OXFORD MATHEMATICAL MONOGRAPHS

Oscillation Theory of Delay Differential Equations

With Applications

I. GYÖRI

and

G. LADAS

OXFORD SCIENCE PUBLICATIONS

Oscillation Theory of Delay Differential Equations

With Applications

I. GYÖRI
Computing Centre of A. Szent-Györgyi Medical University, Szeged, Hungary

and

G. LADAS
Department of Mathematics, University of Rhode Island, USA

CLARENDON PRESS · OXFORD
1991

Oxford University Press, Walton Street, Oxford OX2 6DP
Oxford New York Toronto
Delhi Bombay Calcutta Madras Karachi
Petaling Jaya Singapore Hong Kong Tokyo
Nairobi Dar es Salaam Cape Town
Melbourne Auckland
and associated companies in
Berlin Ibadan

Oxford is a trade mark of Oxford University Press

Published in the United States
by Oxford University Press, New York

A catalogue record of this book is available from the British Library

Library of Congress Cataloging in Publication Data
Györi, I.
Oscillation theory of delay differential equations: with applications / by I. Györi and G. Ladas.
p. cm.—(Oxford mathematical monographs)
Includes bibliographical references and index.
1. Differential equations—Delay equations. 2. Oscillations.
I. Ladas, G. E. II. Title. III. Series.
QA371.G96 1991 515′.35—dc20 90-20897

ISBN 0-19-853582-1

Typeset by Integral Typesetting, Gorleston, Norfolk
Printed in Great Britain by
Courier International Ltd., Tiptree, Essex

OXFORD MATHEMATICAL MONOGRAPHS

A. Belleni-Morante: *Applied semigroups and evolution equations*
I. G. Macdonald: *Symmetric functions and Hall polynomials*
J. W. P. Hirschfeld: *Projective geometries over finite fields*
N. Woodhouse: *Geometric quantization*
A. M. Arthurs: *Complementary variational principles* Second edition
P. L. Bhatnagar: *Nonlinear waves in one-dimensional dispersive systems*
N. Aronszajn, T. M. Creese, and L. J. Lipkin: *Polyharmonic functions*
J. A. Goldstein: *Semigroups of linear operators*
M. Rosenblum and J. Rovnyak: *Hardy classes and operator theory*
J. W. P. Hirschfeld: *Finite projective spaces of three dimensions*
K. Iwasawa: *Local class field theory*
A. Pressley and G. Segal: *Loop groups*
J. C. Lennox and S. E. Stonehewer: *Subnormal subgroups of groups*
D. E. Edmunds and W. D. Evans: *Spectral theory and differential operators*
Wang Jianhua: *The theory of games*
S. Omatu and J. H. Seinfeld: *Distributed parameter systems: theory and applications*
D. Holt and W. Plesken: *Perfect groups*
J. Hilgert, K. H. Hofmann, and J. D. Lawson: *Lie groups, convex cones, and semigroups*
S. Dineen: *The Schwarz lemma*
B. Dwork: *Generalized hypergeometric functions*
R. J. Baston and M. G. Eastwood: *The Penrose transform: its interaction with representation theory*
S. K. Donaldson and P. B. Kronheimer: *The geometry of four-manifolds*
T. Petrie and J. Randall: *Connections, definite forms, and four-manifolds*
I. Györi and G. Ladas: *Oscillation theory of delay differential equations: with applications*

PREFACE

In recent years there has been much research activity concerning the oscillation of solutions of delay differential equations. To a large extent, this is due to the realization that delay differential equations are important in applications. New applications which involve delay differential equations continue to arise with increasing frequency in the modelling of diverse phenomena in physics, biology, ecology, and physiology.

Our aim in this monograph is to present in a systematic way the most important recent contributions in oscillation theory of delay differential equations. We will also apply the oscillation theory to several equations in mathematical biology to obtain the oscillatory character of their solutions.

There is no doubt that some of the recent developments in oscillation theory have contributed a beautiful body of knowledge in the field of differential equations that has enhanced our understanding of the qualitative behaviour of their solutions and has some nice applications in mathematical biology and other fields.

This monograph contains some recent important developments in the oscillation theory of delay differential equations with some applications to mathematical biology. Our intention is to expose the reader to the frontiers of the subject and to formulate some important open problems that remain to be solved in this area.

Chapter 1 contains some basic definitions and results which are used throughout the book. In this sense, this is a self-contained monograph. Chapter 2 deals with the basic oscillation theory of linear scalar delay differential equations. In Chapter 3 we introduce a generalized characteristic equation and then use it to establish comparison theorems, to prove the existence of a positive solution and also to obtain general oscillation results for delay equations with variable coefficients and variable delays. In Chapter 4 we present a linearized oscillation result and then apply it to some models in mathematical biology. Chapter 5 deals with the oscillation of linear and nonlinear systems of delay differential equations. In Chapter 6 we develop the oscillation of solutions of neutral differential equations. Chapter 7 deals with the oscillation of delay difference equations. The oscillation of equations with piecewise constant arguments is treated in Chapter 8. In Chapter 9 we present some oscillation results for integrodifferential equations. Chapter 10 contains some oscillation results for equations of higher order. In Chapter 11 we study the asymptotic behaviour of both the oscillatory and non-oscillatory solutions of some equations, mostly from mathematical biology, and obtain results about the global attractivity of their respective steady states. Finally, Chapter 12 contains some miscellaneous topics, including some results on slowly oscillating periodic solutions, rapidly oscillating solutions, and oscillations of solutions of differential equations with periodic coefficients.

At the end of every chapter we have included some notes and references about the material presented and we briefly discuss other related developments. We also present some open problems which are worth investigating and which will stimulate further interest in this subject.

Szeged I. G.
Kingston G. L.
1991

ACKNOWLEDGEMENTS

This monograph is the outgrowth of seminars and lecture notes which were given at the University of Rhode Island during the last ten years and in particular during the period September 1987 to May 1990. We are thankful to Professors R. D. Driver, D. A. Georgiou, K. Gopalsamy, E. A. Grove, A. Ivanov, V. Lj. Kocic, M. R. S. Kulenovic, L. Pakula, Z. Wang, and J. Yan, and to our graduate students W. Briden, R. C. DeVault, A. M. Farahani, K. Farrel, J. H. Jaroma Jun., S. A. Kuruklis, S. C. Mosher, E. C. Partheniadis, M. R. Petrarca Jun., C. Qian, I. W. Rodrigues, S. W. Schultz, and P. N. Vlahos for their enthusiastic participation and constructive criticism which helped to improve the exposition. Special thanks are due to E. A. Grove, D. A. Georgiou, V. Lj. Kocic, L. Pakula, S. A. Kuruklis, C. Qian, S. W. Schultz, P. N. Vlahos, and Z. Wang for proofreading in detail parts of this monograph. Finally, we express our appreciation to Edit Györi for her excellent typing of the original manuscript.

CONTENTS

1

PRELIMINARIES

Oscillation theory begins in Chapter 2. The aim in this chapter is to present some preliminary results which will be used throughout the book. In this respect, this is almost a self-contained monograph. The reader may glance at the material covered in this chapter and then proceed to Chapter 2.

Section 1.1 contains a detailed description of all possible global existence and uniqueness results that are needed in our treatment of the oscillation theory of delay and neutral delay differential equations. Of particular importance is to understand what we mean by a solution of an equation. See Definitions 1.1.1(c) and 1.1.2(c). In Section 1.2 we prove that the solutions of linear autonomous differential equations are exponentially bounded. We need this property of solutions because in Sections 2.1, 5.1, and 6.3 we take their Laplace transforms. In Section 1.3 we present some basic properties of Laplace transforms. In Sections 2.1, 5.1, and 6.3 we use Laplace transforms to present some basic necessary and sufficient conditions for the oscillation of all solutions of linear autonomous differential equations. In Section 1.4 we introduce the z-transform, which is the discrete analogue of the Laplace transform. This transform is used in Section 7.1 to give a powerful necessary and sufficient condition for the oscillation of all solutions of linear autonomous systems of difference equations. Section 1.5 contains several basic lemmas from analysis that we use on several occasions throughout this monograph. In Section 1.6 we include some basic results on differential inequalities whose use will simplify considerably the proofs of several theorems. Section 1.7 states some useful theorems from analysis that are needed in this monograph.

1.1 Some basic existence and uniqueness theorems

This section is written for the reader who is not familiar with the existence and uniqueness theory of delay and of neutral delay differential equations which is discussed in specialized books such as Bellman and Cooke (1963), Driver (1977), Hale (1977), and Myskis (1972). Here we emphasize the global existence and uniqueness theorems, because in oscillation theory the solutions are assumed to exist on an infinite interval $[t_0, \infty)$. See Definitions 1.1.1(c), 1.1.2(c), and Remark 1.1.1.

Consider the delay differential system

$$\dot{x}(t) + f(t, x(t), x(t - \tau_1(t)), \ldots, x(t - \tau_n(t))) = 0, \tag{1.1.1}$$

where for some $\tilde{t}_0 \in \mathbb{R}$ and some positive integer m,

$$f \in C[[\tilde{t}_0, \infty) \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m, \mathbb{R}^m] \quad \text{and} \quad \tau_i \in C[[\tilde{t}_0, \infty), \mathbb{R}^+] \quad \text{for } i = 1, 2, \ldots, n, \tag{1.1.2}$$

with

$$\lim_{t \to \infty} [t - \tau_i(t)] = \infty, \qquad \text{for } i = 1, 2, \ldots, n. \tag{1.1.3}$$

For every 'initial point' $t_0 \geqslant \tilde{t}_0$ we define $t_{-1} = t_{-1}(t_0)$ to be

$$t_{-1} = \min_{1 \leqslant i \leqslant n} \left\{ \inf_{t \geqslant t_0} \{t - \tau_i(t)\} \right\}. \tag{1.1.4}$$

As we see, t_{-1} depends on the delays $\tau_i(t)$ of the differential equation as well as the point t_0. The interval $[t_{-1}, t_0]$ is called the 'initial interval' associated with the initial point t_0 and the delay differential equation (1.1.1).

With eqn (1.1.1) and with a given initial point $t_0 \geqslant \tilde{t}_0$ one associates an 'initial condition'

$$x(t) = \phi(t) \qquad \text{for } t_{-1} \leqslant t \leqslant t_0 \tag{1.1.5}$$

where ϕ: $[t_{-1}, t_0] \to \mathbb{R}^m$ is a given 'initial function'.

Definition 1.1.1.

(a) *A function x is said to be a solution of eqn* (1.1.1) *on the interval I, where I is of the form $[t_0, T)$, $[t_0, T]$, or $[t_0, \infty)$, with $\tilde{t}_0 \leqslant t_0 < T$, if x: $[t_{-1}, t_0] \cup I \to \mathbb{R}^m$ is continuous, x is continuously differentiable for $t \in I$ and x satisfies eqn* (1.1.1) *for all $t \in I$.*

(b) *A function x is said to be a solution of the initial value problem* (1.1.1) *and* (1.1.5) *on the interval I, where I is of the form $[t_0, T)$, $[t_0, T]$, or $[t_0, \infty)$, if x is a solution of eqn* (1.1.1) *on the interval I and x satisfies* (1.1.5).

(c) *A function x is said to be a solution of eqn* (1.1.1) *if for some $t_0 \geqslant \tilde{t}_0$, x is a solution of eqn* (1.1.1) *on the interval $[t_0, \infty)$.*

(d) *A function x is said to be a solution of the initial value problem* (1.1.1) *and* (1.1.5) *if x is a solution of eqn* (1.1.1) *on the interval $[t_0, \infty)$ and x satisfies* (1.1.5).

Throughout this book, unless otherwise specified, for any m-dimensional vector $x = (x_1, \ldots, x_m)^T \in \mathbb{R}^m$, $\|x\|$ denotes any vector norm. For any $m \times m$ real matrix A, the associated matrix norm is then defined by $\|A\| = \max_{\|x\| = 1} \|Ax\|$.

The following result, which is known as Gronwall's inequality, is needed in the proof of uniqueness theorems. For a proof see Driver (1977).

Lemma 1.1.1 (Gronwall's Inequality). *Let $I = [t_0, T)$ be an interval of real numbers and suppose that*

$$u(t) \leqslant c + \int_{t_0}^{t} v(s)u(s)\,ds \qquad \text{for } t \in I$$

where

$$c \in [0, \infty) \qquad \text{and} \qquad u, v \in C[I, \mathbb{R}^+].$$

Then

$$u(t) \leqslant c \exp\left(\int_{t_0}^{t} v(s)\,ds\right) \qquad \text{for } t \in I.$$

The following result is the basic global existence and uniqueness theorem for delay differential systems.

Theorem 1.1.1. *In addition to conditions* (1.1.2) *and* (1.1.3) *assume that there exists a function $p \in C[[\tilde{t}_0, \infty), \mathbb{R}^+]$ such that for all $t \geqslant \tilde{t}_0$ and for all $x_i, y_i \in \mathbb{R}^m$ for $i = 0, 1, \ldots, n$, the function f satisfies the following global Lipschitz condition:*

$$\|f(t, x_0, x_1, \ldots, x_n) - f(t, y_0, y_1, \ldots, y_n)\| \leqslant p(t) \sum_{i=0}^{n} \|x_i - y_i\|. \quad (1.1.6)$$

Let $t_0 \geqslant \tilde{t}_0$ and $\phi \in C[[t_{-1}, t_0], \mathbb{R}^m]$ be given. Then the initial value problem (1.1.1) *and* (1.1.5) *has exactly one solution in the interval $[t_0, \infty)$.*

Sketch of the Proof. Define the operator T on continuous functions $y\colon [t_{-1}, \infty) \to \mathbb{R}^m$ by

$$(Ty)(t) = \begin{cases} \phi(t), & t_{-1} \leqslant t < t_0 \\ \phi(t_0) + \displaystyle\int_{t_0}^{t} f(s, y(s), y(s - \tau_1(s)), \ldots, y(s - \tau_n(s)))\,ds, & t \geqslant t_0. \end{cases}$$

Clearly $(Ty)(t)$ is a continuous function on $[t_{-1}, \infty)$. We now define the following sequence of functions:

$$x_0(t) = \begin{cases} \phi(t), & t_{-1} \leqslant t < t_0 \\ \phi(t_0), & t \geqslant t_0 \end{cases}$$

and

$$x_{l+1} = Tx_l \qquad \text{for } l = 0, 1, 2, \dots .$$

By using (1.1.6) one can show (as in Picard's method of successive approximations) that

$$\|x_{l+1}(t) - x_l(t)\| \leqslant M(t)\frac{(t-t_0)^l}{l!} \qquad \text{for } t \geqslant t_{-1} \text{ and } l = 0, 1, \dots,$$

where $M(t) = \max_{t_0 \leqslant s \leqslant t} p(s)$. It follows that

$$x(t) = \lim_{l\to\infty} x_l(t) = \sum_{l=0}^{\infty} [x_{l+1}(t) - x_l(t)] + x_0(t)$$

exists for all $t \geqslant t_{-1}$, and x is a solution of the initial value problem (1.1.1) and (1.1.5) on $[t_0, \infty)$.

To show uniqueness, we assume that x and y are two solutions of (1.1.1) and (1.1.5) on the interval $[t_0, \infty)$. Then $x(t) = y(t)$ for $t_{-1} \leqslant t \leqslant t_0$. Set

$$u(t) = \max_{t_0 \leqslant s \leqslant t} \|x(s) - y(s)\|.$$

Clearly

$$u(t) \leqslant (n+1)\int_{t_0}^{t} p(s)u(s)\,ds \qquad \text{for } t \geqslant t_0$$

and by Gronwall's inequality (with $c = 0$) it follows that $u(t) = 0$ for $t \geqslant t_0$. Hence $x(t) = y(t)$ for $t \geqslant t_0$ and the proof is complete.

Corollary 1.1.1. *Consider the linear non-autonomous delay system*

$$\dot{x}(t) + P_0(t)x(t) + \sum_{i=1}^{n} P_i(t)x(t - \tau_i(t)) = 0 \qquad (1.1.7)$$

where for some $\tilde{t}_0 \in \mathbb{R}$ *and some positive integer* m,

$$P_i \in C[[\tilde{t}_0, \infty], \mathbb{R}^{m\times m}] \qquad \textit{for } i = 0, 1, \dots, n,$$

$$\tau_i \in C[[\tilde{t}_0, \infty), \mathbb{R}^+] \qquad \textit{and} \qquad \lim_{t\to\infty}[t - \tau_i(t)] = \infty \textit{ for } i = 1, 2, \dots, n.$$

Let $t_0 \geqslant \tilde{t}_0$ *and* $\phi \in C[[t_{-1}, t_0], \mathbb{R}^m]$ *be given. Then the initial value problem* (1.1.7) *and* (1.1.5) *has exactly one solution in the interval* $[t_0, \infty)$.

Corollary 1.1.2. *Consider the linear autonomous delay system*

$$\dot{x}(t) + P_0x(t) + \sum_{i=1}^{n} P_ix(t - \tau_i) = 0, \qquad (1.1.8)$$

where for $i = 0, 1, \ldots, n$ the coefficients P_i are real $m \times m$ matrices and for $i = 1, 2, \ldots, n$ the delays τ_i are non-negative real numbers. Then for every $t_0 \in \mathbb{R}$ and for every $\phi \in C[[t_{-1}, t_0], \mathbb{R}^m]$, where $t_{-1} = t_0 - \max\{\tau_1, \tau_2, \ldots, \tau_n\}$, the initial value problem (1.1.8) *and* (1.1.5) *has exactly one solution in the interval* $[t_0, \infty)$.

When the function f in eqn (1.1.1) does not depend explicitly on $x(t)$ and when the delays $\tau_i(t)$ are all positive, one can establish a global existence theorem for eqn (1.1.1) without the assumption that the Lipschitz condition (1.1.6) holds. This is accomplished by utilizing the so-called *method of steps*. In fact we shall now utilize the method of steps to obtain a global existence theorem for a differential system of a more general form, namely, for the so-called *neutral delay differential system*

$$\frac{\mathrm{d}}{\mathrm{d}t}[x(t) + g(t, x(t-\tau_1(t)), \ldots, x(t-\tau_l(t)))] + f(t, x(t-\sigma_1(t)), \ldots, x(t-\sigma_n(t))) = 0 \quad (1.1.9)$$

where for some $\tilde{t}_0 \in \mathbb{R}$ and some positive integer m,

$$g \in C[[\tilde{t}_0, \propto) \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m, \mathbb{R}^m], \quad (1.1.10)$$

$$f \in C[[\tilde{t}_0, \infty) \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m, \mathbb{R}^m], \quad (1.1.11)$$

and for $i = 1, \ldots, l$ and $j = 1, \ldots, n$

$$\left.\begin{array}{l} \tau_i \in C[[\tilde{t}_0, \infty), (0, \infty)],\ \sigma_j \in C[[\tilde{t}_0, \infty), (0, \infty)] \text{ and} \\ \lim\limits_{t\to\infty} [t - \tau_i(t)] = \infty = \lim\limits_{t\to\infty} [t - \sigma_j(t)]. \end{array}\right\} \quad (1.1.12)$$

For a given initial point $t_0 \geqslant \tilde{t}_0$, we now define t_{-1} to be

$$t_{-1} = \min\left\{\min_{1\leqslant i\leqslant l}\left\{\inf_{t\geqslant t_0}\{t - \tau_i(t)\}\right\}, \min_{1\leqslant j\leqslant n}\left\{\inf_{t\geqslant t_0}\{t - \sigma_j(t)\}\right\}\right\}. \quad (1.1.13)$$

With eqn (1.1.9) we also associate the initial condition

$$x(t) = \phi(t) \qquad \text{for } t_{-1} \leqslant t \leqslant t_0 \quad (1.1.14)$$

where $\phi\colon [t_{-1}, t_0] \to \mathbb{R}^m$ is a given initial function.

Definition 1.1.2.

(a) *A function x is said to be a solution on the interval I, where I is of the form $[t_0, T)$, $[t_0, T]$, or $[t_0, \infty)$ with $\tilde{t}_0 \leqslant t_0 < T$, if $x\colon [t_{-1}, t_0] \cup I \to \mathbb{R}^m$ is continuous, $x(t) + g(t, x(t-\tau_1(t)), \ldots, x(t-\tau_l(t)))$ is continuously differentiable for $t \in I$ and x satisfies eqn* (1.1.9) *for all $t \in I$.*

(b) *A function x is said to be a solution of the initial value problem* (1.1.9)