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Dedicated to Weixi, Leijun,

Lingzhou and Eric

Preface

This book is designed to introduce some important methods for nonlinear evolution equations. The material in this book has been used in recent years as lecture notes for the graduate students at Fudan University.

Nonlinear evolution equations, i.e., partial differential equations with time t as one of the independent variables, arise not only from many fields of mathematics, but also from other branches of science such as physics, mechanics and material science. For example, Navier-Stokes and Euler equations from fluid mechanics, nonlinear reaction-diffusion equations from heat transfers and biological sciences, nonlinear Klein-Gorden equations and nonlinear Schrödinger equations from quantum mechanics and Cahn-Hilliard equations from material science, to name just a few, are special examples of nonlinear evolution equations. Complexity of nonlinear evolution equations and challenges in their theoretical study have attracted a lot of interest from many mathematicians and scientists in nonlinear sciences. The first question to ask in the theoretical study is whether for a nonlinear evolution equation with given initial data there is a solution at least locally in time, and whether it is unique in the considered class. Generally speaking, this problem has been solved for a wide class of nonlinear evolution equations by two powerful methods in nonlinear analysis, i.e., the contraction mapping theorem and the Leray-Schauder fixed-point theorem. Roughly speaking, since the 1960s, much more attention has been paid to the global existence and uniqueness of a solution, i.e., when a local solution can be extended to become a global one in time. Furthermore, if there is a global solution for a given nonlinear evolution equation, one also wants to know about the asymptotic behavior of the solution as time goes to infinity.

A series of useful methods and theories have been developed, especially since the 1960s. Basic knowledge about the central issues of this subject, i.e., global existence and uniqueness, and long time behavior of a solution as time goes to infinity is extremely important for graduate students and researchers in mathematics, as well as in other branches of science. This book aims to concisely explain in a clear and easily readable manner a wide range of relevant theories and methods for nonlinear evolution equations such as the semigroup method, compactness and monotone operator method, monotone iterative method and invariant regions, global existence and uniqueness theory for small initial data as well as results on convergence to a stationary solution and the existence of global attractors. Moreover, some material in the book consists of the research results of the author, while results on convergence to a stationary solution in higher-space dimensions are updated. Most are only available in various journals, and have not yet been seen in book form. In addition, there are bibliographic comments in each chapter that provide the reader with references and further reading materials, though this book is not intended to be exhaustive.

The book consists of six chapters. Chapter 1 is a preliminary chapter in which we not only describe motivations from other branches of science, but also introduce the concepts of local well-posedness and global well-posedness, and outline the theory of global solutions, which is the main concern of this book. In this chapter we also collect some basic material on PDE and Sobolev spaces for the convenience of the reader. In the next five chapters, we introduce some important methods for nonlinear evolution equations which have been developed from the 1960s to the present day.

Chapter 2 is concerned with the semigroup method. This method was developed from the end of the 1940s to the 1960s, and it is still very useful nowadays. Based on the Hille-Yosida theorem and a theorem by I. Segal, which are introduced in detail, global solvability of linear and semilinear evolution equations as well as regularities are discussed. Examples of applications to nonlinear parabolic equations and nonlinear hyperbolic equations are presented.

Chapter 3 is devoted to the compactness method and monotone operator method. These two methods were developed in the 1960s. Combining the Faedo-Galerkin method with the compactness argument yields a powerful method which allows us to deal with some nonlinear evolution equations. The monotone operator method is very useful to deal with some quasilinear evolution equations. Examples illustrating the applications to concrete nonlinear evolution equations are also shown in detail.

Chapter 4 is concerned with the monotone iterative method and invariant regions. The monotone iterative method was introduced by H. Amann and D. Sattinger around 1971 to deal with nonlinear parabolic equations as well as nonlinear elliptic equations. Based on the comparison principle, one can draw a conclusion about the existence of a solution to these nonlinear PDEs provided that a suitable pair of an upper solution and a lower solution can be found. This method has been significantly developed to deal with certain nonlinear parabolic systems nowadays. It is important to concisely introduce these methods and ideas. A related method, i.e., the invariant regions, was initiated by H. Amann and K. Chueh, C. Conley and J. Smoller around 1977. In this book we mainly follow the line of descriptions by K. Chueh, C. Conley and J. Smoller with some modifications of proof. Applications to not only nonlinear parabolic equations but also some nonlinear hyperbolic equations are shown in this chapter.

During the 1970s, a systematic method of dealing with highly nonlinear evolution equations with small initial data emerged. Chapter 5 of this book introduces this important method by illustrating how it works for the fully nonlinear parabolic equations to which the author of this book is one of the main contributors.

Since the 1980s, convergence of solutions of nonlinear evolution equations to stationary solutions as time goes to infinity and the study of the related infinite-dimensional dynamical system become two of the main concerns in the field of nonlinear evolution equations. The final chapter of this book is devoted to these topics. A lemma in analysis, which was established by the author and his collaborator in 1993, is introduced and the applications to the study of long time behavior of solutions are also shown. Convergence of solution of nonlinear evolution equations to a corresponding stationary solution as time goes to infinity has been a problem of great interest and importance for a long time. For the one-dimension case, significant progress has been made, as can be seen from the work by H. Matano in 1978 and other works later on. For the higher space dimension case, it has been a focus of many researchers in the last twenty years. In a paper published in 1983 L. Simon extended a lemma by S. Lojasiewicz on analytic functions to the infinite dimensional case and developed a method for the study of convergence to the stationary solution when the nonlinear term is analytic. Since then, there have been many research papers on the generalizations, although they have not yet been seen in book form. We will introduce these ideas and methods in a clear and detailed manner in the final chapter of this book. Another aspect of the study of the long time behavior of solutions to nonlinear evolution equations is to investigate whether there exists a global attractor, etc. when initial data vary in any given bounded set. Since the 1980s, this has become a hot topic in research, as three books by R. Temam, J.K. Hale and A.V. Babin & M.I. Vishik indicated. In the final chapter we will also concisely introduce some results on this topic especially for the gradient system.

It is expected that after reading this book, the reader will know the ideas and essences of several important methods in nonlinear evolution equations and their applications. This together with the references and further reading materials provided in the bibliographic comments will enable the reader to prepare for further study and research.

I appreciate the stimulating interactions with graduate students at Fudan University where I have taught this course in recent years. Special thanks go to Hao Wu and Yuming Qin for their careful reading and many suggestions. I would also like to acknowledge the NSF of China for their continued support. Currently, this book project is being supported by the grants No. 19331040 and No. 10371022 from NSF of China. Finally, my deepest gratitude goes to my wife, Weixi Shen, also a mathematician and my collaborator at Fudan University, for her constant encouragement, advice and support in my career.

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Contents

Chapter 1

Preliminaries

In this chapter we first present some examples to show how nonlinear evolution equations naturally arise from other branches of science. We will also give definitions of local and global solutions, and outline the theory of global solution. Since this book is designated for graduate students as well as researchers, some basic materials on partial differential equations and Sobolev's spaces, which will be needed in the remainder of the book, are introduced for convenience. Most results are just recalled without proofs, but the relevant references are given. In the final section of this chapter, some references for further reading are also given.

1.1 Motivations from Other Branches of Science

An evolution equation usually means a partial differential equation with one of the independent variables being time t. The linear wave equation or linear heat equation describing vibration of string or heat conduction are two simple examples of evolution equations. However, there are many nonlinear evolution equations naturally arising from physics, mechanics, biology, chemistry, material science needing to be investigated. In the following text we give some examples.

1. Let us first look at a first-order nonlinear evolution equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0. \tag{1.1.1}$$

This equation is derived when one studies the one-dimensional traffic problem. In this situation, $u(x,t)$ represents the density of cars at place x and time t. Equation (1.1.1) is also often taken as a model equation for the study of gas dynamics with conservation law.

2. In practice we will often encounter second-order nonlinear evolution equations. For instance, if we investigate the heat transfer process in a body with a heat source that produces heat quantity at the unity of time depending on the instant temperature, then we are led to the following nonlinear heat equation:

$$\frac{\partial u}{\partial t} = div(k\nabla u) + f(u). \tag{1.1.2}$$

Considering vibration of a strain with the external force nonlinearly depending on displacement, one is led to the nonlinear wave equation:

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(u). \tag{1.1.3}$$

In quantum mechanics one encounters various nonlinear evolution equations such as the nonlinear Sine-Gorden equation:

$$u_{tt} - \Delta u + sinu = 0, \tag{1.1.4}$$

the Klein-Gorden equation,

$$u_{tt} - \Delta u + mu + \gamma u^3 = 0, \tag{1.1.5}$$

and the nonlinear Schrödinger equation,

$$iu_t - \Delta u + \gamma|u|^2 u = 0, \tag{1.1.6}$$

or even a coupled system of nonlinear evolution equations. Hereafter we use the subscript t or x to denote differentiation with respect to t or x.

3. In addition to the second-order equations, some even higher-order nonlinear evolution equations naturally arise in other branches of science. For instance, in the study of phase transitions of binary alloys such as polymers, glasses etc., the following Cahn-Hilliard equation is derived:

$$u_t + \varepsilon\Delta^2 u = \Delta\phi(u) \tag{1.1.7}$$

where ε is a given small positive constant and typically $\phi(u) = u^3 - u$. Notice that equation (1.1.7) is a fourth-order nonlinear evolution equation. Other examples of a higher-order evolution equation include the following famous KdV equation:

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1.8}$$

which is a third-order evolution equation.

4. When a complicated physical, mechanical or a biological system is investigated, it is quite often that not only one independent variable is not enough, but also a system of dependent variables are needed to describe the whole system. In these circumstances one is led to study a system of nonlinear evolution equations. There are many examples in this aspect. For instance, the following famous Navier-Stokes equations, which describe the flow of viscous incompressible fluids, is a system of this kind.

$$\begin{cases} div\,\vec{u} = 0, \\ \vec{u_t} + (\vec{u}\cdot\nabla)\vec{u} + \frac{1}{\rho}\,grad\,p = \mu\Delta\vec{u} \end{cases} \tag{1.1.9}$$

where $\vec{u}$ denotes the velocity vector, ρ is density and p is pressure. Other examples include the Euler equations describing the flow of viscous and heat-conductive compressible fluids, and nonlinear thermoelastic systems or nonlinear thermoviscoelastic systems. Some systems arise from biology and ecology. For instance, the following reaction-diffusion system is a good example of this kind:

$$\vec{u_t} - D\Delta\vec{u} = \vec{f}(\vec{u}) \tag{1.1.10}$$

where $\vec{u}$ and $\vec{f}$ are vector functions and D is a matrix function. In ecology, prey and predator coexist in an ecological system, and their existence and development is an interactive process that can be described by system (1.1.10). From the mathematical point of view, (1.1.10) is a semilinear parabolic system when D is a positive definite matrix. Another example of this kind is the following Fitzhugh-Nagumo equations, which describes the propagations of pulses in a nerve system:

$$\begin{cases} u_t - \Delta u = f(u) - z, \\ z_t = \sigma u - \gamma z. \end{cases} \tag{1.1.11}$$

This is a system of a nonlinear parabolic equation coupled with an ordinary differential equation, which can be considered as a degenerate parabolic equation.

5. In practice, one will encounter degenerate nonlinear evolution equations that change the type of equation in some regions. For instance, the following porous media equation is a good example of this kind:

$$u_t - \Delta(u^m) = 0. \tag{1.1.12}$$

This equation describes the flow of underground water in porous media. When $u = 0$, the above equation is degenerate. It is expected that many properties, including uniqueness will not be the same as other non-degenerate equations.

6. Sometimes one also encounters nonlinear evolution equations defined on a manifold. For the needs of large-scale weather broadcasting, one is led to the study the nonlinear evolution equations defined on a manifold, i.e., the surface of the earth. In differential geometry, to find a harmonic mapping from an n-dimensional Riemannian manifold M to another m-dimensional Riemannian manifold N, J. Eells and J.H. Sampson in 1964 proposed a new approach by studying the corresponding nonlinear parabolic system defined on a Riemannian manifold.

The above are just a few examples, but enough to exhibit the variety and complexity of nonlinear evolution equations, and consequently, the challenge of their study.

1.2 Local Solutions and Global Solutions

For a given nonlinear evolution equation, two basic problems posed to study are the initial value problem and the initial boundary value problem. For the preceding problem, while the initial data are given at $t = 0$, and the space variables $(x_1, \cdot, x_n)$ vary in R^n or in an n-dimensional Riemannian manifold without boundary, one wants to find the solution for later time $t > 0$. For the later problem, the space variables $(x_1, \cdot, x_n)$ vary in a domain Ω of R^n or an n-dimensional Riemannian manifold with boundary. Then for a solution to be well defined, in addition to the initial conditions at $t = 0$, suitable boundary conditions on the boundary Γ of Ω or a Riemannian manifold also need to be given. In addition to the above two basic problems, finding a travelling-wave solution to a given nonlinear evolution equation is also an interesting and important problem. In contrast to the previous two problems, the initial condition is not posed. Finding a travelling-wave solution is often reduced to solving nonlinear elliptic equations subject to boundary conditions, and we will not go into the detail here since it is beyond the scope of the present book.

For a given nonlinear evolution equation, under general assumptions on initial data and boundary conditions, we very often find that existence and uniqueness of the solution in a small time interval near the origin $t = 0$, i.e., the local well-posedness can be obtained by two fun-