Stability and Control: Theory, Methods and Applications
Volume 20

Stability of Differential Equations with Aftereffect

N. V. Azbelev and
P. M. Simonov

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Stability of Differential Equations with Aftereffect

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Introduction to the series

The problems of modern society are both complex and interdisciplinary. Despite the apparent diversity of problems, tools developed in one context are often adaptable to an entirely different situation. For example, consider the Lyapunov's well known second method. This interesting and fruitful technique has gained increasing significance and has given a decisive impetus for modern development of the stability theory of differential equations. A manifest advantage of this method is that it does not demand the knowledge of solutions and therefore has great power in application. It is now well recognized that the concept of Lyapunov-like functions and the theory of differential and integral inequalities can be utilized to investigate qualitative and quantitative properties of nonlinear dynamic systems. Lyapunov-like functions serve as vehicles to transform the given complicated dynamic systems into a relatively simpler system and therefore it is sufficient to study the properties of this simpler dynamic system. It is also being realized that the same versatile tools can be adapted to discuss entirely different nonlinear systems, and that other tools, such as the variation of parameters and the method of upper and lower solutions provide equally effective methods to deal with problems of a similar nature. Moreover, interesting new ideas have been introduced which would seem to hold great potential.

Control theory, on the other hand, is that branch of application-oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. To control an object implies the influence of its behavior so as to accomplish a desired goal. In order to implement this influence, practitioners build devices that incorporate various mathematical techniques. The study of these devices and their interaction with the object being controlled is the subject of control theory. There have been, roughly speaking, two main lines of work in control theory which are complementary. One is based on the idea that a good model of the object to be controlled is available and that we wish to optimize its behavior, and the other is based on the constraints imposed by uncertainty about the model in which the object operates. The control tool in the latter is the use of feedback in order to correct for deviations from the desired behavior. Mathematically, stability theory, dynamic systems and functional analysis have had a strong influence on this approach.

Due to the increased interdependency and cooperation among the mathematical sciences across the traditional boundaries, and the accomplishments thus far achieved in the areas of stability and control, there is every reason to believe that many breakthroughs await us, offering existing prospects for these versatile techniques to advance further. It is in this spirit that we see the importance of the 'Stability and Control' series, and we are immensely thankful to Taylor & Francis for their interest and cooperation in publishing this series.

Preface

This volume presents the problems of stability and asymptotic behaviour of solutions to the functional differential equation

$$\mathcal{L}x = \mathcal{F}x + q \tag{1}$$

with linear or nonlinear Volterra by Tikhonov (see Azbelev, Maksimov and Rakhmatullina [1, 2], Corduneanu [5], Kurbatov [16, 17], Elsgoltz and Norkin [1]) operators $\mathcal{L}$ and $\mathcal{F}$ respectively, that are defined on the set D_{loc} of locally absolutely continuous functions $x\colon [a,\infty) \to R^n$ (the functions admitting the representation

$$x(t) = \int_a^t \dot{x}(s)\, ds + x(a),$$

where $\dot{x} \in L_{loc}$, and L_{loc} is the space of the classes of the equivalence of locally summable functions on $[a,\infty)$). The equations of this type are generalizations of the ordinary differential equation

$$\frac{dx(t)}{dt} + P(t)x(t) = f(t, x(t)) + q(t),$$

contain the equation with delaying argument

$$\frac{dx(t)}{dt} + \int_a^t d_s R_1(t,s)x(s) = f\left(t, \int_a^t d_s R_2(t,s)x(s)\right) + q(t),$$

the integro-differential equation

$$\frac{dx(t)}{dt} + \int_a^t K(t, s, x(s))\, ds = q(t),$$

and a number of other classes of functional differential equations, important for modern applications (see Azbelev, Maksimov and Rakhmatullina [1,2]).

Stability theory has been in development for a long time in the research direction and studies indicated and initiated by Lyapunov one hundred years ago. However, sometimes for the equations with delaying argument and their generalizations, the classical Lyapunov concepts and methods are not efficient and do not yield the desired results. This is due to the specific character of ordinary differential equations, on which some of Lyapunov ideas are based.

Bellman [1], Krein, M.G. [1], Massera and Schäffer [1], Daletzky and Krein, M.G. [1], Hartman [1] and Barbashin [1] have initiated new trends in stability investigations. Their investigations are based on the fact that the stability properties may be associated with the solvability of the Cauchy problem in special functional spaces. In works by Tyshkevich [1], Kurbatov [3, 5, 8 – 17], Akhmerov and Kurbatov [1, 2] and other authors (see Michel and Miller [1], Kolmanovsky and Nosov [1], Corduneanu [5], Slusarchuk [1], Gusarenko and Domoshnitzky [1], Gusarenko [3, 5], Berezansky [1 – 4], Chistyakov [1], Rekhlitzky [1], Azbelev, Berezansky, Simonov and Chistyakov [1], Domoshnitzky and Sheina [1], Cruz and Hale [1, 2], Hale and Meyer [1], Azbelev, Berezansky and Simonov [1], Azbelev [2, 3, 5, 6, 8, 9], Azbelev, Berezansky and Chistyakov [1], Azbelev and Berezansky [1], Karnishin [1, 2], Berezansky and Larionov [1], Halanay [1], Malygina and Sokolov [1], Drakhlin [1, 2], Zabrejko, Mazel and Tretjakova [1], Nosov [1], Sokolov [1], Abdullayev [1], Simonov [1], Simonov, Fedorenko and Chistyakov [1], Simonov and Chistyakov [2 – 4], Chistyakov and Simonov [1], Azbelev, Yermolayev and Simonov [1], Azbelev, Maksimov and Rakhmatullina [1, 2]) the above mentioned directions have been developed and applied to special classes of functional differential equations. The investigations treat general notions of dichotomy and theorems of the type of Bohl-Perron theorem, stating that for linear equation the boundedness of all solutions for all bounded right-hand sides of q equals under certain conditions to uniform exponential stability. In the above papers, the problem of efficient (expressed via the equation parameters) stability criteria was not undertaken. In the proposed book the Bohl-Perron type theorems are discussed in a separate chapter. However the whole book is very much concentrated on establishing of concrete stability tests.

The ideas of functional analysis and the modern interpretation of functional differential equation lead naturally to new concepts in the problems on asymptotic behaviour of solutions and the performing "$\mathcal{W}$-method" for particular equations. This method and some other techniques discussed in the book allowed the development of a number of new stability tests as well as the simplification and sometimes refining of the known results.

The approach to the study of asymptotic properties of solutions to equation (1) proposed in this book is based on establishing the "D-property" of this equation. Namely, a particular solvability of the Cauchy problem

$$\mathcal{L}x = \mathcal{F}x + q, \qquad x(a) = \alpha \tag{2}$$

in the prescribed functional space D and under the right-hand side q from a prescribed functional space B and under the initial condition α from the finite dimensional space R^n. To put it differently, "D-property" (or "D-stability", that is the same) means the existence, uniqueness and continuous dependence on parameters $q \in B$ and $a \in R^n$ of the solution to problem (2) in the Banach space D of functions $x\colon [0,\infty) \to R^n$ with the prescribed asymptotic properties. Under an appropriate choice of spaces B and D, the D-property ensures this or the stability in the usual sense. Besides, the main problem of the stability investigation of the equation is the construction of a space D with prescribed properties where a specific solvability of problem (2) can be stated. The solution of this problem is based on a choice of "model" linear equation $\mathcal{L}_0 x = z$ and a space B of locally summable functions $z\colon [0,\infty) \to R^n$. In particular, the space D is a linear manifold of all locally absolutely continuous solutions x to model equation $\mathcal{L}_0 x = z$ for all $z \in B$. It is suitable to define the norm in space D as

$$\|x\|_D = \|\mathcal{L}_0 x\|_B + |x(a)|.$$

The volume consists of five chapters.

Chapter 1 provides all the necessary information on functional differential equations. For a detailed presentation of the theory of these equations (see Azbelev, Maksimov and Rakhmatullina [1, 2], Azbelev and Rakhmatullina [5]).

Chapter 2 deals with examples of D-space constructions and D-stability test. For linear functional differential equations with delay, a number of D-stability tests are cited, expressed in terms of these equation parameters.

Chapter 3 treats linear functional differential equation with delay, solved with respect to the derivative. The problem of representing general solution to such equation is scrutinized here as well. Based on the peculiarities of this representation the best possible existence tests are proposed for the exponential estimate of the Cauchy functions for scalar equations and the refining tests for the existence of such estimate for the Cauchy matrices of some classes of systems.

In Chapter 4 the conditions are discussed under which D-stability of linear equation ensures more refined properties of solutions, i.e. a specific solvability of the Cauchy problem in a more narrow space $D_1 \subset D$. The development of this problem is associated with the names of Bohl, Perron, Halanay, Tyshkevich.

Chapter 4 is based on a specific sections of theory of semiordered spaces and may be omitted at first reading.

In Chapter 5, the quasilinear equation of type (1) is studied. An analogue of the Lyapunov theorem on stability with reference to the first approximation is proposed, and the scheme of the proofs is set out for the theorems on existence of solutions to problem (2) satisfying the given a priori estimates. The scheme incorporates the theorems on integral functional inequalities of the type of well-known theorems on integral inequalities.

The volume gives an account of some results that have been obtained during 20 years of study by a large group of mathematicians working at the Perm Seminar in the Scientific Research Centre "Functional Differential Equations" (Perm State Technical University) and in the International Laboratory of Constructive Methods of Dynamical Model Investigations (Perm State University).

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Notations

$\Longrightarrow$ – "implies"
$\stackrel{\text{def}}{=}$ – equal by definition
$\equiv$ – identical
$\Omega \cap \Theta$ – intersection of sets Ω and Θ
$\Omega \cup \Theta$ – union of sets Ω and Θ
$\Omega \subseteq \Theta$ – set Ω is a subset of set Θ
$\Omega \subset \Theta$ – set Ω is a subset of set Θ and does not coincide with it
$\Delta \equiv \Delta^a \stackrel{\text{def}}{=} \{(t,s) \in [a,\infty) \times [a,\infty) \colon s \le t\}$
$\Delta^b \stackrel{\text{def}}{=} \{(t,s) \in \Delta \colon b \le s \le t\}$ for $b \ge a$
$\Delta(a,h) \stackrel{\text{def}}{=} \{(t,s) \in \Delta \colon a \le s \le t \text{ for } h(t) = t \text{ or } a \le s \le h(t) \text{ for } h(t) < t\}$
E is the identity $n \times n$ matrix with columns $E_1, \ldots, E_n$
N is the set of positive integers
Z is the ring of integers
Z_+ is the set of non-negative integers
K is the field of complex numbers
R is the space of real numbers with norm (modulus) $|\cdot|$
R^n is the space of real vector columns $\alpha = \text{col}\{\alpha_1, \ldots, \alpha_n\}$ with norm $|\cdot|$
$R^{n\times n}$ is the algebra of real $n \times n$ matrices with the unit E and norm $|\cdot|$ coordinated with the norm in R^n
$X \times Y$ is the direct (Cartesian) product of linear spaces X and Y
$X \oplus Y$ is the direct sum of linear spaces X and Y

$\alpha \geq 0$ means for the vector $\alpha = \operatorname{col}\{\alpha_1, \ldots, \alpha_n\}$ that $\alpha \geq 0$ for all $i = 1, \ldots, n$

$\alpha \geq \beta$ means that $\alpha - \beta \geq 0$

$|\alpha| \stackrel{\text{def}}{=} \operatorname{col}\{|\alpha_1|, \ldots, |\alpha_n|\}$ is the modulus of the vector α

$\Re\lambda$ is the real part of the complex number λ

mes is Lebesgue measure

χ_Θ is the characteristic function of the set Θ:

$$\chi_\Theta(t) = \begin{cases} 1, & \text{if} \quad t \in \Theta, \\ 0, & \text{if} \quad t \notin \Theta \end{cases}$$

$$\chi^b(t) \stackrel{\text{def}}{=} \chi_{\Delta(b,h)}(t,b) = \begin{cases} 1, & \text{if} \quad h(t) \geq b, \\ 0, & \text{if} \quad h(t) < b \end{cases}$$

$\chi_k^b(t) \stackrel{\text{def}}{=} \chi_{\Delta(b,h_k)}(t,b)$

$h^b(t) \stackrel{\text{def}}{=} h(t)\chi^b(t)$

$$\chi^0(t) \equiv \chi^+(t) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if} \quad h(t) \geq 0, \\ 0, & \text{if} \quad h(t) < 0 \end{cases}$$

$h^0(t) \equiv h^+(t) \stackrel{\text{def}}{=} h(t)\chi^0(t)$

$$h^t(t) \stackrel{\text{def}}{=} \begin{cases} t, & \text{if} \quad h(t) < a, \\ h(t), & \text{if} \quad h(t) \geq a \end{cases}$$

$z \geq 0$ means that $z(t) \geq 0$ for (almost) every $t \geq a$

$z \geq y$ means that $z - y \geq 0$

$|z|(t) \stackrel{\text{def}}{=} |z(t)|$ is the modulus of the (vector-) function z

$z^0(t) \equiv z^+(t) \stackrel{\text{def}}{=} \max\{z(t), 0\}$

$z^-(t) \stackrel{\text{def}}{=} -\min\{z(t), 0\}$

$z^a(t) \stackrel{\text{def}}{=} \max\{z(t), a\}$

$[z]^b \stackrel{\text{def}}{=} \chi_{(b,\infty)}z$

$[z]_b \stackrel{\text{def}}{=} \chi_{[a,b]}z$

$z_\gamma(t) \stackrel{\text{def}}{=} z(t)e^{-\gamma t}$ for $\gamma \in R$, $t \geq a$

$\operatorname*{vrai\,sup}_{t \in \Theta} z(t)$ is the essential supremum of the measurable function $z\colon [a,\infty) \to R$ on the set $\Theta \subset [a,\infty)$

$\operatorname*{vrai\,inf}_{t \in \Theta} z(t)$ is the essential infimum of the measurable function $z\colon [a,\infty) \to R$ on the set $\Theta \subset [a,\infty)$

$z(\infty) \stackrel{\text{def}}{=} \operatorname*{vrai\,lim}_{t\to\infty} z(t)$ is the essential limit of the measurable (vector-) function $z\colon [a,\infty) \to R^n$: $\lim_{b\to\infty} \operatorname*{vrai\,sup}_{t \geq b} |z(t) - z(\infty)| = 0$

$\overline{\text{vrai lim}}_{t\to\infty} z(t) \stackrel{\text{def}}{=} \lim_{b\to\infty} \text{vrai sup}_{t\geq b} z(t)$ is the essential upper limit of the measurable function $z\colon [a,\infty) \to R$

$z_\infty \equiv z_\infty^a \stackrel{\text{def}}{=} \text{col}\left\{\text{vrai sup}_{t\geq a} |z_1(t)|, \dots, \text{vrai sup}_{t\geq a} |z_n(t)|\right\}$ for the measurable function $z\colon [a,\infty) \to R^n$, $z = \text{col}\{z_1,\dots,z_n\}$

$P_\infty \equiv P_\infty^a \stackrel{\text{def}}{=} \left\{\text{vrai sup}_{t\geq a} |p_{ij}(t)|\right\}$ for the measurable (matrix-) function $P\colon [a,\infty) \to R^{n\times n}$, $P = \{p_{ij}\}$, $i, j = 1,\dots,n$

$\text{Var}_{t=a}^{b}\, x(t) \stackrel{\text{def}}{=} \sup\left\{\sum_{i=1}^{m} |x(t_i) - x(t_{i-1})|\colon\ m \in N,\ a = t_0 < t_1 \cdots < t_{m-1} < t_m = b\right\}$ is the variation of the function $x\colon [a,\infty) \to R^n$ on the segment $[a,b]$

$\text{diag}\,\{p_{11},\dots,p_{nn}\}$ is the $n\times n$ matrix with the diagonal $p_{11},\dots,p_{nn}$ and zeros outside the diagonal

$L[a,b]$ is the Banach space of the classes of the equivalence of the measurable and summable functions $z\colon [a,b] \to R^n$ with the norm $\|z\|_{L[a,b]} \stackrel{\text{def}}{=} \int_a^b |z(t)|\,dt$

$M[a,b]$ is the Banach space of the classes of the equivalence of measurable and essentially bounded functions $z\colon [a,b] \to R^n$, with the norm $\|z\|_{M[a,b]} \stackrel{\text{def}}{=} \text{vrai sup}_{t\in[a,b]} |z(t)|$

L_{loc} is a linear space of the classes of the equivalence of the measurable and locally summable functions $z\colon [a,\infty) \to R^n$

$L_{loc}^b \stackrel{\text{def}}{=} \{z \in L_{loc}\colon\ z(t) = 0 \text{ almost everywhere on } [a,b]\}$ for $b \geq a$

$B^b \stackrel{\text{def}}{=} B \cap L_{loc}^b$, where $B \subseteq L_{loc}$, $b > a$

L^p, $1 < p < \infty$, is the Lebesgue space: i.e., the Banach space of the functions $z \in L_{loc}$, summable in power p, and with norm $\|z\|_{L^p} \stackrel{\text{def}}{=} \left(\int_a^\infty |z(t)|^p\,dt\right)^{1/p}$

L is the Banach space of the summable functions $z \in L_{loc}$ with the norm $\|z\|_L \stackrel{\text{def}}{=} \int_a^\infty |z(t)|\,dt$

M is the Banach space of essentially bounded functions $z \in L_{loc}$ with the norm $\|z\|_M \stackrel{\text{def}}{=} \text{vrai sup}_{t\geq a} |z(t)|$

M_l is the subspace of all functions $z \in M$ for which there exists $\text{vrai lim}_{t\to\infty} z(t)$ with the norm $\|z\|_{M_l} \stackrel{\text{def}}{=} \|z\|_M$

M_0 is the subspace of all functions $z \in M_l$ for which there exists $\text{vrai lim}_{t\to\infty} z(t) = 0$, with the norm $\|z\|_{M_0} \stackrel{\text{def}}{=} \|z\|_M$

M_γ, $\gamma \in R$, is the Banach space of all functions $z \in L_{loc}$ for which $z(t) = y(t)e^{-\gamma t}$ for any $y \in M$, with the norm $\|z\|_{M_\gamma} \stackrel{\text{def}}{=} \|y\|_M$

C_{loc} is the linear space of continuous functions $x\colon [a,\infty) \to R^n$

$C^b_{loc} \stackrel{\text{def}}{=} \{x \in C_{loc}\colon\ x(t) = 0 \text{ on } [a,b]\}$ for $b \geq a$

C is the Banach space of continuous bounded functions $x \in C_{loc}$ with the norm $\|x\|_C \stackrel{\text{def}}{=} \sup_{t\geq a} |x(t)|$

$C^b \stackrel{\text{def}}{=} C \cap C^b_{loc}$ is the Banach space of continuous bounded functions with the norm $\|x\|_{C^b} \stackrel{\text{def}}{=} \sup_{t\geq b} |x(t)|$

C_l is the subspace of all functions $x \in C$ for which there exists $x(\infty) \stackrel{\text{def}}{=} \lim_{t\to\infty} x(t)$ with the norm $\|x\|_{C_l} \stackrel{\text{def}}{=} \|x\|_C$

C_0 is the subspace of all functions $x \in C_l$ for which $\lim_{t\to\infty} x(t) = 0$ with the norm $\|x\|_{C_0} \stackrel{\text{def}}{=} \|x\|_C$

C_γ, $\gamma \in R$, is the Banach space of all functions $x \in C_{loc}$ for which $x(t) = y(t)e^{-\gamma t}$ for any $y \in C$, with the norm $\|x\|_{C_\gamma} \stackrel{\text{def}}{=} \|y\|_C$

$C_{(1),loc}$ is the linear space of scalar continuous functions $x\colon [a,\infty) \to R$

$C_{(1)}$ is the Banach space of scalar bounded functions $x \in C_{(1),loc}$ with the norm $\|x\|_{C^1} \stackrel{\text{def}}{=} \sup_{t\geq a} |x(t)|$

D_{loc} is the linear space of the functions $x\colon [a,\infty) \to R^n$, absolutely continuous on every finite interval, i.e. when x is locally absolutely continuous, $\dot{x}$ is locally summable

$D[a,b]$ is the Banach space of the absolutely continuous functions $x\colon [a,b] \to R^n$ with the norm $\|x\|_{D[a,b]} \stackrel{\text{def}}{=} \|\dot{x}\|_{L[a,b]} + |x(a)|$

$D^b_{loc} \stackrel{\text{def}}{=} \{x \in D_{loc}\colon\ x(t) = 0 \text{ on } [a,b]\}$ for $b \geq a$

$D^b \stackrel{\text{def}}{=} D \cap D^b_{loc}$, where $D \subset D_{loc}$, $b \geq a$

W is the Sobolev space, i.e. the Banach space of all functions $x \in D_{loc}$ for which $x \in C$, $\dot{x} \in M$, with the norm $\|x\|_W \stackrel{\text{def}}{=} \|x\|_C + \|\dot{x}\|_M$

W_γ, $\gamma \in R$, is a Banach space of all functions $x \in D_{loc}$ for which $x \in C_\gamma$, $\dot{x} \in M_\gamma$, with the norm $\|x\|_{W_\gamma} \stackrel{\text{def}}{=} \|x\|_{C_\gamma} + \|\dot{x}\|_{M_\gamma}$

$(\mathcal{N}_f x)(t) \stackrel{\text{def}}{=} f(t, x(t))$ is the Nemytzky operator generated by the function f (an operator of the exterior superposition)

$(\mathcal{S}_h x)(t) \stackrel{\text{def}}{=} \begin{cases} x[h(t)], & \text{if } h(t) \geq a, \\ 0, & \text{if } h(t) < a \end{cases}$ is an operator of the interior superposition

$x_h \stackrel{\text{def}}{=} \mathcal{S}_h x$

$\varphi^h(t) \stackrel{\text{def}}{=} \begin{cases} 0, & \text{if} \quad h(t) \geq a, \\ \varphi[h(t)], & \text{if} \quad h(t) < a \end{cases}$

$\mathcal{Q}U \stackrel{\text{def}}{=} \{\mathcal{Q}U_1, \ldots, \mathcal{Q}U_n\}$ is the value of the operator $\mathcal{Q}$ at the $n \times n$-matrix $U\colon [a, \infty) \to R^{n\times n}$ with the columns $U_1, \ldots, U_n$

$\rho(\mathcal{Q})$ is the spectral radius of linear bounded operator $\mathcal{Q}$

$\mathcal{Q}_\gamma$, $\gamma \in R$, is the operator associated with the operator $\mathcal{Q}$ by the equality $(\mathcal{Q}_\gamma z)(t) \stackrel{\text{def}}{=} e^{\gamma t}(\mathcal{Q}z_\gamma)(t)$

ODE – ordinary differential equation
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