theta)$ passes through v_0 exactly once on each of the intervals, and routine use of winding numbers shows that the image of Δ_r intersects the line $v = v_0$ along a single segment. The same reasoning applies to parallels in any direction, and we conclude that the image is convex.

The condition $|f''(0)/f'(0)| \leq 2$ has an interesting geometric interpretation. Consider an arc γ in Δ that passes through the origin and whose image is a straight line. The curvature of γ is measured by $d(\arg dz)/|dz|$. By assumption $d(\arg df) = 0$ along γ so that $d(\arg dz) = -d \arg f'$. The curvature is thus a directional derivative of $\arg f'$, and as such it is at most $|f''/f'|$ in absolute value. We conclude that the curvature at the origin is at most 2.

This result has an invariant formulation. If the curvature at the origin is ≤ 2, the circle of curvature intersects $|z| = 1$. But the circle of curvature is the circle of highest contact. A conformal self-mapping preserves circles and preserves order of contact. Circles of curvature are mapped on circles of curvature, and our result holds not only at the origin, but at any point.

Theorem 1-4 Let γ be a curve in Δ whose image under a conformal mapping on a convex region is a straight line. Then the circles of curvature of γ meet $|z| = 1$.

This beautiful result is due to Carathéodory.

1-4 ANGULAR DERIVATIVES

For $|a| < 1$ and $R < 1$ let $K(a,R)$ be the set of all z such that

$$\left| \frac{z - a}{1 - \bar{a}z} \right| < R.$$

Clearly, $K(a,R)$ is an open noneuclidean disk with center a and radius d such that $R = \tanh (d/2)$.

Let $K_n = K(z_n,R_n)$ be a sequence of disks such that $z_n \to 1$ and

$$\frac{1 - |z_n|}{1 - R_n} \to k \neq 0, \infty. \tag{1-10}$$

We claim that the K_n tend to the *horocycle* K_∞ defined by

$$\frac{|1 - z|^2}{1 - |z|^2} < k. \tag{1-11}$$

The horocycle is a disk tangent to the unit circle at $z = 1$.

The statement $K_n \to K_\infty$ is to be understood in the following sense: (1) If $z \in K_n$ for infinitely many n, then $z \in \bar{K}_\infty$, the closure of K_∞; (2) if $z \in K_\infty$, then $z \in K_n$ for all sufficiently large n. For the proof we observe that $z \in K_n$ is equivalent to

$$\frac{|1 - \bar{z}_n z|^2}{1 - |z|^2} < \frac{1 - |z_n|^2}{1 - R_n^2}. \tag{1-12}$$

If this is true for infinitely many n, we can go to the limit and obtain (1-11) by virtue of (1-10), except that equality may hold. Conversely, if

(1-11) holds, then

$$\lim_{n\to\infty} \frac{|1 - \bar{z}_n z|^2}{1 - |z|^2} < k$$

while

$$\lim_{n\to\infty} \frac{1 - |z_n|^2}{1 - R_n^2} = k,$$

so that (1-12) must hold for all sufficiently large n.

After these preliminaries, let f be analytic and $|f(z)| < 1$ in Δ. Suppose that $z_n \to 1$, $f(z_n) \to 1$, and

$$\frac{1 - |f(z_n)|}{1 - |z_n|} \to \alpha \neq \infty. \tag{1-13}$$

Given $k > 0$ we choose R_n so that $(1 - |z_n|)/(1 - R_n) = k$; this makes $0 < R_n < 1$ provided $1 - |z_n| < k$. With the same notation

$$K_n = K(z_n, R_n)$$

as above, we know by Schwarz's lemma that $f(K_n) \subset K_n' = K(w_n, R_n)$ where $w_n = f(z_n)$. The K_n converge to the horocycle K_∞ with parameter k as in (1-11), and because $(1 - |w_n|)/(1 - R_n) \to \alpha k$, the K_n' converge to K_∞' with parameter αk. If $z \in K_\infty$, it belongs to infinitely many K_n. Hence $f(z)$ belongs to infinitely many K_n' and consequently to $\bar{K}_\infty'$. In view of the continuity it follows that

$$\frac{|1 - z|^2}{1 - |z|^2} \leq k \qquad \text{implies} \qquad \frac{|1 - f(z)|^2}{1 - |f(z)|^2} \leq \alpha k.$$

This is known as *Julia's lemma.*

Since k is arbitrary, the same result may be expressed by

$$\frac{|1 - f(z)|^2}{1 - |f(z)|^2} \leq \alpha \frac{|1 - z|^2}{1 - |z|^2},$$

or by

$$\beta = \sup \left[\frac{|1 - f(z)|^2}{1 - |f(z)|^2} : \frac{|1 - z|^2}{1 - |z|^2} \right] \leq \alpha.$$

In particular, α is never 0, and if $\beta = \infty$, there is no finite α.

Let us now assume $\beta < \infty$ and take $z_n = x_n$ to be real. Then

$$|1 - w_n|^2 < \beta \frac{1 - x_n}{1 + x_n},$$

and the condition $w_n \to 1$ is automatically fulfilled. Furthermore,

$$\beta \geq \frac{|1 - w_n|^2}{1 - |w_n|^2} \frac{1 + x_n}{1 - x_n} \geq \frac{1 + x_n}{1 + |w_n|} \frac{|1 - w_n|}{1 - x_n} \geq \frac{1 + x_n}{1 + |w_n|} \frac{1 - |w_n|}{1 - x_n}$$