Topics in Noncommutative Geometry

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PREFACE

After R. Descartes, I. M. Gelfand, and A. Grothendieck, it became a truism that any commutative ring is a ring of functions on an appropriate space.

Noncommutative algebra resisted geometrization longer. A recent upsurge of activity in this domain was prompted by various developments in theoretical physics, functional analysis, and algebra.

When I was invited to give the Milton Brockett Porter Lectures at Rice University in the fall of 1989, I decided to give an introduction to some current ideas in noncommutative geometry. This book was prepared before the lectures and contains more material than could be presented orally. Still, I tried to preserve some spirit of lecture notes, especially in the first chapter, which intends to be an overview of various points of departure and basic themes. The rest of the book is more specialized. Chapters 2 and 3 are devoted to supersymmetric curves and flag spaces of supergroups, respectively. Chapter 4 develops an approach to quantum groups as symmetry objects in noncommutative geometry initiated in my Montreal lectures [Ma2].

Section 1 of Chapter 1 can be read as an introduction to the entire book. The rest of Chapter 1 gives some definitions, examples, and constructions but contains practically no proofs. It should be considered a guide for further reading.

Starting with Chapter 2, we prove most of the results. The choice of material was dictated by personal interests of the author. Exposition is based upon lecture courses and seminars I have led for several years at Moscow University and elsewhere. Partly it is taken from the papers of participants of these seminars or based upon their notes.

I want to thank many people for friendly collaboration and shared insights, especially A. A. Beilinson, V. G. Drinfeld, D. A. Leites, I. B. Penkov, A. O. Radul, I. A. Skornyakov, A. Yu. Vaintrob, A. A. Voronov, M. Wodzicki.

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CHAPTER 1

An Overview

1. Sources of Noncommutative Geometry

1.1. COMMUTATIVE GEOMETRY. The classical Euclidean geometry studies properties of some special subsets of plane and space: circles, triangles, pyramids, etc. Some of the crucial notions are those of a measure (of an angle, distance, surface, volume) and of "congruence" or equality of geometric objects.

An implicit basic object that only a century ago started to become a subject of independent geometric study is the group of motions. In fact, measures can be introduced as various motion invariants, and equality can be defined in terms of orbits of this group.

Since Descartes, this geometric picture became enriched with an essential algebraic counterpart, centered around the idea of coordinates. Subsets of space can be defined by relations between the coordinates of points, motions by means of functions. Geometry can be systematically translated into algebraic language.

Relations between coordinates are written in the form, say,

$$f_i(x_1,\dots,x_n)=0,\quad i\in I \text{ (or } f_i\neq 0, \text{ or } f_i\geq 0),$$

where the f_i belong to a class $\mathcal{O}$ of functions on the basic space. This class $\mathcal{O}$ depends on the type of geometric properties one wants to study. It may consist of polynomials (algebraic geometry), complex-analytic functions (complex geometry), smooth functions (differential geometry), continuous functions (topology), and measurable functions (measure theory). In certain basic situations, the transcription of geometry into algebraic language is as neat and simple as one can possibly hope. A classical example is Gelfand's theorem, which states (in modern words) that by associating to a locally compact Hausdorff topological space its $*$-algebra of complex-valued continuous functions vanishing at infinity, we get an antiequivalence of categories. (The source category should be considered with proper continuous maps as morphisms; the essential image of the functor consists of commutative $*$-algebras and $*$-homomorphisms.)

Symmetries of a geometric object are traditionally described by its automorphism group, which often is an object of the same geometric class (a topological space, an algebraic variety, etc.). Of course, such symmetries are only a particular type of morphisms, so that Klein's Erlangen program is, in principle, subsumed by the general categorical approach. Still, automorphisms and spaces with large automorphism groups are quite special and cherished objects of study in any geometrical discipline.

Since automorphisms of an object act on any linear space naturally associated to this object (functions, (co)homology, etc.), this explains the role of representation theory.

With traditional pointwise multiplication and addition, many important classes of functions form a commutative ring. This property seems to be very crucial if we want to consider an abstract as a ring of functions.

Grothendieck's algebraic geometry, via the notion of an affine scheme, shows that there is no need, in general, to ask anything more (e.g., absence of nilpotents).

However, one can ask less.

Attempts to build geometric disciplines based upon noncommutative rings are continuing. They have led already to various important, if disparate, developments. In the following, we shall consider below some motivations and approaches, stemming from physics, functional analysis, and algebraic geometry. We make no claims of completeness at all and discuss mostly subjects that appeal to the author personally.

1.2. PHYSICS. The basic symmetry group of Euclidean geometry has a distinctly physical origin. In fact, it is the group of motions of a rigid body, the latter notion being one of the pillars of classical physics, whose remnants can be traced as late as in Einstein's discussion of space–time in terms of rulers and clocks.

It is no surprise, then, that quantum physics supplied its own stock of basic geometries. We shall not be concerned here with already traditional Hilbert space geometry but rather with more recent developments connected with supergeometry and quantum groups.

Supergeometry is a variant of classical (differential, analytic, or algebraic) geometry in which, together with the usual pairwise commuting coordinates, one considers also anticommuting ones. The latter correspond to the internal (spinlike) degrees of freedom of fermions, elementary constituents of matter such as electrons and quarks, whereas commuting coordinates are used to describe their "external," space–time position. Since anticommuting coordinates are nilpotents, supergeometry looks like a very slight extension of the classical geometry. This does not mean at all that it is a trivial extension. It reveals a lot of very concrete new structures whose resemblance to the old ones is beautiful and fascinating. In particular, a fundamental role is played by the "supersymmetry" mixing even and odd coordinates. This notion cannot be made precise in terms of classical group theory; in fact, supersymmetry is described by Lie supergroups, or, infinitesimally, Lie superalgebras. The classification of simple Lie superalgebras, due to V. Kac ([K1]), demonstrated the existence of a quite unexpected extension of the Killing–Cartan theory and drew the attention of mathematicians to supergeometry, whose first constructions were invented by physicists Yu. Golfand, E. Lichtman, A. Volkov, J. Wess, and F. Berezin.

In this book, Chapters 2 and 3 are devoted to supergeometry. We have chosen not to explain foundations and first examples, but to develop two concrete and fairly advanced subjects: a theory of supersymmetric algebraic curves and a theory of Schubert–Bruhat decomposition of superhomogeneous flag spaces. We assume that the reader is familiar with Chapters 3 and 4 of [Ma1] containing an introduction to supergeometry. The exposition in Chapter 3 closely follows [VM].

Quantum groups could have been, but were not, invented in the same way as supergroups, i.e., as symmetry objects of certain "quantum spaces," described by noncommutative rings of functions. Actually, they originated in the work of L. D. Faddeev and his school on the quantum inverse scattering method (cf. [SkTF], [Dr1], and references therein).

An intuitive notion of a quantum space is based upon one of the schemes of quantization of classical Hamiltonian systems. Namely, one replaces the classical algebra of observables, consisting of functions on a phase space, by a quantum Lie algebra of observables consisting of the same functions with the Poisson bracket instead of multiplication, and a unitary representation of this algebra. One can imagine the universal enveloping algebra of this Lie algebra (or its exponentiated form) as a noncommutative coordinate ring of the quantized initial phase space.

Into any concrete description of the quantum commutation rules enters a small parameter h (Planck's constant). (Of course, since it is not dimensionless, its "smallness" means rather that a characteristic action of a system of macroscopic scale is large when measured in units of h.) This means that one algebraic approach to noncommutative spaces and quantum groups is via deformation theory. The most intensively studied quantum groups up to now are represented by the deformations $U_h(\mathfrak{g})$ of the universal enveloping algebras of classical simple (or Kac–Moody) Lie algebras $\mathfrak{g}$.

We present an introduction to this approach to quantum groups in Section 3 of this chapter. Chapter 4 presents quantum groups as symmetry objects of quantum spaces.

Algebraically, there is a subtle difference between supergroups and quantum groups. Formally speaking, quantum groups are Hopf algebras, virtually noncommutative and noncocommutative, while supergroups are Hopf superalgebras, supercommutative if represented by the function rings. A difference in the axioms of these two types of objects is best understood if one writes them down in an abstract tensor category (cf. [DM]) and realizes that quantum groups are constructed over a tensor category of usual vector spaces, whereas supergroups are based upon a different tensor category of $\mathbb{Z}_2$-graded spaces with the twisted permutation isomorphism $S_{(12)} : V \otimes W \to W \otimes V$. One may well combine both variations and define quantum supergroups (cf. Chapter 4).

Section 4 of this chapter is devoted to some basic facts concerning monoidal, tensor, and pseudotensor categories whose role in understanding quantum groups stems also from a generalization of the Tannaka–Krein duality. Namely, a very important role is played by those quantum groups for which braid groups act naturally on tensor powers of representation spaces. Such an action is described by solutions of Yang–Baxter equations that emerged in two-dimensional statistical physics and recently became related also to two-dimensional conformal field theories.

1.3. FUNCTIONAL ANALYSIS. A functional-analytic approach, vigorously pursued in recent years by Alain Connes and his collaborators, starts with two remarks. First, due to Gelfand's theorem, cited earlier, one can take noncommutative C^*-algebras as a natural category for noncommutative topology. Second, there is a supply of quite common geometric situations leading to such algebras.

In [C2] and [C3], Connes suggests the following examples.

(a) Let V be a smooth manifold, F a smooth foliation on V. The leaf space, V/F, of course, exists as a topological space but is very far from being a manifold, and its properties cannot be described by conventional means. It is suggested that its topology is encoded in the C^*-algebra $C^*(V,F)$ defined, e.g., in [C4].

(b) Let Γ be a discrete group. The topology of the reduced dual space $\hat{\Gamma}$ is described by the norm closure of $\mathbb{C}\Gamma$ in the algebra of bounded operators in $l^2(\Gamma)$, that is, by a C^*-algebra $C_r^*(\Gamma)$.

(c) One can treat similarly the topology of quotient spaces V/Γ and V/G, where Γ (resp. G) is a discrete (resp. Lie) group acting upon a smooth manifold V. In these cases, the C^*-algebra in question is a crossed product of a function algebra of V with Γ (resp. G).

What kind of invariants of an algebra A should be qualified as topological invariants of an imaginary noncommutative space corresponding to A?

First, there are K-theoretical invariants. A general principle discovered first in algebraic geometry is that the topology of a usual ("commutative") space is encoded in the category of the vector bundles of this space, which in its turn is equivalent to a category of projective A-modules, where A is an appropriate function ring. K-groups are constructed directly in terms of this category.

Second, there are differential-geometric invariants embodied in the de Rham complex of A. This complex calculates (co)homology and contains some information on homotopical properties. K-theory and (co)homology are connected by the Chern character.

In [C3], Connes develops these ideas in a very broad context and, in particular, investigates the so-called cyclic cohomology groups of a ring, which he introduced in connection with his noncommutative de Rham complex. This very important construction was independently invented by B. L. Tsygan as an additive analog of K-theory (cf. [T], [FT]).

One of the latest most important developments is due to M. Wodzicki. His theory of noncommutative residue was reported in [Kas]; cf. his original work [W1]–[W3].

In Section 2 of this chapter, we review more algebraic parts of Connes's work, referring the reader to the bibliography for further reading.

1.4. ALGEBRAIC GEOMETRY. Turning to algebra-geometric sources of noncommutative geometry, one must confess that although its general influence was very significant, concrete endeavors to lay down foundations of noncommutative algebraic geometry Grothendieck-style were unsuccessful (but see [Ro]). One stumbling block invariably was noncommutative localization. The point is that whereas, say, a smooth manifold is described by its algebra of global smooth functions, an algebraic variety is not described by its algebra of polynomial functions unless it is affine. Hence, we must have functions that are defined only locally, and for this we probably need tangible geometric objects on which such functions are defined as local models. Notice that in Connes's approach, we have no local models: His C^*-algebras are connected with such spaces as V/F in a rather indirect way and are not readily visualized as functions on them.

Since attempts to glue together noncommutative algebraic spaces from affine ones generally fail, we have to resort to more particular cases and to learning lessons of other approaches.

For example, one can define some analogs of affine algebraic groups, following the lead of the theory of quantum groups, and study them as in classical algebraic geometry. In doing so, we discover that there are very special values of deformation parameters that, on the one hand, correspond to nontrivial representation theories and, on the other hand, lead to function rings that are almost commutative, that is, have large commutative subrings. One may try to use these subrings for constructing geometric spectra and localizing with respect to them so that the rest of the algebra becomes encoded in the structure sheaf on an actual space. Lessons of supergeometry may be quite helpful here.

Chapter 4 of this book will develop this viewpoint. For further reading, we recommend a very interesting recent work by B. Parshall and J.-P. Wang, "Quantum Linear Groups," Parts I and II, University of Virginia preprints, 1989.

2. Noncommutative de Rham Complex and Cyclic Cohomology

2.1. CYCLES. Following Connes [3], we shall call an *n-cycle* a triple $(\Omega, d, \int)$, where $\Omega = \bigoplus_{j=0}^{n} \Omega^j$ is a graded C-algebra, d is its graded derivation of degree $1, d^2 = 0$, and $\int : \Omega^n \to C$ is a closed graded (super)trace, i.e., a linear functional satisfying the following identities:

$$\int d\omega = 0 \quad \text{for all} \quad \omega \in \Omega^{n-1};$$

$$\int \omega\omega' = (-1)^{\deg(\omega)\deg(\omega')} \int \omega'\omega.$$

2.2. EXAMPLES. (a) Let X be a compact smooth oriented n-dimensional manifold, $(\Omega(X), d)$ its de Rham complex over C, $\int$ the integral of volume forms over X. It is an n-cycle.

More generally, consider a closed q-current C on X. Then $(\bigoplus_{i=0}^{q} \Omega^i(X), d, \int = \langle C, .\rangle)$ is a q-cycle.

(b) For an associative C-algebra A and a linear functional $\mathrm{tr} : A \to C$ with $\mathrm{tr}([A, A]) = 0$, the triple $(A = \Omega = \Omega^0, \mathrm{tr})$ is a 0-cycle.

(c) By replacing $\int$ by $-\int$ in a cycle, we, by definition, change its *orientation*.

Direct sum of two cycles is defined in an obvious way. The following example describes a functional-analytic situation leading to cycles.

2.3. FREDHOLM MODULES. Let A be an associative C-algebra, $H = H_0 \oplus H_1$ a Z_2-graded separable Hilbert space, endowed with an odd bounded C-linear involution F. Assume that H is also endowed with a structure of the left A-module such that A acts by even bounded operators.

Then (A, H, F) is called an *n-summable Fredholm A-module* if $[F, a] = Fa - aF \in L^n(H)$ for all $a \in A$, where $L^n(H)$ is the so-called *n-th Schatten ideal*, consisting of those bounded operators T, for which $|T|^n$ is of trace class, $|T| = (T^*T)^{1/2}$.

Given such a module, we can construct the following n-cycle. Put $\Omega^0 = A$; Ω^q = closure of the linear span in $L^{n/q}(H)$ of the family of operators $(a^0 + \lambda \cdot \mathrm{id})[F, a^1][F, a^2] \ldots [F, a^q]$ where $a^i \in A$, $\lambda \in C$. Define d by $d\omega = i[F, \omega]$. Finally, for $\omega \in \Omega^n$, put

$$\int \omega = (-1)^n \,\mathrm{trace}(\omega).$$

Axiomatizing essential algebraic features of this situation, Connes arrives at the following abstract construction.

2.4. THE NONCOMMUTATIVE DE RHAM COMPLEX. For a fixed associative C-algebra A, with or without identity, consider all algebra homomorphisms $f : A \to \Omega^0$, where Ω^0 is the 0-component of a differential graded unitary algebra (Ω, d). We do not assume that f transforms the identity of A (if any) into the identity of Ω^0.

These homomorphisms form a category with an initial object. Here is its direct construction.

$\Omega^0 = \hat{A} = A \oplus C1$ (formal adjunction of identity);
$\Omega^n = \hat{A} \otimes A^{\otimes n}$ (tensor products over C);
$d((a^0 + \lambda.1) \otimes a^1 \otimes \ldots \otimes a^n) = 1 \otimes a^0 \otimes a^1 \otimes \ldots \otimes a^n$;
in particular, $da = 1 \otimes a$ for $a \in A$;
the multiplication map $\Omega^m \otimes \Omega^n \to \Omega^{m+n}$ is uniquely defined by the following conditions: for $m = 0$, it is the standard $\hat{A}$- multiplication on the leftmost tensor factor of Ω^n, the Leibniz formula holds.

As an example, let us multiply $\hat{a}^0 \otimes a^1$ by $\hat{b}^0 \otimes b^1$, where $\hat{a}^0 = a^0 + \lambda \cdot 1$, $\hat{b}^0 = b^0 + \mu \cdot 1$:

$$\begin{aligned}(\hat{a}^0 \otimes a^1)(\hat{b}^0 \otimes b^1) &= (\hat{a}^0 da^1)(\hat{b}^0 db^1) = \hat{a}^0(da^1\hat{b}^0)db^1 \\ &= \hat{a}^0[d(a^1\hat{b}^0) - a^1 d\hat{b}^0]db^1 \\ &= \hat{a}^0 d(a^1\hat{b}^0)db^1 - (\hat{a}^0 a^1)db^0 db^1 \\ &= \hat{a}^0 \otimes a^1 b^0 \otimes b^1 - \hat{a}^0 a^1 \otimes b^0 \otimes b^1.\end{aligned}$$

Here is a general formula for right multiplication by A:

$$(\hat{a}^0 \otimes a^1 \otimes \ldots \otimes a^n)b = \sum_{j=0}^{n} (-1)^{n-j} \hat{a}^0 \otimes \ldots \otimes a^j a^{j+1} \otimes \ldots \otimes b.$$

Although this noncommutative de Rham complex has many properties in common with the usual one, it should not be considered as a "final solution." In fact, in Chapter 4, we shall see that in the category of quadratic algebras, for example, a natural substitute for the de Rham complex is one of the four Koszul complexes, having a very different structure.

2.5. TRACES. In order to complete a (truncated) de Rham complex $(\Omega^{\leq q}(A), d)$ to a cycle, we need a trace functional. If such a functional $\int : \Omega^q(A) \to \mathbb{C}$ is given, consider a linear map $\tau : A^{\otimes q} \to A^*$ (here the asterisk means linear dualization),

$$\tau(a^1 \otimes \ldots \otimes a^q)(a^0) = \int a^0 da^1 \ldots da^q.$$

Connes has shown that τ satisfies certain functional equations that can best be expressed by saying that τ is a cocycle of a very important complex, and that this correspondence between traces and cocycles is one-to-one.

To explain this result in a natural generality, we shall digress and start with the notion of a cyclic object of an abstract category.

2.6. CYCLIC OBJECTS. First recall that a simplicial object of a category C is a functor $\Delta^0 \to C$ where Δ is the category of well-ordered finite sets $[n] = \{0, 1, \ldots, n\}$ with nondecreasing maps as morphisms.

Following Connes, we similarly define a *cyclic object* of C as a functor $\Lambda^0 \to C$, where objects of Λ are

$$\{n\} = \text{ roots of unity of any degree } n+1,$$

and morphisms are defined by any of the following equivalent ways.

By writing k instead of $\exp(2\pi ik/(n+1))$, introduce on $\{n\}$ the cyclic order $0 \leq 1 \leq 2 \leq \cdots \leq n \leq 0$.

VARIANT 1. $\mathrm{Hom}_\Lambda(\{n\}, \{m\})$ = the set of homotopy classes of continuous cyclically nondecreasing maps $\varphi : S^1 \to S^1$ such that $\varphi(\{n\}) \subseteq \{m\}$. Here $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, and homotopy is considered in the same class of maps.

VARIANT 2. A morphism $\{n\} \to \{m\}$ is a pair consisting of a set-theoretical map $f : \{n\} \to \{m\}$ and a set σ of total orders, one on each fiber $f^{-1}(i), i \in \{m\}$. They must satisfy the following condition: The cyclic order on $\{n\}$ induced by the standard cyclic order on $\{m\}$, and σ should coincide with the standard cyclic order on $\{n\}$.

The composition rule is: $(g, \tau)(f, \sigma) = (gf, \tau\sigma)$, where $i \leq j$ with respect to $\tau\sigma$ if either $f(i) \leq f(j)$ with respect to τ, or $f(i) = f(j)$ and $i < j$ with respect to σ.

A given f can be extended to a morphism in Λ iff it is cyclically nondecreasing. This extension is unique unless f is constant, in which case there are $n+1$ extensions.

VARIANT 3. Morphisms in Λ are given formally by the following sets of generators and relations.

Generators:

$$\delta_n^i : \{n-1\} \to \{n\}; \quad \sigma_n^i : \{n+1\} \to \{n\}; \quad \tau_n : \{n\} \to \{n\}.$$

Relations:

$$\delta_n^j \delta_{n-1}^i = \delta_n^i \delta_{n-1}^{j-1} \quad \text{for} \quad i \leq j;$$

$$\sigma_n^j \sigma_{n+1}^i = \sigma_n^i \sigma_{n+1}^{j+1} \quad \text{for} \quad i \leq j;$$

$$\sigma_n^j \delta_{n+1}^i = \begin{cases} \delta_n^i \sigma_{n-1}^{j-1} & \text{for} \quad i < j; \\ \mathrm{id} & \text{for} \quad i = j \text{ or } j+1; \\ \delta_n^{i-1} \sigma_{n-1}^j & \text{for} \quad i > j+1; \end{cases}$$

$$\tau_n \delta_n^i = \delta_n^{i-1} \tau_{n-1} \quad \text{for} \quad i = 1, \ldots, n;$$

$$\tau_n \sigma_n^i = \sigma_n^{i-1} \tau_{n+1} \quad \text{for} \quad i = 1, \ldots, n;$$

$$\tau_n^{n+1} = \mathrm{id}\,.$$

In terms of the previous description, δ_n^i omits i; σ_n^i takes the value i twice; $\tau_n(j) = j+1$. Only for σ_0^0, we must fix an order on a fiber: it is $0 < 1$.

Remarkably, Λ is isomorphic to Λ^0. The isomorphism is identical on objects and acts as follows on morphisms: If, in the second description, $(f, \sigma) : \{n\} \to \{m\}$, we define $(f, \sigma)^* = (g, \tau) : \{m\} \to \{n\}$ by

$g(i) = \sigma$ − minimal element of $f^{-1}(j)$ where j is the maximal element of $f^{-1}(j)$ cyclically preceding i.

An additional piece of information is needed only if g is constant. This happens precisely when f is constant. Then τ is defined by the condition that f is the τ-minimal element of $\{n\}$.

2.7. CYCLIC COMPLEXES. Let now $\mathbf{E} = (E_n, d_i^n, s_i^n, t_n)$ be a cyclic object of an *abelian category*, where d, s, t, respectively, correspond to δ, σ, τ. Put

$$d^n = \sum_{i=0}^{n} (-1)^i d_i^n : E_n \to E_{n-1};$$

$$d'^n = \sum_{i=0}^{n-1} (-1)^i d_i^n : E_n \to E_{n-1};$$

$$t = (-1)^n t_n : E_n \to E_n; N = \sum_{i=0}^{n} t^i.$$

First of all, (E_n, d^n) and (E_n, d'^n) are complexes. Since $d(1-t) = (1-t)d'$, they can be combined in the following bicomplex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
d\downarrow & & -d'\downarrow & & d\downarrow & & \\
E_2 & \xleftarrow{1-t} & E_2 & \xleftarrow{N} & E_2 & \xleftarrow{1-t} & \dots \\
d\downarrow & & -d'\downarrow & & d\downarrow & & \\
E_1 & \xleftarrow{1-t} & E_1 & \xleftarrow{N} & E_1 & \xleftarrow{1-t} & \dots \\
d\downarrow & & -d'\downarrow & & d\downarrow & & \\
E_0 & \xleftarrow{1-t} & E_0 & \xleftarrow{N} & E_0 & \xleftarrow{1-t} & \dots
\end{array}$$

Denote by $C \cdot E$ the associated complex and define the *cyclic homology* $HC \cdot (\mathbf{E})$ of the cyclic object $\mathbf{E}$ as $H(C \cdot E)$.

If multiplication by $n+1$ is an isomorphism of E_n for all n, the rows of the bicomplex are exact everywhere except the leftmost column. Thus, in this case, $HC \cdot (\mathbf{E}) = H(E \cdot /(1-t)\mathbf{E} \cdot)$.

A *cocyclic object* of C can be defined dually as a functor $\mathbf{E} : \Lambda \to C$. If C is abelian, one can repeat the previous construction with the arrows reversed. In this way, we obtain the *cyclic cohomology* $HC^{\cdot}(\mathbf{E})$.

2.8. Cyclic Cohomology as a Derived Functor. Suppose that C is the category of k-modules over a commutative ring k. Denote by Λk an object of the category ΛC of cocyclic objects of C, all of whose components are k and all of whose morphisms are identical.

ΛC is an abelian category, and

$$HC^k(\mathbf{E}) = \mathrm{Ext}^k_{\Lambda C}(\Lambda k, \mathbf{E}).$$

One can similarly treat cyclic homology as a certain Tor$\cdot$-functor. These results can be generalized to the abstract categories admitting a unit object 1 (similar to k) and internal Hom of diagrams.

2.9. Connection with the Hochschild Homology. Let A be an algebra over a commutative ring containing $\mathbb{Q}$. We can define a cyclic object $\mathbf{A}$ with the ith component $A^{\otimes(i+1)}$ and the following structure morphisms:

$$\begin{aligned}
d_i^n(a^0 \otimes \ldots \otimes a^n) &= a^0 \otimes \ldots \otimes a^i a^{i+1} \otimes \ldots \otimes a^n, \quad 0 \le i \le n; \\
d_n^n(a^0 \otimes \ldots \otimes a^n) &= a^n a^0 \otimes a^1 \otimes \ldots \otimes a^{n-1}; \\
s_i^n(a^0 \otimes \ldots \otimes a^n) &= a^0 \otimes \ldots \otimes a^i \otimes 1 \otimes a^{i+1} \otimes \ldots \otimes a^n; \\
t_n(a^0 \otimes \ldots \otimes a^n) &= a^n \otimes a^0 \ldots \otimes a^{n-1}.
\end{aligned}$$

The left column of the bicomplex associated to this object in Section 2.7 is called the Hochschild complex $C \cdot (A, A)$.

Let us denote by L the whole bicomplex, by S its endomorphism shifting it by two columns to the right, by K the bicomplex consisting of the first two columns of L and zeroes elsewhere. There is an exact sequence

$$0 \to K \to L \to L/K \to 0$$

and canonical isomorphisms $H \cdot (\Delta K) = H \cdot (A, A)$, $S : L \simeq L/K$. Passing to homology, we get an exact sequence:

$$\ldots \to H_n(A,A) \to HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{\delta} H_{n-1}(A,A) \to \ldots.$$

Here we write $HC_n(A)$ instead of $HC_n(\mathbf{A})$.

For cyclic cohomology, one can obtain in the same way an exact sequence involving Hochschild cohomology with coefficients in A^*:

$$\ldots \to H^n(A, A^*) \to HC^{n-1}(A) \to HC^{n+1}(A) \to h^{n+1}(A, A^*) \to \ldots.$$

2.10. Relative Cyclic Homology of Algebras. Consider a differential graded algebra A over a field k of characteristic zero. It is called *free* if it is isomorphic to the tensor algebra of a graded vector space (nothing is assumed about the differential). More generally, a morphism $A \to B$ is called free if it is isomorphic to a morphism $A \to A * C$, where C is free and $*$ denotes amalgamation. We also say that B is free over A.

The category of associative k-algebras is embedded in the category of differential graded algebras with vanishing components of degree zero.

Let $f : A \to B$ be a morphism of k-algebras. A *resolution* of B over A is a commutative diagram

$$\begin{array}{ccc}
 & & R \\
 & {}^{i}\nearrow & \\
A & & \downarrow \pi \\
 & \searrow {}_{f} & \\
 & & B
\end{array}$$

such that R is free over A and π is a surjective quasiisomorphism. Every f admits a resolution.

Consider a resolution R as a complex. From the Leibniz formula, it follows that $[R,R] + i(A)$ is a subcomplex (here $[R,R]$ is the linear span of supercommutators). Put

$$HC_n(A \to B) = H_{n+1}(R/([R,R] + i(A))).$$

We then have the following results.

(a) These homology groups do not depend on the choice of a resolution and define a covariant functor on the category of morphisms of k-algebras.

(b) Composition of morphisms $A \to B \to C$ generates a functorial exact sequence

$$\dots \to HC_n(A \to B) \to HC_n(A \to C) \to HC_n(B \to C) \to HC_n(A \to B) \to \dots$$

(c) There are functorial isomorphisms $HC_n(A \to 0) = HC_n(A)$.

2.11. CONNECTION WITH THE DE RHAM COHOMOLOGY. Let A be a finitely generated commutative k-algebra, and k a field of characteristic zero. One can define the algebraic de Rham complex $(\Omega A, d)$ in the standard way.

When $k = \mathbb{C}$ and A is the ring of polynomial functions on a nonsingular affine variety X, one can identify $H^{\cdot}(\Omega A)$ with the singular cohomology $H^{\cdot}(X, \mathbb{C})$. In the general case, to calculate $H^{\cdot}(X, \mathbb{C}), X = \mathrm{Spec}(A)(\mathbb{C})$ algebraically, one must first choose an affine embedding of $\mathrm{Spec}(A)$, i.e., a surjection $B \to A$, where B is a polynomial algebra. Let I be the kernel of this surjection. Define the following filtration of the complex ΩB:

$$F^n\Omega^j B = \begin{cases} \Omega^j B & \text{for } n \le j; \\ I^{n-j}\Omega^j B & \text{for } n > j. \end{cases}$$

Put

$$H^{\cdot}_{\mathrm{cris}}(A; n) = H^{\cdot}(\Omega B / F^{n+1}\Omega B).$$

These crystalline cohomology groups do not depend on the choice of $B \to A$. Grothendieck established an isomorphism

$$H^{\cdot}(X, \mathbb{C}) = H^{\cdot}(\varprojlim \Omega B / F^{n+1}\Omega B).$$

Cyclic homology is related with the finite levels of this filtration via functorial morphisms,

$$\chi_{n,i} : HC_n(A) \to H^{n-2i}_{\mathrm{cris}}(A, n-1).$$

If I is generated by a regular sequence, we have the isomorphisms

$$\bigoplus_i \chi_{n,i} : HC_n(A) \to \bigoplus_{0 \le 2i \le n} H^{n-2i}_{\mathrm{cris}}(A; n-i).$$

If $\mathrm{Spec}(A)$ is a reduced smooth scheme, we have

$$H^n_{\mathrm{cris}}(A; m) = \begin{cases} H^n(\Omega A) = H^n_{\mathrm{DR}}(\mathrm{Spec}(A)) & \text{for } n \le m; \\ \Omega^n A / d\Omega^{n-1} A & \text{for } n = m. \end{cases}$$

Thus, in this case cyclic homology is

$$HC_n(A) \simeq \Omega^n A / d\Omega^{n-1} A \oplus \left(\bigoplus_{i \ge 1} H^{n-2i}_{\mathrm{DR}}(\mathrm{Spec}(A)) \right).$$

2.12. TRACES AND CYCLIC COCYCLES. Let us now return to the situation in Section 2.5, where we started with a linear functional $\int : \Omega^{\le q}(A) \to \mathbb{C}$, completing the complex $(\Omega^{\le q}, d)$ to a Connes cycle and constructed an operator $\tau : A^{\otimes q} \to A^*$. Here $\Omega(A)$ is the noncommutative de Rham complex.

One can now explain in what sense cyclic cohomology classifies integrals: *the correspondence $\int \leftrightarrow \tau$ is a bijection between closed graded traces and cyclic cocycles (called characters).*

We can now extend some further notions of topology and differential geometry to the noncommutative case.

2.13. COBORDISM. An $(n+1)$-*chain* in Connes's sense is a quadruple $(\Omega, \partial\Omega, r, \int)$ consisting of the following data.

(a) $\Omega = \bigoplus_{i=0}^{n+1} \Omega^i, \partial\Omega = \bigoplus_{i=0}^{n} (\partial\Omega)^i$ are differential graded algebras, and $r : \Omega \to \partial\Omega$ is a surjective morphism of degree zero.

(b) $\int : \Omega^{n+1} \to \mathbb{C}$ is a trace such that $\int d\omega = 0$ if $r(\omega) = 0$.

Given a chain, its *boundary* is the n-cycle $(\partial\Omega, d, \int')$, where $\int' \omega' = \int d\omega$ if $r(\omega) = \omega'$.

Two cycles Ω', Ω'' are called cobordant if there exists a chain with boundary $\Omega' \oplus \bar{\Omega}''$, where the bar denotes the orientation reversal.

In the same way, one can define the relative notions of cobordism over an algebra A.

Cobordism is an equivalence relation.

It can be determined in terms of cyclic and Hochschild cohomology. Let τ', τ'' be the characters of cycles Ω', Ω''. Then these cycles are cobordant iff the difference of their cohomology classes belongs to the image of the morphism described in Section 2.9.

$$S : H^{n+1}(A, A^*) \to HC^n(A).$$

2.14. CONNECTIONS. Consider a cycle $\rho : A \to \Omega$ over A and a projective right A-module E of finite rank. A *connection* on E is a $\mathbb{C}$-linear map $\nabla : E \to E \otimes_A \Omega^1$ with the following property:

$$\nabla(ea) = (\nabla e)a + a \otimes d(\rho(a)) \quad \text{for all } e \in E, a \in A.$$

We shall assume that A has an identity and put $\mathcal{E} = E \otimes_A \Omega$. We extend ∇ to $\mathcal{E}$ by

$$\nabla(e \otimes \omega) = (\nabla e)\omega + e \otimes d\omega.$$

Consider the graded $\mathrm{End}_A(E)$-algebra $\mathrm{End}_\Omega(\mathcal{E})$. For $T \in \mathrm{End}(\mathcal{E})$ put

$$\delta(T) = \nabla T - (-1)^{\deg(T)} T\nabla.$$

Define a trace functional on $\mathrm{End}_\Omega(\mathcal{E})^n$ as the composition of the matrix trace and $\int : \Omega^n \to \mathbf{C}$.

The triple $(\mathrm{End}_\Omega(\mathcal{E}), \delta, \int \circ \mathrm{tr})$ is not, however, a cycle because $\delta^2 \neq 0$ in general. In fact, ∇^2 on $\mathcal{E}$ is the multiplication by a curvature form θ, and $\delta^2 T = [\theta, T]$.

In noncommutative geometry, there is a universal method of killing curvatures.

Abstractly, consider a system $(\Xi, \delta, \theta, \int)$ consisting of a graded differential algebra (Ξ, δ) with the last component Ξ^n, a closed trace $\int$, and an element $\theta \in \Xi^2$ such that $\delta\theta = 0$ and $\delta T = [\theta, T]$ for all $T \in \Xi$.

We adjoin to Ξ an element X of degree 1 subject to the following relations:

$$X^2 = \theta; \ \omega_1 X \omega_2 = 0 \quad \text{for all} \quad \omega_i \in \Xi.$$

On the resulting algebra Ξ' define d' and $\int'$ by the following rules:

$$d'\omega = \delta\omega + X\omega - (-1)^{\deg(\omega)}\omega X \quad \text{for} \quad \omega \in \Xi;$$
$$d'X = 0;$$

$$\int' (\omega_{11} + \omega_{12}X + X\omega_{21} + X\omega_{22}X) = \int \omega_{11} - (-1)^{\deg(\omega_{11})} \int \omega_{22}\theta,$$

where $\deg(\omega_{11}) = n = \deg(\omega_{12}) + 1 = \deg(\omega_{22}) + 2$.

Connes proves that $(\Xi', d', \int')$ is a cycle.

If we now apply this construction to $(\mathrm{End}_\Omega(\mathcal{E}), \delta, \theta, \int)$, we get a cycle over $\mathrm{End}_A(E)$. Its character depends only on the class of E in $K_0(A)$ and on the character of the initial cycle. Besides, in the spirit of Morita, the cyclic cohomology $H(A)$ of A and $\mathrm{End}_A(E)$ coincide.

Connes proves that the resulting map $K_0(A) \times H(A) \to H(A)$ is biadditive and makes of $H(A)$ a $K_0(A)$-module if A is commutative.

3. Quantum Groups and Yang–Baxter Equations

3.1. Affine Algebraic Groups. An affine algebraic group G over a field k can be defined in the following "naive" way. It is given by an ideal $J \subset k[z_i^j], i, j = 1, \dots, n$ with the following properties.

(a) Let $U = (u_i^j), V = (v_i^j)$ be two $n \times n$-matrices whose coefficients lie in a commutative k-algebra A and satisfy the algebraic equations $f(u_i^j) = f(v_i^j) = 0$ for all $f \in I$ (or for a family of generators of I). Then UV is also a solution of this system.

(b) The identity matrix $I_n = I = (\delta_i^j)$ satisfies J.

(c) If U (as in (a)) satisfies J and is invertible in $M(n, A)$, then U^{-1} also satisfies J.

In more invariant terms, G is determined by its functor of points on the category of commutative k-algebras A,

$$G(A) = \{f \in \mathrm{Hom}(k[z_i^j]/J, A) \mid (f(z_i^j)) \text{ is invertible}\}.$$

Of course, the invertibility condition can be built into the definition of the function ring of G: It suffices to invert formally $Z = (z_i^j)$ first, that is, to introduce new independent variables (y_i^j) forming an $n \times n$ matrix Y, and to put

$$F(G) = k[z_i^j, y_i^j]/(J, \text{ coefficients of } XY - I).$$

Then

$$G(A) = \mathrm{Hom}_{k-\mathrm{alg}}(F(G), A).$$

The initial ring $F(\overline{G}) = k[z_i^j]/J$ represents an *algebraic matrix semigroup* $\overline{G}$. The matrix Z itself, representing the identity morphism of $F(G)$ or $F(\overline{G})$, is called a *generic point of* G or $\overline{G}$. An arbitrary point corresponding to an *injective* morphism $F(G) \to A$(resp. $F(\overline{G}) \to A$) is also called generic.

Looking at conditions (a), (b), and (c) in terms of generic points, we can rewrite them in the following way. For U, V take points $F(G) \to F(G) \otimes F(G)$ corresponding to $z_i^j \to z_i^j \otimes 1$ and $z_i^j \to 1 \otimes z_i^j$, respectively. Then the product corresponds to the point $Z \otimes Z$ with coefficients $(Z \otimes Z)_i^k = \sum z_i^j \otimes z_j^k$ (thus, our tensor product of matrices is not Kronecker's tensor product!). In other words, we get the diagonal map (or comultiplication) $\Delta : F(G) \to F(G) \otimes F(G), \Delta(Z) = Z \otimes Z$. Similarly, condition (b) furnishes a counit map $\varepsilon : F(G) \to k$, and condition (c) furnishes an antipode $i : F(G) \to F(G) : i(Z) = Y = Z^{-1}$.

It is well known that if we explicitly add to this data the multiplication map $m : F(G) \otimes F(G) \to F(G)$ and the unit map $\eta : k \to F(G)$, we shall obtain a particular case of a general notion of Hopf algebra. Let us recall its definition.

3.2. ALGEBRAS, COALGEBRAS, BIALGEBRAS, AND HOPF ALGEBRAS.

(a) *An associative k-algebra with unit* is a linear space E with the structure maps $m : E \otimes E \to E$ and $\eta : k \to E$ such that

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) : E \otimes E \otimes E \to E;$$
$$m \circ (\eta \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes \eta) = \mathrm{id} : k \otimes E = E \otimes k = E \to E.$$

(b) *A coassociative k-algebra with counit* is a linear space E with structure maps $\Delta : E \to E \otimes E$ and $\varepsilon : E \to k$ such that

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta : E \to E \otimes E \otimes E;$$
$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} : E \to E = k \otimes E = E \otimes k.$$

(c) *A bialgebra* is a linear space E endowed with the structures of an algebra (m, ε) and a coalgebra (Δ, η) satisfying a compatibility axiom that can be written in the form: *Δ is a morphism of k-algebras.*

It is assumed here that the multiplication in $E \otimes E$ is given by the usual rule $(e \otimes f)(e' \otimes f') = ee' \otimes ff'$.

This is the main place where the definition of, say, superbialgebra differs from that of bialgebra: A sign enters in the formula for multiplication in $E \otimes E$. For this reason, it is better to write the compatibility axiom in the form

$$(m \otimes m) \circ S_{(23)} \circ (\Delta \otimes \Delta) = \Delta \circ m : E \otimes E \to E \otimes E,$$

where $S_{(23)} : E^{\otimes 4} \to E^{\otimes 4}$ is the morphism interchanging two middle factors. It may become nontrivial in a tensor category different from that of vector spaces, e.g., that of $\mathbb{Z}_2$-graded vector spaces. If we carefully write all the relevant axioms with the necessary permutation morphisms, they will be automatically applicable in more general tensor categories (cf. Section 3.3).

We see also that the bialgebra data and axioms are self-dual with respect to the reversal of all arrows and the replacement of (m, η) by (Δ, ε), and vice versa.

(d) *A Hopf algebra* is a bialgebra $(E, m, \eta, \Delta, \varepsilon)$ endowed with a linear map $i : E \to E$ (*antipode*) such that

$$m \circ (i \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes i) \circ \Delta = \eta \circ \varepsilon : E \to E.$$

The properties of an antipode in a general Hopf algebra can differ considerably from those in an affine algebraic group.

First, it is in general not a morphism of algebras or coalgebras; it reverses both multiplication and comultiplication. Precisely, put $m^{\mathrm{op}} = m \circ S_{(12)}, \Delta^{\mathrm{op}} = S_{(12)} \circ \Delta$. Reversing in a bialgebra either multiplication, comultiplication, or both, we still get a bialgebra. An antipode i, if it exists at all, is a bialgebra morphism $(E, m, \eta, \Delta, \varepsilon) \to (E, m^{\mathrm{op}}, \eta, \Delta^{\mathrm{op}}, \varepsilon)$. If, in addition, it is bijective (which is not always so), then i^{-1} is an antipode for $(E, m^{\mathrm{op}}, \Delta)$ and $(E, m, \Delta^{\mathrm{op}})$, hence i is one for $(E, m^{\mathrm{op}}, \Delta^{\mathrm{op}})$.

E is commutative if $m = m^{\mathrm{op}}$; cocommutative if $\Delta = \Delta^{\mathrm{op}}$.

If an antipode for a bialgebra exists, it is unique (cf. [A] and Chapter 4), but not necessarily bijective. If it is bijective, it may have arbitrary finite or infinite order.

One can now easily prove that an affine algebraic group is (the spectrum of) a finitely generated (as algebra) commutative Hopf algebra, and vice versa. The group itself is commutative iff this Hopf algebra is cocommutative.

Now a tentative definition of an affine group in noncommutative geometry, or quantum group, is obvious: It should be a Hopf algebra, in general noncommutative and noncocommutative, with some finiteness conditions and possibly a condition of bijectivity of the antipode.

One remark is in order now. We have seen that the reversal of arrows transforms a bialgebra into a bialgebra and an antipode into an antipode. This formal duality can be transformed into a linear duality functor $(E, m, \Delta) \to (E', \Delta', m')$, where E' consists of linear functionals on E vanishing on an ideal of finite codimension. For an affine algebraic group, $(F(G))'$ consists of distributions with finite support. In characteristic zero, the distributions supported by identity form the universal enveloping algebra of the corresponding Lie algebra. In general, speaking of a quantum group, we must specify how we imagine a given Hopf algebra: as its function algebra or its distribution algebra. We shall usually prefer the first choice and state our definitions correspondingly. In particular, speaking of *representations*, we shall deal with *comodules* rather than modules (cf. below).

3.3. AFFINE QUANTUM GROUPS IN NONCOMMUTATIVE GEOMETRY. We shall often construct our groups by direct generalization of the data in Section 3.1, therefore we shall start by rephrasing them in the noncommutative situation. Consider a bilateral ideal $J \subset k\langle z_i^j \rangle, i, j = 1, \dots, n$, where $k\langle z_i^j \rangle$ is the free associative algebra. We shall say that J defines a *quantum matrix semigroup* $\overline{G}$, with the function ring $F(\overline{G}) = k\langle z_i^j \rangle / J$, if the following analogs of the conditions 3.1(a) and (b) are valid:

(a′) Let $U = (u_i^j), V = (v_i^j)$ be two $n \times n$ matrices whose coefficients lie in an associative k-algebra A, satisfy the noncommutative polynomial equations $f(u_i^j) = f(v_i^j) = 0$ for all $f \in J$ and furthermore pairwise commute: $[u_i^j, v_k^l] = 0$. Then UV also verifies $f((UV)_i^j) = 0$.

Note the commutation condition: Two points of a quantum (semi)-group can be multiplied only if their coefficients are "simultaneously pairwise observable." In particular, one cannot define a "one-parametric subgroup": one cannot even hope that U^2 or U^{-1} are points of the same group!

(b′) The identity matrix $I_n = I$ satisfies J.

Of course, this is equivalent to the statement that $(F(\overline{G}), \Delta, \varepsilon)$ is a bialgebra, where $\Delta(Z) = Z \otimes Z, \varepsilon(Z) = I$ (Δ and ε are applied coefficientwise).

For the reasons that should be already clear, we must replace 3.1(c) by a more complicated condition if we want to go from a semigroup to a group.

If (E, m, Δ) already admits an antipode, it represents a quantum group. Otherwise, one can argue as follows.

In the category of morphisms of bialgebras $F(\overline{G}) \to H$, where H is a Hopf algebra, there is a universal morphism. The corresponding Hopf algebra represents a quantum group G, and A-points of this group is $\mathrm{Hom}_{k-\mathrm{alg}}(H, A)$.

We shall prove this in Chapter 4; to construct H from $F(\overline{G})$, it is necessary formally to invert $Z, (Z^{-1})^t, (((Z^{-1})^t)^{-1})^t, \ldots$, etc., to infinity.

Therefore, an analog of Section 3.1(c). must ask for invertibility of an infinite set of matrices, and also for the validity of a set of noncommutative polynomial equations for their elements.

Since in our presentation a special role is played by a matrix Z, it is appropriate to clarify its place in the theory. For an affine algebraic group, the choice of Z is equivalent to that of a faithful representation of G in the coordinated vector space k^n. The same is true in general if one replaces representations by comodules.

3.4. CATEGORY OF COMODULES. A left (resp. right) comodule M can be defined over any coalgebra (E, Δ, ε). The data and the axioms can be obtained by reversing arrows from those of a module: M is a k-space endowed with comultiplication $\delta : M \to E \otimes M$ (resp. $M \to M \otimes E$) such that

$$(\Delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta : M \to E \otimes E \otimes M \quad \text{(coassociativity)};$$
$$(\varepsilon \otimes \mathrm{id}) \circ \delta = \mathrm{id} : M \to k \otimes M = M \quad \text{(counit)},$$

and similarly for the right coaction.

A morphism of left comodules is a linear map $r : M \to N$ such that $(\mathrm{id} \otimes r) \circ \delta_M = \delta_N \circ r$. Right comodules are defined in a similar manner. The direct sum is defined in a straightforward way. Left and right are exchanged by going to Δ^{op}: If (M, δ) is a left (right) (E, Δ)-module, $(M, \delta^{\mathrm{op}} = S_{(12)} \circ \delta)$ is a right (left) $(E, \Delta^{\mathrm{op}})$-module. This is an isomorphism of categories. Checking this involves only axioms of the basic tensor category, hence is valid also for supercoalgebras, etc.

In order to define the *tensor product* of two left (resp. right) comodules, we need multiplication. If (E, m, Δ) is a bialgebra, and M, N are two left comodules, we define the coaction map

$$\delta_{M \otimes N} : M \otimes N \to E \otimes M \otimes N$$

as the composition

$$M \otimes N \xrightarrow{\delta_M \otimes \delta_N} E \otimes M \otimes E \otimes N$$
$$\xrightarrow{S_{(23)}} E \otimes E \otimes M \otimes N \xrightarrow{m \otimes \mathrm{id} \otimes \mathrm{id}} E \otimes M \otimes N.$$

Again, this is well defined in view of the general tensor category axioms.

Finally, if M is a left (E, Δ)-comodule, the dual space M^* has a natural structure of the right comodule, which can be transformed into a left $(E, \Delta^{\mathrm{op}})$-comodule. If $(E, \Delta^{\mathrm{op}})$ is (a part of) a Hopf algebra, we can use the antipode $i : (E, \Delta^{\mathrm{op}}) \to (E, \Delta)$ to induce on M^* the structure of (E, Δ) comodule.

Thus, in the category of, say, left comodules over a Hopf algebra, one can define the data of a tensor category, but in general, they will not satisfy the usual axioms, e.g., commutativity of the tensor product. We shall see it in down-to-earth terms after discussing the connection with multiplicative matrices.

Let (E, Δ, ε) be a coalgebra. A matrix $Y \in M(n, E)$ is called *multiplicative*, if

(a) $\Delta(Y) = Y \otimes Y$.

(b) $\varepsilon(Y) = I_n$.

Let (k^n, δ) be a left E-comodule. Define $Y = (y_i^j) \in M(n, E)$ by $\delta(e_i) = \sum_j y_i^j \otimes e_j$, where $\{e_j\}$ is the standard basis of k^n.

3.5. PROPOSITION. *(a) The construction described above establishes a bijection between $n \times n$-multiplicative matrices and structures of left comodule on k^n.*

(b) Let $r : (k^m, \delta_1) \to (k^n, \delta_2)$ be a morphism of E-comodules, given by a matrix $R = (r_i^j) : r(e_i) = \sum r_i^j \otimes e_j'$. Then $Y_1 R = R Y_2$, where Y_i are the multiplicative matrices defining the comodules. Any $R \in M(m \times n, k)$ with this property represents such a morphism.

(c) The direct sum of comodules represented by Y_1, Y_2 is represented by $\begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix}$.

(d) The tensor product of comodules represented by Y_1, Y_2 is represented by the Kronecker product $Y_1 \odot Y_2$;

$$(Y_1 \odot Y_2)_{ij}^{kl} = (Y_1)_i^k (Y_2)_j^l.$$

(e) Y is multiplicative for $(E, \Delta, \varepsilon) \Leftrightarrow Y^t$ is multiplicative for $(E, \Delta^{\mathrm{op}}, \varepsilon)$.

All these statements directly follow from the definitions. Note only that statement (d) changes in the category of superspaces: Some signs must enter in the definition of the Kronecker product (see Chapter 4).

One can say that multiplicative matrices form a category equivalent to that of finite-dimensional left comodules, with morphisms defined by (b).

Consider now the following situation. Let $M_i, i = 1,2.3$ be three left comodules over a bialgebra. Then the canonical isomorphism of the linear spaces $(M_1 \otimes M_2) \otimes M_3 \to M_1 \otimes (M_2 \otimes M_3)$ is also an isomorphism of comodules. This follows from the coassociativity axiom of Δ. However, the canonical isomorphism $M_1 \otimes M_2 \to M_2 \otimes M_1$ in general is not an isomorphism of comodules. In fact, it may well happen that these comodules are not isomorphic at all. For example, one-dimensional comodules correspond to multiplicative (or grouplike) elements that may well form a noncommutative group as in the Hopf algebra of functions on a finite non-abelian group.

Suppose, however, that for a comodule M, there exists a nontrivial isomorphism (or just a morphism) $r : M^{\otimes 2} \to M^{\otimes 2}$. If M is given by a matrix Z and r is given by R, we get from 3.5(b) and (d) a quadratic relation between the coefficients of Z,

$$(3.1) \qquad RZ \odot Z = Z \odot ZR.$$

3.6. Yang–Baxter Equations and Braid Groups. The standard isomorphism $S_{(12)} : M \otimes M \to M \otimes M$ can be used to define a representation of the symmetric group S_n on $M^{\otimes n}$: One decomposes a permutation into a product of transpositions and then takes the product of the corresponding linear operators. The result does not depend on the choice of the decomposition because the Coxeter relations

$$S_{(12)}S_{(23)}S_{(12)} = S_{(23)}S_{(12)}S_{(23)},$$
$$S^2_{(12)} = \mathrm{id}$$

form a presentation of S_3. The similar relations for an arbitrary linear operator $R \in \mathrm{End}(M \otimes M)$ (where $R_{12} = R \otimes \mathrm{id}$, etc.)

$$(3.2) \qquad R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} : M^{\otimes 3} \to M^{\otimes 3}$$

are called the (quantum) Yang–Baxter equation. An invertible solution of this equation allows us to define an action of the Artin braid group Bd_n on $M^{\otimes n}$. If in addition

$$(3.3) \qquad R^2_{12} = \mathrm{id}$$

("unitarity condition"), this action reduces to that of S_n.

L. D. Faddeev and his collaborators use the relations (3.2) and (3.1) as a basic definition for the class of quantum groups they consider (see [ReTF]). Starting from a Yang–Baxter operator R, they construct a bialgebra

$$E = k\langle z_i^j \rangle / (\text{coefficients of } RZ \odot Z - Z \odot ZR).$$

(The last stage, transition from E to its Hopf envelope, is discussed in [ReTF] in a less general setting, where the noncommutative localization can be replaced by a commutative one.)

It is worth mentioning that the coefficients of $RZ \odot Z - Z \odot ZR$ generate a coideal with respect to $\Delta(Z) = Z \otimes Z$ for any R, so that in the construction of E, the Yang–Baxter property plays no role at all.

3.7. Some Variations. Since the Yang–Baxter operators play the role of the structure constants of quantum (semi)groups with nice tensorial properties of the representation category, it is natural to discuss here various approaches to their classification.

We shall briefly comment upon some directions of recent research.

3.7.a. Classical Yang–Baxter Equations. Consider a Yang–Baxter operator R close to $S_{(12)}$, say, $R = S_{(12)} + hS_{(12)}r + 0(h^2)$, where h is a small parameter. By inserting this into Eq. (3.2) and considering the equation modulo h^3, we get the following *classical YB-equations for a linear operator* $r \in \mathrm{End}(E \otimes E) = \mathrm{End}(E)^{\otimes 2}$:

$$(3.4) \qquad [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

Here, say, $r_{12} = r \otimes \mathrm{id}$, and the commutator of r_{12} and r_{13} refers to their common first factor in $\mathrm{End}(E)^{\otimes 3}$.

V. G. Drinfeld considered Eq. (3.4) as an abstract equation in a Lie algebra $\mathfrak{g}$, related it to various beautiful structures in the classical Lie group theory (Poisson-Lie groups; isotropic triples), and discussed the quantization of a classical solution, i.e., its extension modulo growing powers of h (see [Dr1] and references therein).

3.7.b. Yang–Baxter Equations with Parameters. The initial discovery of Yang–Baxter operators was connected with one-dimensional quantum mechanics and led to a slightly different algebraic structure. Imagine a system of vector spaces $V(t)$ depending on a parameter t, e.g., fibers of a vector bundle over a space T. Suppose that an operator $R(t_1, t_2) \in GL(V(t_1) \otimes V(t_2))$ is given for generic points t_1, t_2 in such a way that

$$(R(t_1,t_2) \otimes \mathrm{id})(\mathrm{id} \otimes R(t_1,t_3))(R(t_2,t_3) \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes R(t_2,t_3))(R(t_1,t_3) \otimes \mathrm{id})(\mathrm{id} \otimes R(t_1,t_2)).$$

This means that the two ways to rearrange $V(t_1) \otimes V(t_2) \otimes V(t_3)$ in reverse order with the help of R coincide. Physically, R may correspond to the scattering operator of two particles moving with momenta t_1, t_2; $V(t_1)$ is an inner state space; momenta are conserved after the scattering.

One usually identifies all $V(t)$ (for example, by trivializing the vector bundle). A new feature of the situation is that one can now consider, for example, a meromorphic dependence of R on t_1, t_2; an important class of constructions leads to solutions with a pole at $t_1 = t_2$ so that one cannot rearrange fibers at the same point but only at different ones. Belavin and Drinfeld gave a classification of an important class of such solutions parametrized by an algebraic curve and having a pole of the first order on the diagonal.

They lead to bialgebras generated by families of multiplicative matrices $Z(t)$, with the commutation relations

$$R(t_1,t_2)Z(t_1) \odot Z(t_2) = Z(t_1) \odot Z(t_2)R(t_1,t_2),$$

which play an important role in two-dimensional physics.

3.7.c. Yang–Baxter Categories. A natural generalization of a Yang–Baxter operator and simultaneously a version of Yang–Baxter equations with parameters is given by the notion of a category with tensor product on tensor powers of objects on which a functorial action of braid groups is defined. We shall discuss this in some detail in the next section.

3.8. AN EXAMPLE: QUANTUM GL(2). As in the classical context, quantum GL(2) is a basic example and a germ of practically all aspects of the general theory.

3.8.a. Bialgebra $M_q(2)$. By definition, it is generated as an algebra by the coefficients of a matrix $Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ subject to the commutation relations

$$(3.5) \qquad \begin{aligned} &ab = q^{-1}ba; \quad ac = q^{-1}ca; \quad bd = q^{-1}db; \quad cd = q^{-1}dc; \\ &bc = cb; \quad ad - da = (q^{-1} - q)bc. \end{aligned}$$

Here $q \in k^*$ is an arbitrary parameter.

This algebra has a bialgebra structure uniquely defined by the condition that Z is multiplicative. A direct proof is possible but cumbersome. For a conceptual proof of this and other properties, see [Ma2,4], and Section 4.2

Note that $(M_q(2), m^{\rm op}\Delta^{\rm op})$ is isomorphic to $M_{q^{-1}}(2)$.

3.8.b. Quantum Determinant. Put

$$(3.6) \qquad D = \mathrm{DET}_q(Z) = ad - q^{-1}bc = da - qcb.$$

This is a multiplicative element: $\Delta(D) = D \otimes D, \varepsilon(D) = I$. It commutes with a, b, c, d. Moreover, if q is not a root of unity, D generates the center of $M_q(2)$.

3.8.c. Adjugate Matrix. We have

$$(3.7) \quad \begin{pmatrix} d & qb \\ -q^{-1}c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & qb \\ -q^{-1}c & a \end{pmatrix} = \begin{pmatrix} D & O \\ O & D \end{pmatrix}.$$

3.8.d. Hopf Algebra $\mathrm{GL}_q(2)$. By definition, it is $M_q(2)[D^{-1}]$ endowed with the obvious diagonal map and the antipodal map derived from Eq. (3.7):

$$\begin{aligned} &i(a) = D^{-1}d; \quad i(b) = -qD^{-1}b; \\ &i(c) = -q^{-1}D^{-1}c; \quad i(d) = D^{-1}a. \end{aligned}$$

Notice that the coefficients of $Z^{-1} = i(Z)$ satisfy the commutation relations of $M_q^{-1}(2)$, i.e., of the opposite bialgebra, as it should be, by general principles. A somewhat mysterious property of $\mathrm{GL}_q(2)$ is the following generalization.

3.8.e. "One-Parametric Subgroup" Passing via Z. The coefficients of Z^n satisfy Eq. (3.5) q^n for all integer n.[1]

In particular, if q is a root of unity, $M_q(2)$ and $\mathrm{GL}_q(2)$ contain a large commutative subring generated by the coefficients of Z^n. In fact, they are even finitely generated as modules over their centers.

3.8.f. Comodules. Put $\mathrm{SL}_q(2) = \mathrm{GL}_q(2)/(D-1)$. This is a Hopf algebra. Its category of, say, left comodules is semisimple precisely when q is not a root of unity. Simple comodules are classified by highest weights, as in the classical theory.

3.8.g. The Yang–Baxter Operator. Relations $(3.5)_q$ can be written in the form $RZ \odot Z = Z \odot ZR$, where R is the Yang–Baxter operator

$$R = \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

In fact, they were first discovered in this way. In Chapter 4, we reproduce another interpretation given in [Ma2].

We finish here our brief introduction to quantum groups based upon the ideas that originated in the work of the Leningrad school, Drinfeld, and Jimbo.

[1]This was remarked by Yu. Kobyzev in 1986 after my talk at a seminar (unpublished), and rediscovered recently by E. Corrigan, D. B. Fairlie, P. Fletcher, R. Sasaki (Preprint DTP-89-29, University of Durham, July 1989) and S. Vokos, B. Zumino, J. Wess (Preprint LAPP-TH-253/89, June 1989).