A. Bertram

Elasticity and Plasticity of Large Deformations

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Professor Dr. Albrecht Bertram
Otto-von-Guericke-Universität
Institut für Mechanik
Universitätsplatz 2
Gebäude N
39106 Magdeburg
Germany
bertram@mb.uni-magdeburg.de

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Preface

This book is based on the lecture notes of courses given by the author over the last decade at the Otto-von-Guericke University of Magdeburg and the Technical University of Berlin. Since the author is concerned with researching material theory and, in particular, elasto-plasticity, these courses were intended to bring the students close to the frontiers of today's knowledge in this particular field, an opportunity now offered also to the reader.

The reader should be familiar with vectors and matrices, and with the basics of calculus and analysis. Concerning mechanics, the book starts right from the beginning without assuming much knowledge of the subject. Hence, the text should be generally comprehensible to all engineers, physicists, mathematicians, and others.

At the beginning of each new section, a brief *Comment on the Literature* contains recommendations for further reading. Throughout the text we quote only the important contributions to the subject matter. We are far from being complete or exhaustive in our references, and we apologise to any colleagues not mentioned in spite of their important contributions to the particular items.

It is intended to indicate any corrections to this text on our website

http://www.uni-magdeburg.de/ifme/l-festigkeit/elastoplasti.html

along with remarks from the readers, who are encouraged to send their frank criticisms, comments and suggestions to

bertram@mb.uni-magdeburg.de.

All the author's royalties from this issue will be donated to charitable organisations like *Terres des Hommes*.

Acknowledgment. The author would like to thank his teachers RUDOLF TROSTEL, ARNOLD KRAWIETZ, and PETER HAUPT who taught him *Continuum Mechanics* in the early seventies, and since then have continued to give much helpful advice.

Many colleagues and friends also made useful comments and suggestions to improve this book, including ENRICO BROSCHE, CARINA BRÜGGEMANN, SAMUEL FOREST, SVEN KASSBOHM, THOMAS KLETSCHKOWSKI, JOHN KINGSTON, WOLFGANG LENZ, GERRIT RISY, MANUELA SCHILDT, MICHAEL SCHURIG, GABRIELE SCHUSTER, BOB SVENDSEN and, most of all, THOMAS BÖHLKE and ARNOLD KRAWIETZ, who gave countless valuable comments. The author is grateful to all of them.

List of Frequently Used Symbols

Sets

$\mathscr{B}$	body (-manifold)
$\mathscr{B}_0$	domain of the body in the reference placement
$\mathscr{B}_t$	domain of the body in the current placement
$\mathscr{E}$	EUCLIDean point space
$\mathscr{Elap}$	elastic range
$\mathscr{Inv}$	set of invertible tensors (general linear group)
$\mathscr{Lin}$	space of linear mappings from $\mathscr{V}$ to $\mathscr{V}$ (*2*nd-order tensors)
$\underline{\mathscr{Lin}}$	space of hardening variables in Chap. 10
$\mathscr{Orth}$	set of orthogonal *2*nd-order tensors (general orthogonal group)
$\mathscr{Psym}$	set of symmetric and positive-definite *2*nd-order tensors
$\mathscr{R}$	space of real numbers
$\mathscr{Sym}$	space of symmetric *2*nd-order tensors
$\mathscr{Skw}$	space of antisymmetric or skew *2*nd-order tensors
$\mathscr{Unim}$	set of *2*nd-order tensors with determinant ± 1 (general unimodular group)
$\mathscr{V}$	space of vectors (EUCLIDean shifters)

A superimposed + such as $\mathscr{Inv}^+$ means: with positive determinant. $\mathscr{R}^+$ denotes the positive reals.

Variables and Abbreviations

$\mathbf{a} = \mathbf{r}^{\bullet\bullet}$	$\in \mathscr{V}$	acceleration
$\mathbf{A}$	$\in \mathscr{Unim}^+$	symmetry transformation
$\mathbf{B} = \mathbf{F}\,\mathbf{F}^T$	$\in \mathscr{Psym}$	left CAUCHY-GREEN tensor
$\mathbf{b}$	$\in \mathscr{V}$	specific body force
$\mathbf{c}$	$\in \mathscr{V}$	translational vector in the EUCLIDean transform.
$\mathbf{c}_i$	$\in \mathscr{V}$	lattice vector in Chap. 10.5
$\mathbf{C} = \mathbf{F}^T\,\mathbf{F}$	$\in \mathscr{Psym}$	right CAUCHY-GREEN tensor
$\mathbf{C}_e = \mathbf{P}^T\,\mathbf{C}\,\mathbf{P}$	$\in \mathscr{Psym}$	transformed right CAUCHY-GREEN tensor in Chap. 10
COOS		coordinate system

$\mathbf{d}_O = \int_{\mathscr{B}_t} \mathbf{r}_O \times \mathbf{v}\, dm$	$\in \mathscr{V}$	moment of momentum with respect to O
$\mathbf{d}_\alpha$		directional vector of a slip system in Chap. 10.5
$\mathbf{D} = sym(\mathbf{L})$	$\in \mathscr{Sym}$	stretching tensor, rate of deformation tensor
da		element of area in current placement
da_0		element of area in reference placement
$\mathbf{da}$		vectorial element of area in current placement
$\mathbf{da}_0$		vectorial element of area in reference placement
Div, div		material and spatial divergence operator
dm		element of mass
dv		element of volume in current placement
dv_0		element of volume in reference placement
$\mathbf{dx}$		vectorial line element in current placement
$\mathbf{dx}_0$		vectorial line element in reference placement
$\mathbf{e}_i$	$\in \mathscr{V}$	basis vector of ONB
$\mathbf{E} = sym(\mathbf{H})$	$\in \mathscr{Sym}$	linear strain tensor
$\mathbf{E}^a = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$	$\in \mathscr{Sym}$	ALMANSI's strain tensor
$\mathbf{E}^b = \mathbf{I} - \mathbf{V}^{-1}$	$\in \mathscr{Sym}$	spatial BIOT's strain tensor
$\mathbf{E}^B = \mathbf{U} - \mathbf{I}$	$\in \mathscr{Sym}$	material BIOT's strain tensor
$\mathbf{E}^G = \frac{1}{2}(\mathbf{C} - \mathbf{I})$	$\in \mathscr{Sym}$	GREEN's strain tensor
$\mathbf{E}^{gen}$	$\in \mathscr{Sym}$	spatial generalised strain tensor
$\mathbf{E}^{Gen}$	$\in \mathscr{Sym}$	material generalised strain tensor
$\mathbf{E}^h = ln\,\mathbf{V}$	$\in \mathscr{Sym}$	spatial HENCKY's strain tensor
$\mathbf{E}^H = ln\,\mathbf{U}$	$\in \mathscr{Sym}$	material HENCKY's strain tensor
E	$\in \mathscr{R}$	internal energy
$\mathbf{f}$	$\in \mathscr{V}$	resulting force acting on the material body
$\mathbf{F} = Grad\,\boldsymbol{\chi}$	$\in \mathscr{Inv}^+$	deformation gradient
$\mathbf{g}_i, \mathbf{g}^i$	$\in \mathscr{V}$	basis vectors of dual bases
$\mathbf{g} = grad\,\theta$	$\in \mathscr{V}$	spatial temperature gradient
$\mathbf{g}_0 = Grad\,\theta$	$\in \mathscr{V}$	material temperature gradient
$Grad$, $grad$		material and spatial gradient operator
G_{ik}, G^{ik}	$\in \mathscr{R}$	metric coefficients
$\mathbf{H} = Grad\,\mathbf{u}$	$\in \mathscr{Lin}$	displacement gradient
$\mathbf{I}$	$\in \mathscr{Psym}$	unit tensor of 2nd-order

$\boldsymbol{I}$		unit tensor of *4*th-order
$\boldsymbol{I}^S$		symmetriser of *4*th-order
$\boldsymbol{I}^A$		antimetriser of *4*th-order
$J = det\,\mathbf{F}$	$\in \mathscr{R}^+$	JACOBIan, determinant of the deformation gradient
$K = \frac{1}{2} \int_{\mathscr{B}_t} \mathbf{v}^2\, dm$	$\in \mathscr{R}$	kinetic energy
$\mathbf{K} = Grad(\kappa_0\ \underline{\kappa}_0^{-1})$	$\in \mathscr{Inv}^+$	local change of the reference placement
$\mathbf{L} = grad\,\mathbf{v} = \mathbf{F}^\bullet\,\mathbf{F}^{-1}$	$\in \mathscr{Lin}$	velocity gradient
$l = \rho^{-1}\,\mathbf{T}\cdot\mathbf{L}$	$\in \mathscr{R}$	specific stress power
$\mathbf{m}_O$	$\in \mathscr{V}$	moment with respect to O
m	$\in \mathscr{R}^+$	mass
$\mathbf{n}$	$\in \mathscr{V}$	normal vector
$\mathbf{o}$	$\in \mathscr{V}$	zero vector
$\mathbf{0}$	$\in \mathscr{Sym}$	zero tensor
ONB		orthonormal basis
p	$\in \mathscr{R}$	pressure
$\mathbf{p} = \int_{\mathscr{B}_t} \mathbf{v}\, dm$	$\in \mathscr{V}$	linear momentum
$\mathbf{P}$	$\in \mathscr{Inv}$	plastic transformation in Chap. 10
PFI	Principle of form invariance	
PISM	Principle of invariance under superimposed rigid body motions	
PMO	Principle of material objectivity	
$\mathbf{q}$	$\in \mathscr{V}$	spatial heat flux vector
$\mathbf{q}_0 = J\,\mathbf{F}^{-1}\,\mathbf{q}$	$\in \mathscr{V}$	material heat flux vector
Q	$\in \mathscr{R}$	heat supply
$\mathbf{Q}$	$\in \mathscr{Orth}^+$	versor in EUCLIDean transformation
$\mathbf{r}$	$\in \mathscr{V}$	position vector
r	$\in \mathscr{R}$	specific heat source
$\mathbf{R}$	$\in \mathscr{Orth}^+$	rotation tensor
$\mathbf{S} = \mathbf{F}^{-1}\,\mathbf{T}\,\mathbf{F}^{-T}$	$\in \mathscr{Sym}$	material stress tensor
$\mathbf{S}^P = -\,\mathbf{C}\,\mathbf{S}\,\mathbf{P}^{-T}$	$\in \mathscr{Lin}$	plastic stress tensor in Chap. 10
$\mathbf{T}$	$\in \mathscr{Sym}$	CAUCHY′s stress tensor
$\mathbf{T}^{\mathrm{B}} = sym(\mathbf{T}^{\mathrm{2PK}}\,\mathbf{U})$	$\in \mathscr{Sym}$	BIOT′s stress tensor
$\mathbf{T}^{\mathrm{K}} = J\,\mathbf{T}$	$\in \mathscr{Sym}$	KIRCHHOFF′s stress tensor
$\mathbf{T}_{\mathrm{k}} = \mathbf{F}^T\,\mathbf{T}\,\mathbf{F}$	$\in \mathscr{Sym}$	convected stress tensor

$\mathbf{T}^{\mathrm{M}} = \mathbf{C}\,\mathbf{T}^{2\mathrm{PK}}$	$\in \mathscr{Lin}$	MANDEL's stress tensor
$\mathbf{T}_{\mathrm{r}} = \mathbf{R}^T\,\mathbf{T}\,\mathbf{R}$	$\in \mathscr{Sym}$	relative stress tensor
$\mathbf{T}^{1\mathrm{PK}} = J\,\mathbf{T}\,\mathbf{F}^{-T}$	$\in \mathscr{Lin}$	*1*st PIOLA-KIRCHHOFF stress tensor
$\mathbf{T}^{2\mathrm{PK}} = J\,\mathbf{F}^{-1}\,\mathbf{T}\,\mathbf{F}^{-T}$	$\in \mathscr{Sym}$	*2*nd PIOLA-KIRCHHOFF stress tensor
$\mathbf{t} = \mathbf{T}\,\mathbf{n}$	$\in \mathscr{V}$	stress vector on a surface
t	$\in \mathscr{R}$	time
$\mathbf{U} = \sqrt{\mathbf{C}}$	$\in \mathscr{Psym}$	right stretch tensor
$\mathbf{u}$	$\in \mathscr{V}$	displacement vector
$\mathbf{V} = \sqrt{\mathbf{B}}$	$\in \mathscr{Psym}$	left stretch tensor
V	$\in \mathscr{R}^+$	volume in the current placement
V_0	$\in \mathscr{R}^+$	volume in the reference placement
$\mathbf{v} = \mathbf{r}^\bullet$	$\in \mathscr{V}$	velocity
$\mathbf{w}$	$\in \mathscr{V}$	spin vector (axial vector of $\mathbf{W}$)
$\mathbf{W} = skw(\mathbf{L})$	$\in \mathscr{Skw}$	spin tensor
w	$\in \mathscr{R}$	specific elastic energy
$\mathbf{x}$	$\in \mathscr{V}$	position vector in space
$\mathbf{x}_0$	$\in \mathscr{V}$	position vector in the reference placement
$\mathbf{Y}_p = \mathbf{C}_e^\bullet - \mathbf{P}^T\,\mathbf{C}^\bullet\,\mathbf{P}$	$\in \mathscr{Sym}$	incremental plastic variable in Chap. 10
$\mathbf{Z}$	$\in \underline{\mathscr{Lin}}$	hardening variables in Chap. 10

Greek Letters

χ		motion
δ		variation, virtual
$\delta_i^{\ j}$, $\delta^j_{\ i}$, δ_{ij}, δ^{ij}		KRONECKER symbols
δ	$\in \mathscr{R}$	specific dissipation
ε	$\in \mathscr{R}$	specific internal energy
ε_{ij}		linear strains (components of $\mathbf{E}$)
ε_{ijk}		permutation or LEVI-CIVITA-symbol
η	$\in \mathscr{R}$	specific entropy
φ^i		spatial coordinates
φ		yield criterion in Chap. 10
Φ_p		yield criterion for an elastic range in Chap. 10
κ		placement
κ_0		reference placement
λ	$\in \mathscr{R}^+$	plastic consistency-parameter in Chap. 10
$\lambda_i = \mu_i^2$	$\in \mathscr{R}^+$	eigenvalue of $\mathbf{C}$ and $\mathbf{B}$
μ_i	$\in \mathscr{R}^+$	eigenvalue of $\mathbf{U}$ and $\mathbf{V}$

θ	$\in \mathscr{R}^+$	temperature
ρ	$\in \mathscr{R}^+$	mass density in the current placement
ρ_0	$\in \mathscr{R}^+$	mass density in the reference placement
τ	$\in \mathscr{R}$	process time
ψ	$\in \mathscr{R}$	specific free energy
Ψ^i		material coordinates
$I_\mathbf{T}$, $II_\mathbf{T}$, $III_\mathbf{T}$	$\in \mathscr{R}$	principal invariants of a tensors **T**
$[A_{pj}]$		matrix with components A_{pj}

Compositions

$\cdot$	inner or scalar product
$\times$	between vectors: vector or cross product
$\times$	between sets: Cartesian product
$\otimes$	tensor product or dyadic product
$*$	RAYLEIGH product

Logical Symbols

$\in$	"element of"
$\forall$	"for all"
$\wedge$	"and"
$=$	equality
$:=$	equal by definition
$\equiv$	identification ("is set equal in this place")
$\approx$	approximate equality ("almost the same")
$\Leftrightarrow$	equivalence ("if and only if")
$\Rightarrow$	implication ("thus")

List of Selected Scientists

Emilio **Almansi** (1869-1948)
Johann **Bauschinger** (1834-1893)
Maurice Anthony **Biot** (1905-1985)
Ludwig **Boltzmann** (1844-1904)
Augustin Louis **Cauchy** (1789-1857)
Arthur **Cayley** (1821-1895)
Elwin Bruno **Christoffel** (1829-1900)
Rudolf Julius Emmanuel **Clausius** (1822-1888)
Gaspard Gustave de **Coriolis** (1792-1843)
Eugène Maurice Pierre **Cosserat** (1866-1931)
Francois **Cosserat** (1852-1914)
Maurice **Couette** (1858-1943)
Daniel C. **Drucker** (1918-2001)
Pierre-Maurice-Marie **Duhem** (1861-1916)
Jerald Laverne **Ericksen** (1924-)
Euclid (ca. 320-260 a. c.)
Leonhard **Euler** (1707-1783)
Joseph **Finger** (1841-1925)
Jean Baptiste Joseph de **Fourier** (1768-1830)
Maurice Rene **Fréchet** (1878-1973)
Galileo **Galilei** (1564-1642)
René-Eugène **Gâteaux** (1887-1914)
Carl Friedrich **Gauß** (1777-1855)
Josiah Willard **Gibbs** (1839-1903)
George **Green** (1793-1841)
Georg **Hamel** (1877-1954)
William Rowan **Hamilton** (1805-1865)
Hermann Ludwig Ferdinand von **Helmholtz** (1821-1894)
Heinrich **Hencky** (1885-1951)
Rodney **Hill** (1921-)
Robert **Hooke** (1635-1703)
Maksymilian Tytus **Huber** (1872-1950)
Carl Gustav Jacob **Jacobi** (1804-1851)
James Prescott **Joule** (1818-1889)
Gustav Robert **Kirchhoff** (1824-1887)
Joseph Louis **Lagrange** (1736-1813)
Gabriel **Lamé** (1795-1870)
Edmund **Landau** (1877-1938)
Max von **Laue** (1879-1960)
Erastus Henry **Lee** (1916-)
Gottfried Wilhelm **Leibniz** (1646-1716)
Tullio **Levi-Civita** (1873-1941)
Marius Sophus **Lie** (1842-1899)

Jean **Mandel** (1907-1982)
Robert Julius **Mayer** (1814-1878)
Richard von **Mises** (1883-1953)
Paul M. **Naghdi** (1924-1994)
Claude Louis Marie Henri **Navier** (1785-1836)
Carl Gottfried **Neumann** (1832-1925)
Isaac **Newton** (1643-1727)
Walter **Noll** (1925-)
James G. **Oldroyd** (1921-1982)
Gabbrio **Piola** (1791-1850)
Max Karl Ernst Ludwig **Planck** (1858-1947)
William **Prager** (1903-1980)
Ludwig **Prandtl** (1875-1953)
Lord **Rayleigh** (John William Strutt) (1842-1919)
Markus **Reiner** (1886-1976)
Endre (A.) **Reuss** (1900-1968)
Osborne **Reynolds** (1842-1912)
Hans **Richter** (1912-1978)
Bernhard Georg Friedrich **Riemann** (1826-1866)
Friedrich **Riesz** (1880-1956)
Ronald **Rivlin** (1915-)
Olinde **Rodrigues** (1794-1851)
Wilhelm Conrad **Röntgen** (1845-1923)
Adhémar Jean Claude Barré **de Saint-Venant** (1797-1886)
Erich **Schmid** (1896-1983)
Hermann Amandus **Schwarz** (1843-1921)
George Gabriel **Stokes** (1829-1903)
James Joseph **Sylvester** (1814-1897)
Geoffry Ingram **Taylor** (1886-1975)
Henri Edouard **Tresca** (1814-1885)
Clifford Ambrosius **Truesdell** (1919-2000)
Francois **Viète** (1540-1603)
Woldemar **Voigt** (1850-1919)

Contents

Introduction

While several text books on non-linear elasticity have already been published, the subject of finite plasticity appears not yet to have gained the maturity of a textbook science, in spite of its paramount importance for practical applications in, e.g., metal-forming simulations. One of the reasons for this surprising fact is that even the fundamental concepts of plasticity still lack a rational introduction based on clear physical and mathematical reasoning. One of the aims of this book is to at least partially reduce such shortcomings of plasticity theory (see the last chapter).

We intend to introduce plasticity in a *rational* way, i.e., based on axiomatic assumptions with clear physical and mathematical meanings. We are far from believing that this has already been successfully completed. Too many questions are still to be answered and need further concretisation. So the part of this book on plasticity can be considered as an essay, which shall stimulate and encourage further research work.

In our opinion, it takes a solid grounding in finite elasticity to properly understand finite plasticity. Therefore, we will introduce the fundamental concepts of elasticity in some detail. Before we do so, however, a careful description of non-linear continuum mechanics shall be given. As almost all quantities in continuum mechanics are described by tensors or tensor fields of different order, the book starts with an introduction to tensor algebra and analysis. This is necessary because many different notations of tensor calculus are used in the literature.

In mechanics, like any other precise science, it is impossible in principal to *define* all concepts. Certain concepts and laws have to be introduced as primitive ones, relying on an *a priori* empirical understanding of the reader. Based on these primitive concepts, we can then derive other concepts *by definition*. And only if a certain structure of the theory has been constructed, can we try and assess the validity of our axioms. Such an axiomatic approach is aimed at in the present context. We try to give all the concepts used a certain mathematical rigour, without revelling in pure formalisms.

The author considers himself as following in the tradition of the *Berlin school of continuum mechanics*, which was made famous by the likes of GEORG HAMEL, ISTVAN SZABO, WALTER NOLL, RUDOLF TROSTEL, HUBERTUS WEINITSCHKE, ARNOLD KRAWIETZ, and PETER HAUPT. It seems that this school no longer exists in Berlin, some of its members have already passed away, while others left to spread these ideas across the world.

A word on notation. It should be mentioned in passing that one of the early habits of this school was to use a direct notation for vectors and tensors, long before it became fashionable. Of course, we will adopt this elegant notation here whenever it is feasible.

It is practically impossible to give each symbol a unique meaning, without drowning in indices, tildes, primes, etc. The reader will therefore be confronted with some double meanings in the book, but we have always tried to avoid confusion. For this purpose, a list of the symbols repeatedly used in different parts of the book has been included at the beginning of the book. We have always tried to use notations that are common in today's literature, which has been hugely influenced by the masterpiece of TRUESDELL/ NOLL (1964), which included elasticity, but unfortunately not plasticity. Only on a few occasions do we prefer different notations. As an example, all scalar products in this text are denoted by a dot, regardless of the rank of tensors involved.

1 Mathematical Preparation

A Comment on the Literature. Although this mathematical preparation is rather detailed in the field of tensor calculus, basic knowledge of mathematics is required. For further reading we recommend the books by LOOMIS/ STERNBERG (1968), CHOQUET-BRUHAT et al. (1977), and FLEMING (1977).

In continuum mechanics physical quantities can be

- **scalars** or **reals**, like time, energy, power,
- **vectors**, like position vectors, velocities, or forces,
- **tensors**, like deformation and stress measures.

Since we can also interpret scalars as *0*th-order tensors, and vectors as *1*st-order tensors, all continuum mechanical quantities can generally be considered as tensors of different orders.

They can either be defined for the whole body as a **global variable** (like the resulting force acting on a body), or as **local** or **field variables**, i.e. defined in every point of a body (like its velocity).

Our short mathematical preparation shall make us familiar with the mathematical concept of tensors. We will start with the *algebra of tensors*. Thereafter, we will consider tensor functions or tensor fields, on which we can perform calculus or *tensor analysis*. Finally, we will consider integrals over such fields, which will give us resulting tensors or mean values of fields.

Notations. The standard notation of scalars (reals) is by italic letters like α, β, ... or a, b, ... For vectors we will use small Latin letters in bold like $\mathbf{a}$, $\mathbf{b}$, ... *2*nd-order tensors are notated by large Latin letters in bold like $\mathbf{A}$, $\mathbf{B}$, ... For *4*th-order tensors we use larger and italic letters like $\boldsymbol{\mathcal{A}}$, $\boldsymbol{\mathcal{B}}$ ... . Sets and spaces are notated in cursive, like

$\mathscr{R}$	the set of real numbers
$\mathscr{R}^+$	the set of positive real numbers
$\mathscr{R}^n$	the set of n ordered real numbers (n-tuples).

If $\mathscr{V}$ and $\mathscr{W}$ are sets, a **mapping** or **function**

$$f : \mathscr{V} \rightarrow \mathscr{W}$$

assigns to each element of its **domain** $\mathscr{V}$ uniquely one element of its **range** $\mathscr{W}$. If we want to give the variables names, we write

$$f : v \mapsto w \, .$$

Clearly, the **argument** is $v \in \mathscr{V}$ and its **value** or **image** under f is $w = f(v) \in \mathscr{W}$.