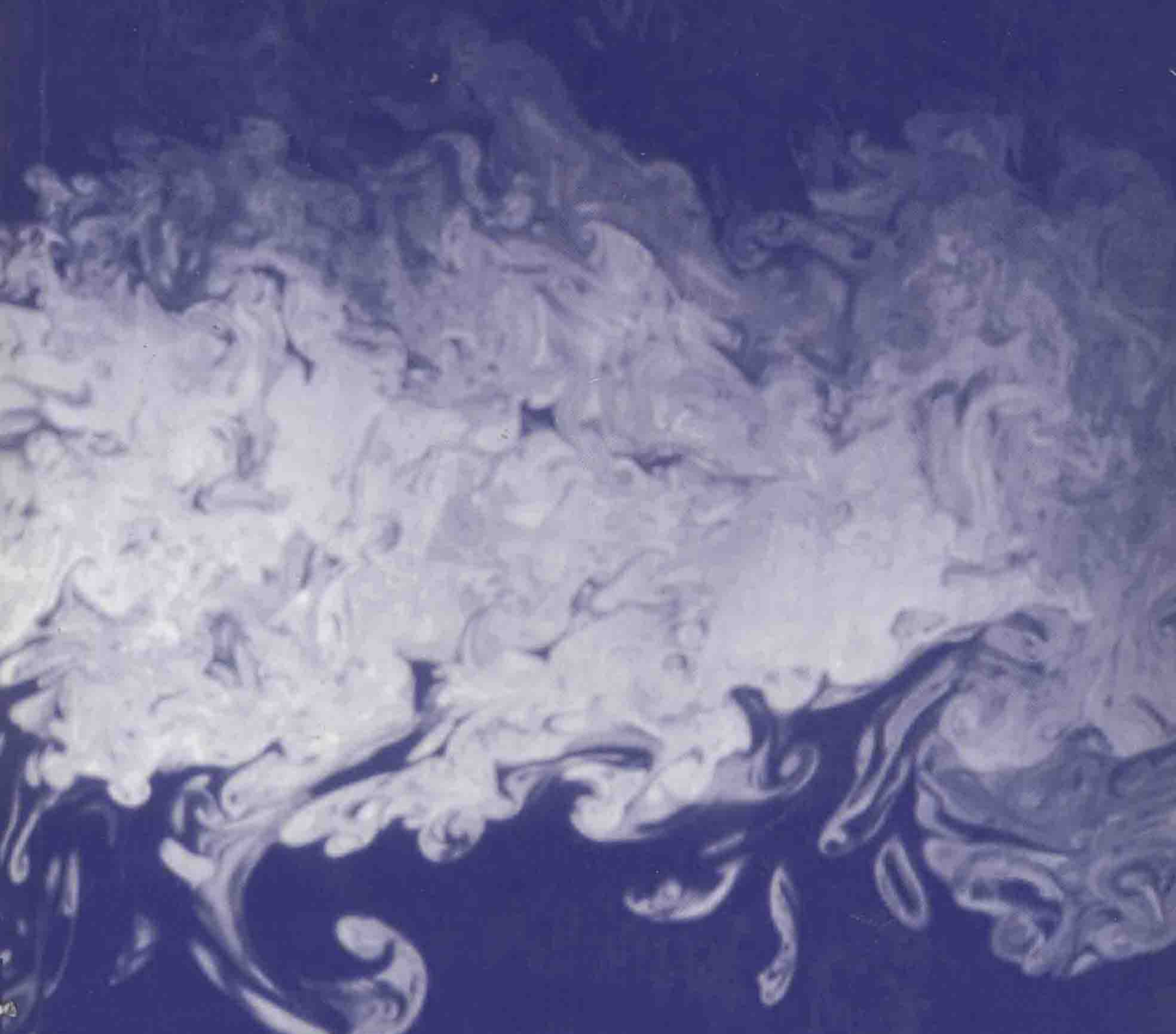

Turbulent Flows

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Turbulent Flows

This a graduate text on turbulent flows, an important topic in fluid dynamics. It is up to date, comprehensive, designed for teaching, and based on a course taught by the author at Cornell University for a number of years.

The book consists of two parts followed by a number of appendices. Part I provides a general introduction to turbulent flows, how they behave, how they can be described quantitatively, and the fundamental physical processes involved. The topics covered include: the Navier–Stokes equations; the statistical representation of turbulent fields; mean-flow equations; the behavior of simple free shear and wall-bounded flows; the energy cascade; turbulence spectra; and the Kolmogorov hypotheses. Part II is concerned with various approaches for modelling or simulating turbulent flows. The approaches described are: direct numerical simulation (DNS); turbulent-viscosity models (e.g., the k–ε model); Reynolds-stress models; probability-density-function (PDF) methods; and large-eddy simulation (LES). There are numerous appendixes in which the necessary mathematical techniques are presented.

This book is primarily intended as a graduate-level text in turbulent flows for engineering students, but it may also be valuable to students in applied mathematics, physics, oceanography, and atmospheric sciences, as well as researchers, and practicing engineers.

STEPHEN B. POPE is the Sibley College Professor in the Sibley School of Mechanical and Aerospace Engineering at Cornell University. Over the past 25 years he has performed research on turbulent flows and turbulent combustion, covering a broad range of approaches. He pioneered the application of PDF methods to turbulent flows; and his work on DNS won First Prize in the 1990 IBM Supercomputing Competition. Prior to joining Cornell University in 1982, Pope held positions at the Massachussetts Institute of Technology, the California Institute of Technology and Imperial College, London, from which he received his PhD and DSc. He is a Fellow of the American Physical Society, Division of Fluid Dynamics, an Overseas Fellow of Churchill College, Cambridge, and holds a visiting appointment at Delft University of Technology as J.M. Burgers Center Professor. He is a member of the editorial boards of several journals, including *Physics of Fluids* and *Flow, Turbulence and Combustion.*

To my wife Linda
and to our children
Sarah and Sam

Preface

This book is primarily intended as a graduate text on turbulent flows for engineering students, but it may also be valuable to students in atmospheric sciences, applied mathematics, and physics, as well as to researchers and practicing engineers.

The principal questions addressed are the following.

(i) How do turbulent flows behave?
(ii) How can they be described quantitatively?
(iii) What are the fundamental physical processes involved?
(iv) How can equations be constructed to simulate or model the behavior of turbulent flows?

In 1972 Tennekes and Lumley produced a textbook that admirably addresses the first three of these questions. In the intervening years, due in part to advances in computing, great strides have been made toward providing answers to the fourth question. Approaches such as Reynolds-stress modelling, probability-density-function (PDF) methods, and large-eddy simulation (LES) have been developed that, to an extent, provide quantitative models for turbulent flows. Accordingly, here (in Part II) an emphasis is placed on understanding how model equations can be constructed to describe turbulent flows; and this objective provides focus to the first three questions mentioned above (which are addressed in Part I). However, in contrast to the book by Wilcox (1993), this text is not intended to be a practical guide to turbulence modelling. Rather, it explains the concepts and develops the mathematical tools that underlie a broad range of approaches.

There is a vast literature on turbulence and turbulent flows, with many worthwhile questions addressed by many different approaches. In a one-semester course, or in a book of reasonable length, it is possible to cover only a fraction of the topics, and then with only a few of the possible approaches. The present selection of topics and approaches has evolved over the 20 years I have been teaching graduate courses on turbulence at MIT and Cornell. The emphasis on turbulent flows – rather than on the theory of homogeneous turbulence – is appropriate to applications in engineering, atmospheric sciences, and elsewhere. The emphasis on quantitative theories and models is consistent with the scientific objective – of developing a tractable, quantitatively accurate theory of the phenomenon – and is ideal for providing a solid understanding of computational approaches to turbulent flows, e.g., turbulence models and LES.

With the exceptions of LES and direct numerical simulation (DNS), the theories and models presented stem from the *statistical* approach, pioneered by Osborne Reynolds, G. I. Taylor, Prandtl, von Kármán, and Kolmogorov. A sizable fraction of the academic research work in the last 25 years has emphasized a more *deterministic* viewpoint: for example experiments on coherent structures, and models based on low-dimensional dynamical systems (e.g., Holmes, Lumley, and Berkooz (1996)). At this stage, this alternative approach has not led to a generally applicable quantitative model, neither – for better or for worse – has it had a major impact on the statistical approaches. Consequently, the deterministic viewpoint is neither emphasized nor systematically presented.

The book consists of two parts followed by a number of appendices. Part I provides a general introduction to turbulent flows, including the Navier–Stokes equations, the statistical representation of turbulent fields, mean-flow equations, the behavior of simple free-shear and wall-bounded flows, the energy cascade, turbulence spectra, and the Kolmogorov hypotheses. In the first five chapters, the focus is first on the mean velocity fields, and how they are affected by the Reynolds stresses. The concept of 'turbulent viscosity' is introduced with a thorough discussion of its deficiencies. The focus then shifts to the turbulence itself, in particular to the production and dissipation of turbulent kinetic energy. This sets the stage for a description (in Chapter 6) of the energy cascade and the Kolmogorov hypotheses. The spectral description of homogeneous turbulence in terms of Fourier modes in wavenumber space is developed in some detail. This provides an alternative perspective on the energy cascade; and it is also used in subsequent chapters in the descriptions of DNS, LES, and rapid distortion theory (RDT).

Simple wall-bounded flows are described in Chapter 7, starting with the mean velocity fields and proceeding to the Reynolds stresses. The exact transport equations for the Reynolds stresses are introduced, and their balances in turbulent boundary layers are examined.

The simulation and modelling approaches described in Part II are: DNS, turbulent viscosity models (e.g., the k–ε model), Reynolds-stress models, PDF methods, and LES. It is natural to consider DNS first (in Chapter 9) since it is conceptually the most straightforward approach. However, its restriction to simple, low-Reynolds-number flows motivates the consideration of other approaches. The most widely used turbulence models are the turbulent-viscosity models described in Chapter 10. Reynolds-stress models (Chapter 11) provide a more satisfactory connection to the physics of turbulence. The Reynolds-stress balance equations can be obtained from the Navier–Stokes equations, and the various contributions to this balance have been measured in experiments and simulations. Rapid-distortion theory is introduced to shed light on the effects that mean velocity gradients have on the Reynolds stresses. In developing and presenting modelled Reynolds-stress equations, the emphasis is on the fundamental concepts and principles, rather than on the detailed forms of particular models.

Chapter 12 deals with PDF methods. The primary object of study is the (one-point, one-time, Eulerian) joint probability density function (PDF) of velocity. The first moments of this PDF are the mean velocities; the second moments are the Reynolds stresses. For several reasons it is both natural and advantageous to proceed from the Reynolds stress to the PDF level of description: in the PDF equation, convection (by both mean and fluctuating velocity) appears in closed form, and hence does not have to be modelled; the effect of rapid distortions on turbulence can (in a limited sense) be treated exactly; and PDF methods are becoming widely used for turbulent reactive flows (e.g., turbulent combustion) because they are able to treat reaction exactly – without modelling assumptions.

Essential ingredients in PDF methods are stochastic Lagrangian models, such as the Langevin model for the velocity following a fluid particle. These models are also described in the context of turbulent dispersion (where they originated with G. I. Taylor's 1921 classic paper).

The final chapter describes LES, in which the large-scale turbulent motions are directly represented, while the effects of the smaller, subgrid-scale motions are modelled. Many of the concepts and techniques developed in Chapters 9–12 find application in the modelling of the subgrid-scale processes.

I use this book in a one-semester course, taught to students who previously have taken one or more graduate courses in fluid mechanics and applied mathematics. For most students, there is a good deal of new material, but I find that they can successfully master it, provided that it is clearly and fully explained. Accordingly there are many appendixes that provide the necessary development and explanation of mathematical techniques and results used in the text. In my experience, it is best not to rely upon the students' prior knowledge of probability theory, and consequently the necessary material is provided in the text (e.g., Sections 3.2–3.5).

For a less demanding pace, Parts I and II can be covered in two semesters – there is ample material. Alternatively, if a coverage of modelling is not required, Part I by itself provides a reasonably complete introduction to turbulent flows.

Many of the exercises ask the reader to 'show that ...,' and thereby introduce additional results and observations. Consequently, it is recommended that all the exercises be read, even if they are not performed. The book is designed to be a self-contained text, but sufficient references are given to provide an entry into the research literature.

However much care is taken in the preparation of a book of this nature, it is inevitable that there will be errors in the first printing. A list of known corrections is given at `http://mae.cornell.edu/~pope/TurbulentFlows`. The reader is asked to report any further corrections to the author at `pope@mae.cornell.edu`.

I am profoundly grateful to many people for their help in the preparation of this work. For their support and technical input I thank my colleagues at Cornell, David Caughey, Sidney Leibovich, John Lumley, Dietmar Rempfer, and Zellman Warhaft. For their valuable suggestions based on reading draft chapters, I am grateful to Peter Bradshaw, Paul Durbin, Rodney Fox, Kemo Hanjalić, Charles Meneveau, Robert Moser, Blair Perot, Ugo Piomelli, P. K. Yeung, and Norman Zabusky. Similarly, I am grateful to the following Cornell graduates for their feedback on drafts of the book: Bertrand Delarue, Thomas Dreeben, Matthew Overholt, Paul Van Slooten, Jun Xu, Cem Albukrek, Dawn Chamberlain, Timothy Fisher, Laurent Mydlarski, Gad Reinhorn, Shankar Subramaniam, and Walter Welton. The first five mentioned are also thanked for their assistance in producing the figures. Most of the typescript was prepared by June Meyermann, whose patience, accuracy, and enthusiasm are greatly appreciated. The accuracy of the bibliography has been much improved by the careful checking performed by Sarah Pope. Above all, I wish to thank my wife, Linda, for her patience, support, and encouragement during this project and over the years.

Nomenclature

The notation used is given here in the following order: upper-case Roman, lower-case Roman, upper-case Greek, lower-case Greek, superscripts, subscripts, symbols, and abbreviations. Then the symbols $\mathcal{O}(\ )$, $o(\ )$, and $\sim$ that are used to denote the order of a quantity are explained.

Upper-case Roman

A^+	van Driest constant (Eq. (7.145))
$\mathcal{A}$	control surface bounding $\mathcal{V}$
B	log-law constant (Eq. (7.43))
B_1	constant in the velocity-defect law (Eq. (7.50))
B_2	Loitsyanskii integral (Eq. (6.92))
B_2	log-law constant for fully-rough walls (Eq. (7.120))
$\tilde{B}(s/\delta_\nu)$	log-law constant for rough walls (Eq. (7.121))
C	Kolmogorov constant related to $E(\kappa)$ (Eq. (6.16))
C_0	coefficient in the Langevin equation (Eqs. (12.26) and (12.100))
C_1	Kolmogorov constant related to $E_{11}(\kappa_1)$ (Eq. (6.228))
C_1'	Kolmogorov constant related to $E_{22}(\kappa_1)$ (Eq. (6.231))
C_2	Kolmogorov constant related to D_{LL} (Eq. (6.30))
C_2	constant in the IP model (Eq. (11.129))
C_3	constant in the model equation for ω^* (Eq. (12.194))
C_{E}	LES dissipation coefficient (Eq. (13.285))
C_{f}	skin-friction coefficient $(\tau_{\mathrm{w}}/(\frac{1}{2}\rho\bar{U}^2))$
C_{R}	Rotta constant (Eq. (11.24))
C_{S}	Smagorinsky coefficient (Eq. (13.128))
C_{s}	constant in Reynolds-stress transport models (Eq. (11.147))

C_ε	constant in the model equation for ε (Eq. (11.150))
$C_{\varepsilon 1}$, $C_{\varepsilon 2}$	constants in the model equation for ε (Eq. (10.53))
C_μ	turbulent-viscosity constant in the k–ε model (Eq. (10.47))
C_ν	LES eddy-viscosity coefficient (Eq. (13.286))
C_ϕ	constant in the IEM mixing model (Eq. (12.326))
C_Ω	constant in the definition of Ω (Eq. (12.193))
$C_{\omega 1}$, $C_{\omega 2}$	constants in the model equation for ω (Eq. (10.93))
$\mathcal{C}_0$	Kolmogorov constant (Eq. (12.96))
$\mathcal{C}_{ij}^o$	cross stress (Eq. (13.101))
D	pipe diameter
D_{ij}	second-order velocity structure function (Eq. (6.23))
$D_{\mathrm{L}}(s)$	second-order Lagrangian structure function (Eq. (12.95))
D_{LL}	longitudinal second-order velocity structure function
D_{LLL}	longitudinal third-order velocity structure function (Eq. (6.86))
D_{NN}	transverse second-order velocity structure function
$D_n(r)$	nth-order longitudinal velocity structure function (Eq. (6.304))
$\mathrm{D}/\mathrm{D}t$	substantial derivative $(\partial/\partial t + \boldsymbol{U}\cdot\nabla)$
$\bar{\mathrm{D}}/\bar{\mathrm{D}}t$	mean substantial derivative $(\partial/\partial t + \langle \boldsymbol{U}\rangle\cdot\nabla)$
$\overline{\mathrm{D}}/\overline{\mathrm{D}}t$	substantial derivative based on filtered velocity
E	Cartesian coordinate system with basis vectors $\boldsymbol{e}_i$
$\overline{E}$	Cartesian coordinate system with basis vectors $\overline{\boldsymbol{e}}_i$
$E(\boldsymbol{x},t)$	kinetic energy $(\frac{1}{2}\boldsymbol{U}\cdot\boldsymbol{U})$
$\bar{E}(\boldsymbol{x},t)$	kinetic energy of the mean flow $(\frac{1}{2}\langle\boldsymbol{U}\rangle\cdot\langle\boldsymbol{U}\rangle)$
$\dot{E}(x)$	kinetic energy flow rate of the mean flow
$E(\kappa)$	energy-spectrum function (Eq. (3.166))
$E_{ij}(\kappa_1)$	one-dimensional energy spectrum (Eq. (6.206))
$\overline{E}(\kappa)$	energy-spectrum function of filtered velocity (Eq. (13.62))
$E(\omega)$	frequency spectrum (defined for positive frequencies, Eq. (3.140))
$\check{E}(\omega)$	frequency spectrum (defined for positive and negative frequencies, Eq. (E.31))
F	determinant of the normalized Reynolds stress (Eq. (11.52))
$F(V)$	cumulative distribution function (CDF) of U (Eq. (3.7))
$F_{\mathrm{D}}(y/\delta)$	velocity-defect law (Eq. (7.46))
$\mathcal{F}$	Fourier transform (Eq. (D.1))
$\mathcal{F}^{-1}$	inverse Fourier transform (Eq. (D.2))

$\mathcal{F}_{\boldsymbol{\kappa}}$	Fourier integral operator (Eq. (6.116))
G_{ij}	coefficient in the GLM (Eqs. (12.26) and (12.110))
$G(r)$	LES filter function
$\widehat{G}(\kappa)$	LES filter transfer function
H	shape factor (δ^*/θ)
$H(x)$	Heaviside function (Eq. (C.33))
$\mathbf{I}$	identity matrix
$I(\boldsymbol{x}, t)$	indicator function for intermittency (Eq. (5.299))
I_s, II_s, III_s	principal invariants of the second-order tensor $\boldsymbol{s}$ (Eqs. (B.31)–(B.33))
K	kurtosis of the longitudinal velocity derivative
K_ϕ	kurtosis of ϕ
Kn	Knudsen number
$K_\nu(z)$	modified Bessel function of the second kind
L	lengthscale ($k^{\frac{3}{2}}/\varepsilon$)
$\bar{L}$	lengthscale (u'^3/ε)
L_{11}	longitudinal integral lengthscale (Eq. (3.161))
L_{22}	lateral integral lengthscale (Eq. (6.48))
$\mathcal{L}$	characteristic lengthscale of the flow
$\mathcal{L}$	length of side of cube in physical space
$\mathcal{L}_{ij}$	resolved stress (Eq. (13.252))
$\mathcal{L}^{\mathrm{o}}_{ij}$	Leonard stress (Eq. (13.100))
$\dot{M}(x)$	momentum flow rate of the mean flow
M_{ij}	scaled composite rate-of-strain tensor (Eq. (13.255))
M_n	normalized nth moment of the longitudinal velocity derivative (Eq. (6.303))
Ma	Mach number
$\mathcal{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$\mathcal{O}(h)$	quantity of big order h
$o(h)$	quantity of little order h
P	pressure (Eq. (2.32))
$P(A)$	probability of event A
$P(\boldsymbol{x}, t)$	particle pressure (Eq. (12.225))
$P_{jk}(\boldsymbol{\kappa})$	projection tensor (Eq. (6.133))
$\mathcal{P}$	production: rate of production of turbulent kinetic energy (Eq. (5.133))
$\mathcal{P}_{ij}$	rate of production of Reynolds stress (Eq. (7.179))
$\mathcal{P}_r$	rate of production of residual kinetic energy (Eq. (13.123))
$\mathcal{P}_\phi$	rate of production of scalar variance (Eq. (5.282))

R	pipe radius
$R(s)$	autocovariance (Eq. (3.134))
$R_{ij}(\boldsymbol{r}, \boldsymbol{x}; t)$	two-point velocity correlation (Eq. (3.160))
$\hat{R}_{ij}(\boldsymbol{\kappa})$	Fourier coefficient of two-point velocity correlation (Eq. (6.152))
$\mathrm{R_T}$	turbulent Reynolds number (Eq. (5.85))
R_λ	Taylor-scale Reynolds number (Eq. (6.63))
Re	Reynolds number
Re	Reynolds number ($2\bar{U}\delta/\nu$)
Re_0	Reynolds number ($U_0\delta/\nu$)
Re_L	turbulence Reynolds number ($k^{1/2}L/\nu = k^2/(\varepsilon\nu)$)
$\mathrm{Re_T}$	turbulence Reynolds number ($u'L_{11}/\nu$)
Re_x	Reynolds number ($U_0 x/\nu$)
Re_δ	Reynolds number ($U_0\delta/\nu$)
Re_{δ^*}	Reynolds number ($U_0\delta^*/\nu$)
Re_θ	Reynolds number ($U_0\theta/\nu$)
Re_τ	Reynolds number based on friction velocity ($u_\tau\delta/\nu$)
$\mathcal{R}_{ij}$	pressure–rate-of-strain tensor (Eq. (7.187))
$\mathcal{R}^{\mathrm{o}}_{ij}$	SGS Reynolds stress (Eq. (13.102))
$\mathcal{R}^{(\mathrm{a})}_{ij}$	redistribution term (anisotropic part of Π_{ij}, Eq. (11.6))
$\mathcal{R}^{\mathrm{c}}_{ij}(\boldsymbol{v}, \boldsymbol{x}, t)$	conditional pressure–rate-of-strain tensor (Eq. (12.20))
$\mathcal{R}^{(\mathrm{e})}_{ij}$	redistribution term used in elliptic-relaxation model (Eq. (11.198))
$\mathcal{R}^{(\mathrm{r})}_{ij}$	rapid pressure–rate-of-strain tensor (Eq. (11.13))
$\mathcal{R}^{(\mathrm{s})}_{ij}$	slow pressure–rate-of-strain tensor
S	spreading rate of a free shear flow
S	velocity-derivative skewness (Eq. (6.85))
$S(\phi)$	chemical source term (Eq. (12.321))
S'	velocity structure function skewness (Eq. (6.89))
S_{ij}	rate-of-strain tensor ($\frac{1}{2}(\partial U_i/\partial x_j + \partial U_j/\partial x_i)$)
$\bar{S}_{ij}$	mean rate-of-strain tensor ($\frac{1}{2}(\partial\langle U_i\rangle/\partial x_j + \partial\langle U_j\rangle/\partial x_i)$)
$\widehat{S}_{ij}$	normalized mean rate-of-strain tensor ($(k/\varepsilon)\bar{S}_{ij}$)
$\overline{S}_{ij}$	filtered rate-of-strain tensor (Eq. (13.73))
$\widetilde{\overline{S}}_{ij}$	doubly filtered rate-of-strain tensor
$\bar{S}_{ijk}(\boldsymbol{r}, t)$	two-point triple velocity correlation (Eq. (6.72))
S_ϕ	skewness of ϕ
S_ω	mean source of turbulence frequency (Eq. (12.184))
$\mathcal{S}$	characteristic mean strain rate $(2\bar{S}_{ij}\bar{S}_{ij})^{\frac{1}{2}}$ ($\mathcal{S} = \partial\langle U_1\rangle/\partial x_2$ in simple shear flow)

$\overline{\mathcal{S}}$	filtered rate-of-strain invariant $(2\overline{S}_{ij}\overline{S}_{ij})^{\frac{1}{2}}$
$\overline{\widetilde{\mathcal{S}}}$	doubly filtered rate-of-strain invariant $(2\overline{\widetilde{S}}_{ij}\overline{\widetilde{S}}_{ij})^{\frac{1}{2}}$
$\mathcal{S}(\kappa)$	sphere in wavenumber space of radius κ
$\mathcal{S}_\lambda$	principal mean strain rate: largest eigenvalue of $\bar{S}_{ij}$
T	time interval
T	turbulent timescale defined by Eq. (11.163)
$\hat{T}(\boldsymbol{\kappa})$	rate of energy transfer to Fourier mode of wavenumber $\boldsymbol{\kappa}$ from other modes (Eq. 6.162)
T_{kij}	flux of Reynolds stress (Eq. (7.195))
$T_{kij}^{(p)}$	flux of Reynolds stress due to fluctuating pressure (Eq. (7.193))
$T_{kij}^{(p')}$	isotropic flux of Reynolds stress due to fluctuating pressure (Eq. (11.140))
$T_{kij}^{(u)}$	flux of Reynolds stress due to turbulent convection $(\langle u_k u_i u_j \rangle)$
$T_{kij}^{(v)}$	diffusive flux of Reynolds stress (Eq. (7.196))
T_{L}	Lagrangian integral timescale (Eq. (12.93))
$\mathcal{T}(\ell)$	rate of transfer of energy from eddies larger than ℓ to those smaller than ℓ
$\mathcal{T}_{\mathrm{EI}}$	rate of transfer of energy from large eddies to small eddies
$\mathcal{T}_{\mathrm{DI}}$	rate of transfer of energy into the dissipation range $(\ell < \ell_{\mathrm{DI}})$ from larger scales
$U(t)$	random process
$\boldsymbol{U}(\boldsymbol{x}, t)$	Eulerian velocity
$U(x, y, z)$	x component of velocity
$U(x, r, \theta)$	x component of velocity
$\bar{U}$	bulk velocity in channel (Eq. (7.3)) and pipe flow (Eq. (7.94))
$\boldsymbol{U}^+(t)$	fluid-particle velocity
$\boldsymbol{U}^*(t)$	model for the fluid-particle velocity
$\overline{\boldsymbol{U}}(\boldsymbol{x}, t)$	filtered (resolved) velocity field
U_0	mean centerline velocity in channel and pipe flow
$U_0(x)$	mean centerline velocity in a jet
$U_0(x)$	freestream velocity
$U_{\mathrm{c}}(x)$	characteristic convective velocity
U_{J}	jet-nozzle velocity
U_{h}	velocity of high-speed stream in a mixing layer
U_{l}	velocity of low-speed stream in a mixing layer
$U_s(x)$	characteristic velocity difference

$\mathcal{U}$	characteristic velocity scale of the flow
V	sample space variable corresponding to U
$\boldsymbol{V}$	sample space variable corresponding to velocity $\boldsymbol{U}$
$V(x, r, \theta)$	r component of velocity
$V(x, y, z)$	y component of velocity
$\mathcal{V}$	control volume in physical space bounded by $\mathcal{A}$
$W(t)$	Wiener process
$\boldsymbol{W}(t)$	vector-valued Wiener process
$W(x, r, \theta)$	θ component of velocity
$W(x, y, z)$	z component of velocity
$\boldsymbol{X}^+(t, \boldsymbol{Y})$	fluid-particle position: position at time t of fluid particle that is at $\boldsymbol{Y}$ at the reference time t_0
$\boldsymbol{X}^*(t)$	model for fluid-particle position (Eq. (12.108))
$\boldsymbol{Y}$	fluid particle position at the reference time t_0

Lower-case Roman

a	drift coefficient of a diffusion process (Eq. (J.27))
a_{ij}	anisotropic Reynolds stresses ($\langle u_i u_j \rangle - \frac{2}{3} k \delta_{ij}$)
a_{ij}	direction cosines (Eq. (A.11))
a_{f}	LES filter constant (Eq. (13.77))
b^2	diffusion coefficient of a diffusion process (Eq. (J.27))
b_{ij}	normalized Reynolds-stress anisotropy ($a_{ij}/(2k)$)
c_{f}	skin-friction coefficient ($\tau_{\mathrm{w}}/(\frac{1}{2}\rho U_0^2)$)
c_{S}	Smagorinsky coefficient (Eq. (13.253))
d	jet-nozzle diameter
$\hat{\boldsymbol{e}}(t)$	unit wavevector (Eq. (11.84))
$\boldsymbol{e}_i$	unit vector in the i-coordinate direction
f	friction factor (Eq. (7.97))
f, $\bar{f}$	self-similar mean axial velocity profile
$f(r, t)$	longitudinal velocity autocorrelation function (Eq. (6.45))
$f(V)$	probability density function (PDF) of U (Eq. (3.14))
$f(\boldsymbol{V}; \boldsymbol{x}, t)$	Eulerian PDF of velocity (Eq. (3.153))
$f'(\boldsymbol{V}; \boldsymbol{x}, t)$	fine-grained Eulerian PDF of velocity (Eq. (H.1))
$f^*(\boldsymbol{V}; \boldsymbol{x}, t)$	modelled Eulerian PDF of velocity (Eq. (12.116))
$f^*(\boldsymbol{V} \mid \boldsymbol{x}; t)$	conditional PDF of particle velocity (Eq. (12.205))
$\overline{f}(\boldsymbol{V}; \boldsymbol{x}, t)$	filtered density function (Eq. (13.287))
$\bar{f}(\boldsymbol{V}, \theta; \boldsymbol{x}, t)$	joint PDF of velocity and turbulence frequency
$\hat{f}(\boldsymbol{V}, \boldsymbol{\psi}; \boldsymbol{x}, t)$	velocity–composition joint PDF
$f_{2\mid 1}(V_2 \mid V_1)$	PDF of U_2 conditional on $U_1 = V_1$ (Eq. (3.95))

$f_{\mathrm{L}}(\boldsymbol{V}, \boldsymbol{x}; t\|\boldsymbol{Y})$	Lagrangian velocity–position joint PDF (Eq. (12.76))
$f_{\mathrm{L}}^{*}(\boldsymbol{V}, \boldsymbol{x}; t)$	joint PDF of $\boldsymbol{U}^{*}(t)$ and $\boldsymbol{X}^{*}(t)$
$f_{\mathrm{N}}(\psi; \boldsymbol{x}, t)$	non-turbulent conditional PDF of scalar $\phi(\boldsymbol{x}, t)$
$f_{\mathrm{T}}(\psi; \boldsymbol{x}, t)$	turbulent conditional PDF of scalar $\phi(\boldsymbol{x}, t)$
$f_{\mathrm{w}}(y^{+})$	law of the wall (Eq. (7.37))
$f_{X}(\boldsymbol{x}; t\|\boldsymbol{Y})$	PDF of fluid-particle position
$f_{X}^{*}(\boldsymbol{x}; t)$	PDF of $\boldsymbol{X}^{*}(t)$
f_{μ}	damping function in k–ε model (Eq. (11.155))
$f_{\phi}(\psi; \boldsymbol{x}, t)$	PDF of scalar $\phi(\boldsymbol{x}, t)$
$f_{\omega}(\theta; \boldsymbol{x}, t)$	PDF of turbulence frequency
g, $\bar{g}$	self-similar shear-stress profile in a free shear flow
g	gravitational acceleration
$\boldsymbol{g}$	gravitational force per unit mass
$g(r, t)$	transverse velocity autocorrelation function (Eq. (6.45))
$g(\boldsymbol{v}; \boldsymbol{x}, t)$	Eulerian PDF of the fluctuating velocity
h, $\bar{h}$	self-similar mean lateral velocity profile
h	grid spacing
k	turbulent kinetic energy ($\frac{1}{2}\langle \boldsymbol{u} \cdot \boldsymbol{u} \rangle$)
$\bar{k}(r, t)$	longitudinal two-point triple correlation (Eq. (6.73))
k_{r}	residual kinetic energy (Eq. (13.92))
$k_{(\kappa_{\mathrm{a}}, \kappa_{\mathrm{b}})}$	turbulent kinetic energy in the wavenumber range $(\kappa_{\mathrm{a}}, \kappa_{\mathrm{b}})$
l	lengthscale defined as ν_{T}/u'
ℓ	lengthscale
ℓ	characteristic eddy size
ℓ_0	lengthscale of the largest eddies
ℓ_{DI}	demarcation lengthscale between the dissipation range ($\ell < \ell_{\mathrm{DI}}$) and the inertial subrange ($\ell > \ell_{\mathrm{DI}}$)
ℓ_{EI}	demarcation lengthscale between the energy-containing range of eddies ($\ell > \ell_{\mathrm{EI}}$) and smaller eddies ($\ell < \ell_{\mathrm{EI}}$)
ℓ_{m}	mixing length (Eq. (7.91))
ℓ_{m}^{+}	mixing length in wall units ($\ell_{\mathrm{m}}/\delta_{\nu}$)
ℓ_{S}	Smagorinsky lengthscale (Eq. (13.128))
$\ell_{\mathrm{w}}(\boldsymbol{x})$	distance between $\boldsymbol{x}$ and the nearest solid surface
$\dot{m}(x)$	mass flow rate of the mean flow
$\boldsymbol{n}$	unit normal vector
$o(h)$	small order h (Eq. (J.34))
p	exponent in power-law spectrum (Eq. (G.5))
$p(\boldsymbol{x}, t)$	modified pressure
$p'(\boldsymbol{x}, t)$	fluctuating (modified) pressure

$p^{(h)}(\boldsymbol{x}, t)$	harmonic pressure (Eq. (2.49))
$p^{(r)}(\boldsymbol{x}, t)$	rapid pressure (Eq. (11.11))
$p^{(s)}(\boldsymbol{x}, t)$	slow pressure (Eq. (11.12))
$p_0(x)$	freestream pressure
$p_w(x)$	wall pressure
q	exponent in power-law structure function (Eq. (G.6))
r	radial coordinate
$r_{1/2}(x)$	half-width of jet or wake
s	time interval
s	lengthscale of wall roughness
s_{ij}	fluctuating rate-of-strain tensor ($\frac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$)
t	time
u	x component of fluctuating velocity
$u(\ell)$	characteristic velocity of an eddy of size ℓ
$\boldsymbol{u}(\boldsymbol{x}, t)$	fluctuating velocity
$\hat{\boldsymbol{u}}(\boldsymbol{\kappa}, t)$	Fourier coefficient of velocity (Eq. (6.102))
u'	r.m.s. velocity
$\boldsymbol{u}^*(t)$	fluctuating component of particle velocity (Eq. (12.207))
u^+	mean velocity normalized by the friction velocity
$\boldsymbol{u}'(\boldsymbol{x}, t)$	residual (SGS) velocity field (Eq. (13.3))
$u'_0(x)$	r.m.s. axial velocity
u_0	velocity scale of the largest eddies
u_e	propagation velocity of the viscous superlayer
u_η	Kolmogorov velocity (Eq. (5.151))
u_τ	friction velocity ($\sqrt{\tau_w/\nu}$)
v	y or r component of fluctuating velocity
$\mathbf{v}$	sample space variable corresponding to $\boldsymbol{u}$
w	z or θ component of fluctuating velocity
$w(y/\delta)$	law of the wake function (Eq. (7.149))
$\boldsymbol{x}$	position
x	Cartesian or polar cylindrical coordinate
x_0	virtual origin
y	Cartesian coordinate
y^+	distance from the wall normalized by the viscous lengthscale, δ_ν
$y_{0.1}(x)$	cross-stream location in mixing layer (also $y_{0.9}(x)$ etc., see Eq. (5.203))
$y_{1/2}(x)$	half-width of jet or wake
y_p	distance from the wall at which wall functions are applied