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Vladimir I. Arnold

Lectures on Partial Differential Equations

偏微分方程讲义

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Translated by Roger Cooke

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Preface to the Second Russian Edition

In the mid-twentieth century the theory of partial differential equations was considered the summit of mathematics, both because of the difficulty and significance of the problems it solved and because it came into existence later than most areas of mathematics.

Nowadays many are inclined to look disparagingly at this remarkable area of mathematics as an old-fashioned art of juggling inequalities or as a testing ground for applications of functional analysis. Courses in this subject have even disappeared from the obligatory program of many universities (for example, in Paris). Moreover, such remarkable textbooks as the classical three-volume work of Goursat have been removed as superfluous from the library of the University of Paris-7 (and only through my own intervention was it possible to save them, along with the lectures of Klein, Picard, Hermite, Darboux, Jordan, ...).

The cause of this degeneration of an important general mathematical theory into an endless stream of papers bearing titles like "On a property of a solution of a boundary-value problem for an equation" is most likely the attempt to create a unified, all-encompassing, superabstract "theory of everything."

The principal source of partial differential equations is found in the continuous-medium models of mathematical and theoretical physics. Attempts to extend the remarkable achievements of mathematical physics to systems that match its models only formally lead to complicated theories that are difficult to visualize as a whole, just as attempts to extend the geometry of second-order surfaces and the algebra of quadratic forms to objects of higher degrees quickly leads to the detritus of algebraic geometry with its discouraging hierarchy of complicated degeneracies and answers that can be computed only theoretically.

The situation is even worse in the theory of partial differential equations: here the difficulties of commutative algebraic geometry are inextricably bound up with noncommutative differential algebra, in addition to which the topological and analytic problems that arise are profoundly nontrivial.

At the same time, general physical principles and also general concepts such as energy, the variational principle, Huygens' principle, the Lagrangian, the Legendre transformation, the Hamiltonian, eigenvalues and eigenfunctions, wave-particle duality, dispersion relations, and fundamental solutions interact elegantly in numerous highly important problems of mathematical physics. The study of these problems motivated the development of large areas of mathematics such as the theory of Fourier series and integrals, functional analysis, algebraic geometry, symplectic and contact topology, the theory of asymptotics of integrals, microlocal analysis, the index theory of (pseudo-)differential operators, and so forth.

Familiarity with these fundamental mathematical ideas is, in my view, absolutely essential for every working mathematician. The exclusion of them from the university mathematical curriculum, which has occurred and continues to occur in many Western universities under the influence of the axiomaticist/scholastics (who know nothing about applications and have no desire to know anything except the "abstract nonsense" of the algebraists) seems to me to be an extremely dangerous consequence of Bourbakization of both mathematics and its teaching. The effort to destroy this unnecessary scholastic pseudoscience is a natural and proper reaction of society (including scientific society) to the irresponsible and self-destructive aggressiveness of the "superpure" mathematicians educated in the spirit of Hardy and Bourbaki.

The author of this very short course of lectures has attempted to make students of mathematics with minimal knowledge (linear algebra and the basics of analysis, including ordinary differential equations) acquainted with a kaleidoscope of fundamental ideas of mathematics and physics. Instead of the principle of maximal generality that is usual in mathematical books the author has attempted to adhere to the principle of minimal generality, according to which every idea should first be clearly understood in the simplest situation; only then can the method developed be extended to more complicated cases.

Although it is usually simpler to prove a general fact than to prove numerous special cases of it, for a student the content of a mathematical theory is never larger than the set of examples that are thoroughly understood. That is why it is examples and ideas, rather than general theorems and axioms, that form the basis of this book. The examination problems at the end of the course form an essential part of it.

Particular attention has been devoted to the interaction of the subject with other areas of mathematics: the geometry of manifolds, symplectic and contact geometry, complex analysis, calculus of variations, and topology. The author has aimed at a student who is eager to learn, but hopes that through this book even professional mathematicians in other specialties can become acquainted with the basic and therefore simple ideas of mathematical physics and the theory of partial differential equations.

The present course of lectures was delivered to third-year students in the Mathematical College of the Independent University of Moscow during the fall semester of the 1994/1995 academic year, Lectures 4 and 5 having been delivered by Yu. S. Il'yashenko and Lecture 8 by A. G. Khovanskiĭ. All the lectures were written up by V. M. Imaĭkin, and the assembled lectures were then revised by the author. The author is deeply grateful to all of them.

The first edition of this course appeared in 1995, published by the press of the Mathematical College of the Independent University of Moscow. A number of additions and corrections have been made in the present edition.

Contents

Lecture 1

The General Theory for One First-Order Equation

In contrast to ordinary differential equations, there is no unified theory of partial differential equations. Some equations have their own theories, while others have no theory at all. The reason for this complexity is a more complicated geometry. In the case of an ordinary differential equation a locally integrable vector field (that is, one having integral curves) is defined on a manifold. For a partial differential equation a subspace of the tangent space of dimension greater than 1 is defined at each point of the manifold. As is known, even a field of two-dimensional planes in three-dimensional space is in general not integrable.

Example. In a space with coordinates x, y, and z we consider the field of planes given by the equation $\mathrm{d}z = y\,\mathrm{d}x$. (This gives a linear equation for the coordinates of the tangent vector at each point, and that equation determines a plane.)

Problem 1. Draw this field of planes and prove that it has no integral surface, that is, no surface whose tangent plane at every point coincides with the plane of the field.

Thus integrable fields of planes are an exceptional phenomenon.

An *integral submanifold* of a field of tangent subspaces on a manifold is a submanifold whose tangent plane at each point is contained in the subspace of the field. If an integral submanifold can be drawn, its dimension usually does not coincide with that of the planes of the field.

In this lecture we shall consider a case in which there is a complete theory, namely the case of one first-order equation. From the physical point of view this case is the duality that occurs in describing a phenomenon using waves or particles. The field satisfies a certain first-order partial differential equation, the evolution of the particles is described by ordinary differential equations, and there is a method of reducing the partial differential equation to a system of ordinary differential equations; in that way one can reduce the study of wave propagation to the study of the evolution of particles.

We shall write everything in a local coordinate system: $x = (x_1, \ldots, x_n)$ are the coordinates (independent variables), $y = u(x)$ is an unknown function of the coordinates. The letter y by itself denotes the coordinate on the axis of values, and we denote the partial derivatives by the letter p: $p_i = \frac{\partial u}{\partial x_i} = u_{x_i}$.

The general first-order partial differential equation has the form

$$F(x_1, \ldots, x_n, y, p_1, \ldots, p_n) = 0 .$$

Examples.

$$\frac{\partial u}{\partial x_1} = 0 ; \tag{1.1}$$

$$\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 = 1 \tag{1.2}$$

(the eikonal equation in geometric optics);

$$u_t + uu_x = 0 \tag{1.3}$$

(Euler's equation).

Consider a convex closed curve in the plane with coordinates x_1, x_2. Outside the region bounded by the curve we consider the function u whose value at each point is the distance from that point to the curve. The function u is smooth.

Theorem 1. *The function u satisfies the equation* (1.2).

PROOF. Equation (1.2) says that the square-norm of the gradient of u equals 1. We recall the geometric meaning of the gradient. It is a vector pointing in the direction of maximal rate of increase of the function, and its length is that maximal rate of increase. The assertion of the theorem is now obvious. □

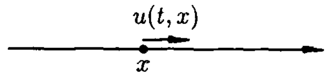
Problem 2. a) Prove that any solution of the equation (1.2) is locally the sum of a constant and the distance to some curve.

b) Understand where the wave-particle duality occurs in this situation. (In case of difficulty see below p. 14, Fig. 2.2.)

Consider the field $u(t, x)$ of velocities of particles moving freely along a line (Fig. 1.1). The law of free motion of a particle has the form $x = \varphi(t) = x_0 + vt$, where v is the velocity of the particle. The function φ satisfies Newton's equation $\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = 0$. We now give a description of the motion in terms of the velocity field u: by definition $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = u(t, \varphi(t))$. We differentiate with respect to t, obtaining the Euler equation:

$$\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = u_t + u_x u = 0 .$$

Fig. 1.1. A particle on a line



Conversely, Newton's equation can be derived from Euler's, that is, these descriptions of the motion using Euler's equation for a field and Newton's equation for particles are equivalent. We shall also construct a procedure for the general case that makes it possible to reduce equations for waves to equations for the evolution of particles. First, however, we consider some simpler examples of linear equations.

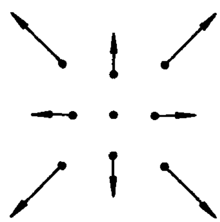
1. Let $v = v(x)$ be a vector field on a manifold or in a region of Euclidean space. Consider the equation $L_v(u) = 0$, where the operator L_v denotes the derivative in the direction of the vector field (the Lie derivative).

In coordinates this equation has the form $v_1 \frac{\partial u}{\partial x_1} + \cdots + v_n \frac{\partial u}{\partial x_n} = 0$; it is called a *homogeneous linear first-order partial differential equation.*

For the function u to be a solution of this equation it is necessary and sufficient that u be constant along the phase curves of the field v. Thus *the solutions of our equation are the first integrals of the field.*

For example, consider the field $v = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$ in Fig. 1.2. Let us solve the equation $L_v(u) = 0$ for this field v. The phase curves are the rays $x = e^t x_0$ emanating from the origin. The solution must be constant along each such ray. If we require continuity at the origin, we find that the only solutions are constants. The constants form a one-dimensional vector space. (The solutions of a linear equation necessarily form a vector space.)

Fig. 1.2. An Eulerian field



In contrast to this example, the solutions of a linear partial differential equation in general form an infinite-dimensional space. For example, for the equation $\frac{\partial u}{\partial x_1} = 0$ the solution space coincides with the space of functions of $n-1$ variables:

$$u = \varphi(x_2, \ldots, x_n) .$$

It turns out that the same is true for an equation in general position in a neighborhood of a regular point.

The Cauchy Problem. Consider a smooth hypersurface Γ^{n-1} in x-space. The *Cauchy problem* is the following: *find a solution of the equation* $L_v(u) = 0$ *that coincides with a given function on this hypersurface* (Fig. 1.3).

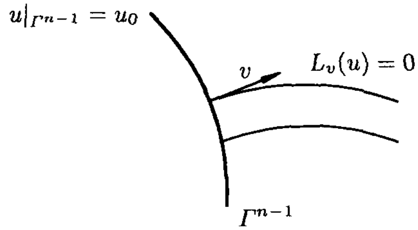


Fig. 1.3. The Cauchy problem

A point of the hypersurface is called *noncharacteristic* if the field v is transversal to the surface at that point.

Theorem 2. *The Cauchy problem has a unique solution in a neighborhood of each noncharacteristic point.*

PROOF. Using a smooth change of variables we can rectify the vector field and convert Γ into the hyperplane $x_1 = 0$. Then in a small neighborhood of a noncharacteristic point we obtain the following problem:

$$\frac{\partial u}{\partial x_1} = 0 \ , \qquad u\Big|_{0,x_2,\ldots,x_n} = u_0(x_2, \ldots, x_n) \ ,$$

which has a unique solution. □

2. Consider the Cauchy problem for a more general, *inhomogeneous linear equation*:

$$L_v(u) = f \ , \qquad u\Big|_{\Gamma^{n-1}} = u_0 \ .$$

The solutions of such a problem form an *affine* space. (The general solution of the inhomogeneous equation is the sum of the general solution of the homogeneous equation and a particular solution of the inhomogeneous equation.)

By a smooth change of variable the problem can be brought into the form

$$\frac{\partial u}{\partial x_1} = f(x_1, x_2, \ldots, x_n) \ , \qquad u\Big|_{0,x_2,\ldots,x_n} = u_0(x_2, \ldots, x_n) \ .$$

This problem has a unique solution:

$$u(x_1, \ldots) = u_0(\ldots) + \int_0^{x_1} f(\xi, \ldots)\,\mathrm{d}\xi \ .$$

3. An equation that is linear with respect to the derivatives is *quasilinear*. In coordinates a first-order quasilinear equation has the form

$$a_1(x,u)\frac{\partial u}{\partial x_1} + \cdots + a_n(x,u)\frac{\partial u}{\partial x_n} = f(x,u)\,. \qquad (*)$$

We remark that in the first two cases the field v is invariantly (independently of the coordinates) connected with the differential operator. How can a geometric object be invariantly connected with a quasilinear equation?

Consider the space with coordinates $(x_1, \ldots, x_n, y)$, the space of *0-jets of functions* of $(x_1, \ldots, x_n)$, which we denote $J^0(\mathbb{R}^n, \mathbb{R})$ or, more briefly, J^0.

We recall that the *space of k-jets* of functions of $(x_1, \ldots, x_n)$ is the space of Taylor polynomials of degree k.

We note that the argument of $(x_1, \ldots, x_n, y, p_1, \ldots, p_n)$ in a first-order equation is the 1-jet of the function. Thus a first-order equation can be interpreted as a hypersurface in the space $J^1(\mathbb{R}^n, \mathbb{R})$ of 1-jets of functions. The space of 1-jets of real-valued functions of n variables can be identified with a $(2n+1)$-dimensional space: $J^1(\mathbb{R}^n, \mathbb{R}) \approx \mathbb{R}^{2n+1}$. For example, for functions on the plane we obtain a five-dimensional space of 1-jets.

The solution of the equation $(*)$ can be constructed using its characteristics (curves of a special form in J^0). The word "characteristic" in mathematics always means "invariantly connected." For example, the characteristic polynomial of a matrix is invariantly connected with an operator and independent of the basis in which the matrix is formed. The characteristic subgroups of a group are those that are invariant with respect to the automorphisms of the group. The characteristic classes in topology are invariant with respect to suitable mappings.

The vector field v (in the space of independent variables) is called the *characteristic field* of the linear equation $L_v(u) = f$.

Definition. The *characteristic field* of the quasilinear equation $(*)$ is the vector field A in J^0 with components $(a_1, \ldots, a_n, f)$.

Claim. *The direction of this field is characteristic.*

Indeed, let u be a solution. Its graph is a certain hypersurface in J^0. This hypersurface is tangent to the field A, as the equation itself asserts. The converse is also true: if the graph of a function is tangent to the field A at each point, then the function is a solution.

The method of solving a quasi-linear equation becomes clear from this. We draw the phase curves of the characteristic field in J^0. They are called characteristics. If a characteristic has a point in common with the graph of a solution, it lies entirely on that graph. Thus the graph is composed of characteristics.

The *Cauchy problem for a quasilinear equation* is stated in analogy with the preceding cases. To be specific, suppose given a smooth hypersurface Γ^{n-1}

in x-space and an initial function u_0 defined on the hypersurface. The graph of this function is a surface $\widehat{\Gamma}$ in J^0, which we regard as the initial submanifold of Fig. 1.4.

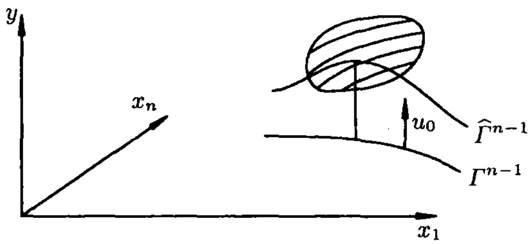


Fig. 1.4. Characteristics of a quasilinear equation passing through the initial manifold $\widehat{\Gamma}^{n-1}$

If the characteristics are not tangent to the hypersurface $\widehat{\Gamma}$, the graph of the solution is composed locally of them.

In this case two conditions are needed for a point to be noncharacteristic: the field A must not be tangent to $\widehat{\Gamma}^{n-1}$ and, in order for an actual graph to result, the vector of the field must not be vertical, that is, the component a must not be zero.

Points where $a = 0$ are singular; at these points the differential equation vanishes, becoming an algebraic equation.

Example. For the Euler equation $u_t + uu_x = 0$ the equation of the characteristics is equivalent to Newton's equation: $\dot{t} = 1$, $\dot{x} = u$, $\dot{u} = 0$.

Let us now pass to the general first-order equation.

Consider the space of 1-jets $J^1(\mathbb{R}^n, \mathbb{R})$. Instead of $\mathbb{R}^n$ one can consider an n-dimensional manifold B^n; in that case we obtain the space $J^1(B^n, \mathbb{R})$. Let (x, y, p) be local coordinates in this space.

A *first-order partial differential equation* is a smooth surface in J^1: $\Gamma^{2n} \subset J^1$.

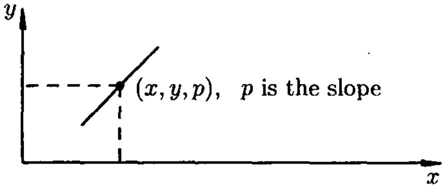
For example, when $n = 1$ we obtain an implicit ordinary differential equation (not solved with respect to the derivative).

It turns out that there is a remarkable geometric structure in our space J^1, an invariantly defined distribution of $2n$-dimensional hypersurfaces. For example, when $n = 1$ we obtain a field of planes in three-dimensional space. The structure arises only because the space is a space of 1-jets. An analogous structure appears in spaces of jets of higher order, where it is called a *Cartan distribution.*

Each function in the space of k-jets has a *k-graph.* For a 0-jet this is the usual graph – the set of 0-jets of the function: $\Gamma_u = \{j^0_x u : x \in \mathbb{R}^n\} = \{(x, y) : y = u(x)\}$. In the case of 1-jets a point of the *1-graph* consists of the argument, the value of the function, and the values of the first-order partial derivatives: $\{j^1_x u : x \in \mathbb{R}^n\} = \{(x, y, p) : y = u(x), p = \frac{\partial u}{\partial x}\}$ (see Fig. 1.5

for $n = 1$). We remark that the 1-graph is a section of the bundle over the domain of definition.

Fig. 1.5. A point of the space of 1-jets



Remark. The surface of the 1-graph is diffeomorphic to the domain of definition of the function, x is the n-dimensional coordinate on this surface. The smoothness of this surface is 1 less than the smoothness of the function, but smoothness is preserved for an infinitely differentiable or analytic function.

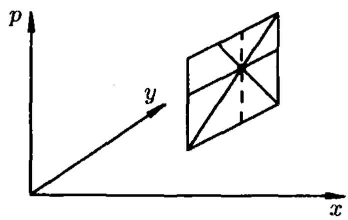
Consider the tangent plane to the 1-graph. This is an n-dimensional plane in a $(2n+1)$-dimensional space.

Theorem 3. *All tangent planes of all* 1*-graphs at a given point lie in the same hyperplane.*

PROOF. Along any tangent plane we have $\mathrm{d}y = \sum \frac{\partial u}{\partial x_i}\,\mathrm{d}x_i = \sum p_i\,\mathrm{d}x_i$, or $\mathrm{d}y = p\,\mathrm{d}x$. Since p is fixed at a given point of the space of 1-jets, we obtain an equation for the components of the tangent vector that determines the hyperplane. Thus the tangent plane to any 1-graph lies in this hyperplane. □

For example, when $n = 1$ the equation $\mathrm{d}y = p\,\mathrm{d}x$ defines a vertical plane in space with the coordinates x, y, p. The tangents to the 1-graphs are the non-vertical lines lying in this plane (Fig. 1.6).

Fig. 1.6. The contact plane in the space of 1-jets



In this case one can see that the hypersurface itself is the closure of the union of the tangents to all 1-graphs passing through the given point.

Problem 3. Prove that this is true for any dimension.