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An Introduction to Nonlinear Analysis

MARTIN SCHECHTER

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Martin Schechter
University of California, Irvine

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An Introduction to Nonlinear Analysis

The techniques that can be used to solve nonlinear problems are very different from those used to solve linear problems. Many courses in analysis and applied mathematics attack linear cases simply because they are easier to solve and do not require a large theoretical background in order to approach them. Professor Schechter's book is devoted to nonlinear methods using the least background material possible and the simplest linear techniques.

An understanding of the tools for solving nonlinear problems is developed while demonstrating their application to problems first in one dimension, and then in higher dimensions. The reader is guided using simple exposition and proof, assuming a minimal set of prerequisites. To complete, a set of appendices covering essential basics in functional analysis and metric spaces is included, making this ideal as a text for an upper-undergraduate or graduate course, or even for self-study.

MARTIN SCHECHTER is Professor of Mathematics at the University of California, Irvine.

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To my wife, Deborah, our children,
our grandchildren (all twenty one of them so far)
and our extended family.
May they enjoy many happy years.

Preface

The techniques that can be used to solve nonlinear problems are very different from those that are used to solve linear problems. Most courses in analysis and applied mathematics attack linear problems simply because they are easier to solve. The information that is needed to solve them is not as involved or technical in nature as that which is usually required to solve a corresponding nonlinear problem. This applies not only to the practical material but also to the theoretical background.

As an example, it is usually sufficient in dealing with linear problems in analysis to apply Riemann integration to functions that are piecewise continuous. Rarely is more needed. In considering the convergence of series, uniform convergence usually suffices. In general, concepts from functional analysis are not needed; linear algebra is usually sufficient. A student can go quite far in the study of linear problems without being exposed to Lebesgue integration or functional analysis.

However, there are many nonlinear problems that arise in applied mathematics and sciences that require much more theoretical background in order to attack them. If we couple this with the difficult technical details concerning the corresponding linear problems that are usually needed before one can apply the nonlinear techniques, we find that the student does not come in contact with substantive nonlinear theory until a very advanced stage in his or her studies. This is unfortunate because students having no more background in mathematics beyond that of second year calculus are often required by their disciplines to study such problems. Moreover, such students can readily understand most of the methods used in solving nonlinear problems.

During the last few years, the author has been giving a class devoted to nonlinear methods using the least background material possible and the simplest linear techniques. This is not an easy tightrope to walk. There are times when theorems from Lebesgue integration are required together with theorems from functional analysis. There are times when exact estimates for the linear problem are needed. What should one do?

My approach has been to explain the methods using the simplest terms. After I apply the methods to the solving of problems, I then prove them. True, I will need theorems from functional analysis and Lebesgue integration. At such times I explain the background theorems used. Then, the students have two options: either to believe me or to consult the references that I provide.

This brings me to the purpose of the text. I was unable to find a book that contained the material that I wanted to cover at the level that I wanted it presented. Moreover, I wanted to include a concise presentation (without proofs) of all of the background information that was needed to understand the techniques used in the body of the text. The writing of this book gave me the opportunity to accomplish both. If I think the students can handle it, I do prove background material in the body of the text. Otherwise, I explain it in four appendices in the back of the book. This also applies to topics that require a whole course to develop. This approach is intended to accommodate students at all levels. If they do not wish to see proofs of background materials, they can skip these sections. If they are familiar with functional analysis and Lebesgue integration, they can ignore the appendices.

The purpose of the course is to teach the methods that can be used in solving nonlinear problems. These include the contraction mapping theorem, Picard's theorem, the implicit function theorem, the Brouwer degree and fixed point theorem, the Schauder fixed point theorem, the Leray–Schauder degree, Peano's theorem, etc. However, the student will not appreciate any of them unless he or she observes them in action. On the other hand, if the applications are too complicated, the student will be bogged down in technical details that may prove to be extremely discouraging. This is another tightrope.

What surprised me was the amount of advanced background information that was needed to understand the methods used to attack even the easiest of nonlinear problems. I quote in the appendices only that material that is needed in the text. And yet, an examination of these appendices will reveal the substantial extent of this background knowledge. If we waited until the student had learned all of this, we would not be able to cover the material in the book until the student was well advanced. On the other hand, students with more modest backgrounds can understand the statements of the background theorems even though they have not yet learned the proofs. In fact, this approach can motivate such students to learn more advanced topics once they see the need for the material. In essence, I am advocating the cart before the horse. I want the student to appreciate the horse because it can be used to transport all of the items in the cart.

Equipping the student with the tools mentioned above is the main purpose of the book. Of course, one can first present a theorem and then give some applications. Most books function this way. However, I prefer to pose a problem and then introduce a tool that will help solve the problem. My choice of problems could be vast, but I tried to select those that require the least background. This outlook affected my choice: differential equations. I found them to require the least preparation. Moreover, most students have a familiarity with them. I picked them as the medium in which to work. The tools are the main objects, not the medium. True, the students do not know where they are headed, but neither does a research scientist searching for answers. It is also true that no matter what medium I pick, the nonlinear problems that the students will encounter in the future will be different. But as long as they have the basic tools, they have a decent chance of success.

I begin by posing a fairly modest nonlinear problem that would be easy to solve in the linear case. (In fact, we do just that; we solve the linear problem.) I then develop the tools that we use in solving it. I do this with two things in mind. The first is to develop methods that will be useful in solving many other nonlinear problems. The second is to show the student why such methods are useful. At the same time I try to keep the background knowledge needed to a minimum. In most cases the nonlinear tools require much more demanding information concerning the corresponding linear problem than the techniques used in solving the linear problem itself. I have made a concerted effort to choose problems that keep such required information to a minimum. I then vary the problem slightly to demonstrate how the techniques work in different situations and to introduce new tools that work when the original ones fail.

I then introduce new problems and new techniques that are used to solve them. The problems and techniques become progressively more difficult, but again I attempt to minimize the background material without ignoring important major nonlinear methods. My goal is to introduce as many nonlinear tools as time permits. I know that the students will probably not be confronted with the problems I have introduced, but they will have a collection of nonlinear methods and the knowledge of how they can be used.

In the first chapter we confront a seemingly simple problem for periodic functions in one dimension and go about solving it. The approach appears not to be related to the problem. We then fit the technique to the problem. (It is not at all obvious that the technique will work.) The student then sees how the techniques solve the problem. The chapter deals with the differentiation of functionals, Fourier series, finding minima of functionals and Hilbert space methods. I try to explain why each technique is used.

In the second chapter we consider the same problem for the cases when the functionals used in the first chapter have no minima. We begin with a simple algebraic problem in two dimensions. I introduce methods that can be used to solve it by producing saddle points, and then generalize the methods to arbitrary Hilbert spaces. We then apply them to the original problem. The tools used include the contraction mapping principle, Picard's theorem in a Banach space, extensions of solutions of differential equations in a Banach space and the sandwich theorem.

The third chapter leaves periodicity and deals with boundary value problems. I introduce mollifiers and test functions. As expected, different and stronger techniques are required.

The fourth chapter studies saddle points of functionals using such properties as convexity and lower semi-continuity. Conditions are given which produce saddle points, and these are applied to various problems. Partial differentiation is introduced, and the implicit function theorem is proved.

The fifth chapter discusses the calculus of variations, the Euler equations and the methods of obtaining minima. Necessary and sufficient conditions are given for the existence of minima. Many examples are presented.

In the sixth chapter I cover degree theory and its applications. Topics include the Brouwer and Schauder fixed point theorems, Sard's theorem, Peano's theorem and the Leray–Schauder degree. Applications are given.

The seventh chapter is devoted to constrained minima, of both the integral (iso-perimetric) and finite (point-wise) types. The Lagrange multiplier rule is proved and a more comprehensive type of differentiation is introduced.

The eighth chapter discusses mini-max techniques and gives examples.

In the ninth chapter I present the method of solving semi-linear equations which are sub-linear at infinity. We solve them by relating them to the Dancer–Fućík spectrum.

The tenth chapter is by far the largest; it is the first to tackle problems in higher dimensions. Even so, I limit the discussions to periodic functions. We consider spaces of distributions, Sobolev inequalities and Sobolev spaces. As expected, a lot of preparation is needed, and the proofs are more difficult. We generalize the one-dimensional results to higher dimensions.

There are four appendices. The first assembles the definitions and theorems (without proofs) from functional analysis that are needed in the text. I was surprised that so much was required. The second appendix deals with the theorems concerning Lebesgue integration required by the text. Again, proofs are omitted with one exception. We prove that Carathéodory functions are measurable. This theorem is not well known and hard to find in the literature. The third appendix describes what is needed concerning metric spaces. The fourth shows how pseudo-gradients can be used to strengthen some of the theorems in the text.

It is hoped that this volume will fill a need and will allow students with modest backgrounds to tackle important nonlinear problems.

TVSLB″O

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1
Extrema

1.1 Introduction

One of the most powerful tools in dealing with nonlinear problems is **critical point theory**. It originates from the fact in calculus that the derivative of a smooth function vanishes at extreme points (maxima and minima). In order to apply this basic reasoning, the given problem must be converted to one in which we look for points where the derivative of a function vanishes (i.e., critical points). This cannot always be arranged. But when it can, one has a very useful method. The easiest situation is when the function has extrema. We discuss this case in the present chapter. We present a problem and convert it into the desired form. We then give criteria that imply that extrema exist. The case when extrema do not exist will be discussed in the next chapter.

1.2 A one dimensional problem

We now consider the problem of finding a solution of

$$-u''(x) + u(x) = f(x, u(x)), \quad x \in I = [0, 2\pi], \tag{1.1}$$

under the conditions

$$u(0) = u(2\pi), \ u'(0) = u'(2\pi). \tag{1.2}$$

We assume that the function $f(x, t)$ is continuous in $I \times \mathbb{R}$ and is periodic in x with period 2π. If $u(x)$ is a solution, then we have

$$(u', v') + (u, v) = (-u'' + u, v) = (f(\cdot, u), v)$$

for all $v \in C^1(I)$ satisfying (1.2). Here,

$$(u, v) = \int_0^{2\pi} u(x)v(x)\,dx,$$