Daniel D. Joseph

# Fluid Dynamics of Viscoelastic Liquids

With 154 Figures in 206 Parts

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# Applied Mathematical Sciences
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## Applied Mathematical Sciences

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This volume is dedicated to Shifra Chana, Mike, Chuck and Samuel.

# Preface

This book is about two special topics in rheological fluid mechanics: the elasticity of liquids and asymptotic theories of constitutive models. The major emphasis of the book is on the mathematical and physical consequences of the elasticity of liquids; seventeen of twenty chapters are devoted to this. Constitutive models which are instantaneously elastic can lead to some hyperbolicity in the dynamics of flow, waves of vorticity into rest (known as shear waves), to shock waves of vorticity or velocity, to steady flows of transonic type or to short wave instabilities which lead to ill-posed problems. Other kinds of models, with small Newtonian viscosities, give rise to perturbed instantaneous elasticity, associated with smoothing of discontinuities as in gas dynamics.

There is no doubt that liquids will respond like elastic solids to impulses which are very rapid compared to the time it takes for the molecular order associated with short range forces in the liquid, to relax. After this, all liquids look viscous with signals propagating by diffusion rather than by waves. For small molecules this time of relaxation is estimated as $10^{-13}$ to $10^{-10}$ seconds depending on the fluids. Waves associated with such liquids move with speeds of $10^5$ cm/s, or even faster. For engineering applications the instantaneous elasticity of these fluids is of little interest; the practical dynamics is governed by diffusion, say, by the Navier-Stokes equations. On the other hand, there are other liquids which are known to have much longer times of relaxation. Polymers mixed in Newtonian solvents and polymer melts, like high viscosity silicone oils or molten plastics, are examples. The longest times of relaxation for these liquids are of practical interest; times we can read on our clock, of the order of milliseconds to minutes, or longer. The study of hyperbolic dynamics is complicated by the presence of many relaxation times. The limiting wave speed is determined by the fastest rather than by the slowest relaxation so that the instantaneous elastic response has already begun before the slow relaxation has begun. The fast relaxation of small molecules gives rise to an effective viscosity which smooths slow waves. If the total viscosity is much greater than the effective viscosity we may consider the theory of perturbed elasticity, with relatively small effective rigidities associated with the long lasting relaxations. The effective wave speeds are slow, ranging roughly from 1 to 1000 cm/sec.

It follows from what has been said that the models which are instantaneously elastic and give rise to hyperbolicity and change of type are precise only for times too short for applications. For the applications, the effective theory appears to work well but not all issues have been resolved. One question is what type of theory may be developed when the effective viscosity is not just a small part of the total. A second question is to what extent we may expect robust values of the effective quantities which are not dependent on flow conditions. It is probable that the successful resolution of these issues will depend more on experiments than on theory.

The contents of the seventeen chapters on the elasticity of liquids is taken from relatively recent papers not before collected into one volume.

The three chapters on asymptotic theory treat some well-known things in a new way. In 15 I review theories of fading memory and show how different theories will lead to different types of constitutive equations. Various types of perturbation theories are considered in detail in 16. In 17 I deal with second order theory emphasizing features which I consider fundamental like the balance of inertia and normal stress effects, the persistence of normal stress, the correlation between extensional viscosity and the intensity of secondary motion, the importance of nonelastic contributions to extensional and secondary motions and the general rheometrical problem of determining values of the quadratic constants.

I have tried to avoid repeating things which are well expressed in other books listed in the references. Since only special topics are treated, this book cannot be used as a general reference, but for many of the special topics treated it is effectively the only reference. Some complementary results for wave propagation in viscoelastic materials and many results about existence and uniqueness of solutions for one-dimensional models can be found in the book by Renardy, Hrusa, and Nohel [1987].





Finally I want to acknowledge the help I have received from Michael Renardy and Jean Claude Saut in joint works, separate works, and discussions, and Edward Fraenkel for his contributions to the solution of the problem of flow over a flat plate. Mark Ahrens, Kangping Chen, Howard Hu, Amitabh Narain, Luigi Preziosi, Oliver Riccius, and Claude Verdier helped me in different ways but especially doing the research reported in this book. Special thanks are due to Verdier and Hu for proofreading and to Hu for his help with the calculation in §5.8. Eric Scouten did the initial word processing of the manuscript and Lee Reynolds carried through revisions and formatted the text as it appears. My work has been supported for many years by the division of mathematics of the Army Research Office and division of fluid mechanics of the National Science Foundation.

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CHAPTER 1

# Models Like Maxwell's and Boltzmann's



## 1.1 The Maxwell element

We can build a very sophisticated nonlinear invariant theory from springs and dashpots. A spring and dashpot in series is called a Maxwell element (Figure 1.1). The spring constant is called G and the force $\sigma_1$ in the spring is $G\gamma_1$ where $\gamma_1$ is the displacement of the spring. The force $\sigma_2$ in the dashpot is $\eta(\partial\gamma/\partial t)_2$ where $\eta$ is the viscosity and $(\partial\gamma/\partial t)_2$ is the velocity, the time rate of change of $\gamma_2$, and because they are in series, $\sigma_1=\sigma_2=\sigma$.

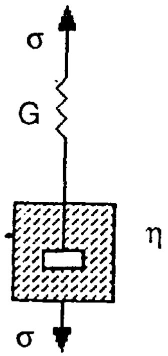


Figure 1.1. Maxwell element. The force $\sigma_1$ in the spring is the same as $\sigma_2$, the force in the dashpot.

We also have that the time rate of change of the total displacement is

$$\frac{\partial\gamma}{\partial t}=\left(\frac{\partial\gamma}{\partial t}\right)_1+\left(\frac{\partial\gamma}{\partial t}\right)_2=\frac{(\partial\sigma/\partial t)_1}{G}+\frac{\sigma_2}{\eta}=\frac{(\partial\sigma/\partial t)}{G}+\frac{\sigma}{\eta}. \quad (1)$$

Hence,

$$\lambda\frac{\partial\sigma}{\partial t}+\sigma=\eta\frac{\partial\gamma}{\partial t} \quad (2)$$

where $\lambda=\eta/G$ is a relaxation time. Another expression for $\sigma$ is

$$\sigma=\frac{\eta}{\lambda}\int_{-\infty}^{t}\exp\left(\frac{-(t-\tau)}{\lambda}\right)\frac{\partial\gamma(\tau)}{\partial\tau}\,d\tau. \quad (3)$$

Obviously (2) is a differential equation model for the relation between force and deformation and (3) is an integral model showing that the present value of the force $\sigma(t)$ is determined by the history of $\gamma$.



## 1.2 Stress relaxation and instantaneous elasticity

Equation (2) is the constitutive equation for the Maxwell element. It will support a force with constant deformation $\gamma\neq0$, $\partial\gamma/\partial t=0$ and

$$\sigma=\sigma(0)\exp(-t/\lambda). \quad (4)$$

Eventually $\sigma$ relaxes to zero. We can regard $\sigma(x,t)$ as a stress and $\gamma(x,t)$ as a strain. We say that the Maxwell element has instantaneous elasticity. At $t=0$, we increase $\gamma$ suddenly, producing a stress. Then we keep $\gamma$ constant and the stress relaxes.

## 1.3 A one-dimensional model in the linearized case

Let

$$\gamma=\frac{\partial\xi(x,t)}{\partial x} \quad (5)$$

be the strain, where

$$\frac{\partial\xi}{\partial t}=u \quad (6)$$

is the velocity. Then

$$\frac{\partial\gamma}{\partial t}=\frac{\partial u(x,t)}{\partial x} \quad (7)$$

and

$$\lambda\frac{\partial\sigma}{\partial t}+\sigma=\eta\frac{\partial u}{\partial x}. \quad (8)$$

The equation of motion is

$$\rho\frac{\partial u}{\partial t}=\frac{\partial\sigma}{\partial x}. \quad (9)$$

We may combine (8) and (9)

$$\frac{\partial^2 u}{\partial t^2} + \frac{1}{\lambda}\frac{\partial u}{\partial t} = \frac{\eta}{\rho\lambda}\frac{\partial^2 u}{\partial x^2} . \tag{10}$$

Equation (10) is a telegraph equation. Recall that $G=\eta/\lambda$. The telegraph equation is a wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{11}$$

perturbed by

$$\frac{1}{\lambda}\frac{\partial u}{\partial t} \tag{12}$$

where

$$c = \sqrt{G/\rho} \tag{13}$$

is the wave speed. If $\partial u/\partial x$ is a longitudinal strain, c is the sound speed. If $\partial u/\partial x$ is a shear strain, c is the speed of a shear wave. For incompressible fluids, the sound speeds are infinite. In liquids, it is more interesting to consider shear waves.

Fluids with long times of relaxation are very nearly elastic; those with short times of relaxation are very nearly viscous.

Suppose $\lambda \to 0$, then

$$\sigma = \eta \frac{\partial u}{\partial x} \tag{14}$$

and

$$\rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial x^2} , \tag{15}$$

which are the equations of a Newtonian fluid. If, on the other hand, $\lambda \to \infty$, then

$$\frac{\partial \sigma}{\partial t} = G \frac{\partial u}{\partial x} \quad \text{or} \quad \sigma = G \frac{\partial \xi}{\partial x} \tag{16}$$

is elastic.



## 1.4 Hyperbolicity, characteristics

Equation (15) is parabolic and equation (11) is hyperbolic. Hyperbolic equations allow wave propagation of discontinuities without smoothing. Parabolic problems smooth discontinuities by diffusion.

The telegraph equation is hyperbolic. To see this, look at the system (8) and (9)

$$\begin{aligned} \lambda\sigma_t - \eta u_x &= -\sigma , \\ \rho u_t - \sigma_x &= 0 \end{aligned} \tag{17}$$

and ask if there are lines $\phi(x,t)=0$ across which the derivatives of u and $\sigma$ are discontinuous with continuity for u and $\sigma$. We form equations for the jumps of the derivatives across these lines and get

$$\begin{aligned} \lambda[\![\sigma_t]\!] - \eta[\![u_x]\!] &= -[\![\sigma]\!] , \\ \rho[\![u_t]\!] - [\![\sigma_x]\!] &= 0 . \end{aligned} \tag{18}$$

According to our set-up, $\sigma$ and u are continuous on $\phi=0$; hence,

$$[\![u]\!] = [\![\sigma]\!] = 0 \quad \text{on} \quad \phi(x,t) = 0 . \tag{19}$$

All the tangential derivatives of $[\![u]\!]=0$ and $[\![\sigma]\!]=0$ also vanish and the only derivatives of $[\![u]\!]=0$ and $[\![\sigma]\!]=0$ are normal to $\phi$. Hence, for example,

$$[\![\sigma_t]\!] = [\![\sigma_\phi]\!]\phi_t .$$

Since $d\phi = \phi_t dt + \phi_x dx = 0$, we may reduce (18) to

$$\begin{aligned} \lambda[\![\sigma_\phi]\!]\frac{dx}{dt} + \eta[\![u_\phi]\!] &= 0 , \\ [\![\sigma_\phi]\!] + \rho[\![u_\phi]\!]\frac{dx}{dt} &= 0 . \end{aligned} \tag{20}$$

We can solve (20) for $[\![\sigma_\phi]\!]$ and $[\![u_\phi]\!]$ if

$$\det \begin{vmatrix} \lambda \frac{dx}{dt} & \eta \\ 1 & \rho \frac{dx}{dt} \end{vmatrix} = \lambda\rho \left(\frac{dx}{dt}\right)^2 - \eta = 0 \,. \tag{21}$$

There are two families of characteristics defined by (21)

$$x \pm ct = \text{const} \tag{22}$$

where c is the wave speed (13).

## 1.5 Linearized Maxwell models

We make a tensorial equation of (17) by declaring that $\sigma$ is a symmetric tensor field

$$\sigma(\mathbf{x},t) = \sigma^T(\mathbf{x},t) \,, \tag{23}$$

$\mathbf{u}(\mathbf{x},t)$ is a solenoidal vector field satisfying

$$\operatorname{div}\mathbf{u} = 0 \tag{24}$$

and $\partial u/\partial x$ now stands for

$$2\mathbf{D}[\mathbf{u}] \overset{\text{def}}{=} \mathbf{A}_1[\mathbf{u}]$$

where $\mathbf{D}[\mathbf{u}]$ stands for the symmetric part of $\mathbf{L}(\mathbf{x},t) \overset{\text{def}}{=} \nabla\mathbf{u}$ ($L_{ij}=\partial u_i/\partial x_j$). The constitutive equation of the rate type generalizing (17) is

$$\sigma = -p\mathbf{1} + \tau \,, \tag{25}$$

$$\lambda \frac{\partial \tau}{\partial t} + \tau = 2\eta\, \mathbf{D}[\mathbf{u}] \tag{26}$$

where $\tau$ is called the extra stress and p is the reaction pressure. The equation expressing the balance of mass is (24) and the balance of momentum is

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \operatorname{div} \tau \tag{27}$$



where the reaction pressure $p=p[\mathbf{u}]$ may be determined as a functional of $\mathbf{u}$ by the solution of the four equations (27) and (24) for the three components of $\mathbf{u}$ and $p[\mathbf{u}]$. The reaction pressure is not determined by a constitutive expression, it varies from problem to problem, as in an incompressible Newtonian fluid.

## 1.6 Nonlinear Maxwell models

By declaration, we shall call fluids which obey a constitutive equation of the rate type

$$\lambda \frac{D\tau}{Dt} + \tau = 2\eta\mathbf{D}[\mathbf{u}] \tag{28}$$

Maxwell models. These models are not unique; they differ in that various invariant derivatives $D\tau/Dt$ can be defined. Up to now we did not consider invariance. In fact all the invariant nonlinear derivatives reduce to partial time derivatives when linearized at states of rest,

$$\frac{D\tau}{Dt} \to \frac{\partial \tau}{\partial t} \,. \tag{29}$$

## 1.7 Form invariance and frame indifference

Many people believe that because constitutive equations characterize the response of the material they ought to be independent of the observer: two observers on different planets, or on different turntables on the same planet should come up with, say, the same equation relating stress and deformation and their equation should not depend on the frame. There are actually two requirements stated here. The first is that equations should be form invariant under Euclidean transformations (30) representing the change of frame. The second is that the invariant form should be independent of frame.

Suppose that the frame of the star observer at $\mathbf{x}^*$ is in a rigid motion so that

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x} + \mathbf{b}(t) \tag{30}$$

where $\mathbf{b}(t)$ is a time dependent constant and $\mathbf{Q}(t)$ is an orthogonal tensor which rotates vectors*; $\mathbf{Q}^{-1}(t)=\mathbf{Q}^T(t)$, $\mathbf{Q}\mathbf{Q}^T=\mathbf{1}$ and $\mathbf{x}$ is the point you see in your frame.

Our * observer studies some material which we think satisfies (28), and we will think him clever if he finds that

$$\lambda \frac{D^*\boldsymbol{\tau}^*}{D^*t} + \boldsymbol{\tau}^* = 2\eta \mathbf{D}^*[\mathbf{u}^*] \tag{31}$$

where the * objects have the same form as in (28). For example,

$$2\mathbf{D}^* = \nabla^*\mathbf{u}^* + (\nabla^*\mathbf{u}^*)^T$$

$$\left( 2D^*_{ij} = \frac{\partial u_i^*}{\partial x_j^*} + \frac{\partial u_j^*}{\partial x_i^*} \right). \tag{32}$$

Now we know that the rate of strain is an *indifferent* tensor: under a change of frame (30), it transforms like an ordinary tensor

$$\mathbf{D}^*[\mathbf{u}^*] = \mathbf{Q}(t)\mathbf{D}[\mathbf{u}]\mathbf{Q}^T(t) . \tag{33}$$

Actually,

$$\mathbf{L}^*(\mathbf{x}^*,t) = \mathbf{Q}(t)\mathbf{L}(\mathbf{x},t)\mathbf{Q}^T(t) + \boldsymbol{\Omega}[\mathbf{u}]$$

where

$$\boldsymbol{\Omega} = \frac{d\mathbf{Q}(t)}{dt}\mathbf{Q}^T(t) = -\boldsymbol{\Omega}^T \tag{34}$$

is an antisymmetric tensor (because $\mathbf{Q}\mathbf{Q}^T=\mathbf{1}$), does not transform as an indifferent tensor; but $\mathbf{L}+\mathbf{L}^T$ does. We are going to *assume* that the extra stress is an indifferent tensor

---

* If $\mathbf{Q}(t)$ is a rotation around, say, $x_3$, then the components of $\mathbf{Q}(t)$ in coordinates $(x_1,x_2,x_3)$ are given by

$$[\mathbf{Q}(t)] = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

where $\theta=\theta(t)$.



$$\boldsymbol{\tau}^*(\mathbf{x}^*,t) = \mathbf{Q}(t)\boldsymbol{\tau}(\mathbf{x},t)\mathbf{Q}^T(t) . \tag{35}$$

Hence, our constitutive equation will be form invariant if we can find some invariant derivatives satisfying

$$\frac{D^*\boldsymbol{\tau}^*}{Dt} = \mathbf{Q}(t)\frac{D\boldsymbol{\tau}}{Dt}\mathbf{Q}^T(t) \tag{36}$$

If (35) and (36) hold, then

$$\lambda \frac{D^*\boldsymbol{\tau}^*}{Dt} + \boldsymbol{\tau}^* - 2\eta \mathbf{D}^*[\mathbf{u}^*]$$
$$= \mathbf{Q}(t)\left\{\lambda\frac{D\boldsymbol{\tau}}{Dt} + \boldsymbol{\tau} - 2\eta\mathbf{D}[\mathbf{u}]\right\}\mathbf{Q}^T(t) \tag{37}$$
$$= 0$$

and all the observers are in agreement about the form of the constitutive equation. Indifferent constitutive equations are form invariant but they may depend on the frame (cf. Exercise 1.13).

The belief that constitutive equations should not depend on the frame is called the "principle of material frame indifference." Perhaps it is stated as a principle because many people believe in it strongly even though they can't prove it. It seems that the exact circumstances under which dynamics give rise to constitutive equations which are independent of frame is a largely unexplored topic at the foundation of continuum mechanics. For a fuller discussion of these issues, see Joseph and Preziosi, "Addendum to the paper 'Heat Waves.'"

## 1.8 Frame independent invariant derivatives

A properly invariant tensor rate can be obtained if we interpret the rate operator as the time derivative $\partial/\partial t$ with respect to a reference frame suitably fixed to the body* Different choices of body-fixed frames in this interpretation yield different invariant tensor rates. We present two examples in Equations (40) and (46) below.

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* Oldroyd, J.G. On the formulation of rheological equations of state. *Proc. R. Soc.*, London A**200**, 523-41. A complete exposition of Oldroyd's approach to constitutive modeling is developed by Lodge [1974].

## 1.9 Upper convected invariant derivatives

To facilitate the presentation of the body-fixed frames in these examples, we cover the body with a convected coordinate system $\xi^i$. (A convected coordinate system deforms with the body so that the coordinates $(\xi^1,\xi^2,\xi^3)$ associated with a particular material point do not change with time.) Base vectors $\mathbf{g}_i$ at a material point are defined by

$$\mathbf{g}_i(\xi^1,\xi^2,\xi^3,t) = \frac{\partial}{\partial \xi^i}\mathbf{x}(\xi^1,\xi^2,\xi^3,t) . \tag{38}$$

where $\mathbf{x}(\xi^1,\xi^2,\xi^3,t)$ is the position vector at time t of the material point with convected coordinates $(\xi^1,\xi^2,\xi^3)$. A 3-D tensor $\mathbf{M}$ may be expressed with respect to the basis $\mathbf{g}_i \otimes \mathbf{g}_j$

$$\mathbf{M}(\xi,t) = M^{ij}(\boldsymbol{\xi},t)\, \mathbf{g}_i(\boldsymbol{\xi},t) \otimes \mathbf{g}_j(\boldsymbol{\xi},t) \tag{39}$$

where $\boldsymbol{\xi}$ is shorthand for the convected label $(\xi^1,\xi^2,\xi^3)$ of the material point.

The frame at the material point with basis $\{\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3\}$ rotates and deforms with the neighborhood of the point. As our first choice of invariant tensor rate, we define the rate of tensor $\mathbf{M}$ to be the time derivative of $\mathbf{M}$ with respect to this body fixed frame; explicitly,

$$\frac{D}{Dt}\mathbf{M} \overset{\text{def}}{=} \left[\frac{\partial}{\partial t}M^{ij}(\boldsymbol{\xi},t)\right]\mathbf{g}_i(\boldsymbol{\xi},t) \otimes \mathbf{g}_j(\boldsymbol{\xi},t) . \tag{40}$$

The rate defined in Equation (40) differs from the material derivative $\frac{\partial \mathbf{M}}{\partial t}$ since it ignores the change of the base vectors $\mathbf{g}_i$ with time.

We now remove the dependence on convected coordinates from the rate definition, Equation (40). Taking the material derivative of Equation (39) gives

$$\frac{d\mathbf{M}}{dt} \overset{\text{def}}{=} \frac{\partial}{\partial t}\mathbf{M}(\boldsymbol{\xi},t) = \left[\frac{\partial}{\partial t}M^{ij}(\boldsymbol{\xi},t)\right]\mathbf{g}_i(\boldsymbol{\xi},t) \otimes \mathbf{g}_j(\boldsymbol{\xi},t)$$
$$+ M^{ij}(\boldsymbol{\xi},t)\left[\frac{\partial}{\partial t}\mathbf{g}_i(\boldsymbol{\xi},t)\right] \otimes \mathbf{g}_j(\boldsymbol{\xi},t) + M^{ij}(\boldsymbol{\xi},t)\, \mathbf{g}_i(\boldsymbol{\xi},t) \otimes \left[\frac{\partial}{\partial t}\mathbf{g}_j(\boldsymbol{\xi},t)\right] . \tag{41}$$

We note that



$$\frac{d\mathbf{g}_i}{dt} \overset{\text{def}}{=} \frac{\partial}{\partial t}\mathbf{g}_i(\boldsymbol{\xi},t) = \frac{\partial}{\partial t}\left[\frac{\partial}{\partial \xi^i}\mathbf{x}(\boldsymbol{\xi},t)\right] = \frac{\partial}{\partial \xi^i}\left[\frac{\partial}{\partial t}\mathbf{x}(\boldsymbol{\xi},t)\right]$$
$$= \frac{\partial}{\partial \xi^i}\mathbf{v}(\boldsymbol{\xi},t) = \frac{\partial}{\partial \mathbf{x}}\mathbf{v}(\mathbf{x},t)\,\frac{\partial}{\partial \xi^i}\mathbf{x}(\boldsymbol{\xi},t) = \mathbf{L}\mathbf{g}_i . \tag{42}$$

Use of this result and the definition (40), in (41) gives, after some arranging,

$$\frac{D}{Dt}\mathbf{M} = \frac{d\mathbf{M}}{dt} - M^{ij}\mathbf{L}\mathbf{g}_i \otimes \mathbf{g}_j - M^{ij}\mathbf{g}_i \otimes \mathbf{L}\mathbf{g}_j$$
$$= \frac{d\mathbf{M}}{dt} - \mathbf{L}(M^{ij}\mathbf{g}_i \otimes \mathbf{g}_j) - (M^{ij}\mathbf{g}_i \otimes \mathbf{g}_j)\mathbf{L}^T \tag{43}$$
$$= \frac{d\mathbf{M}}{dt} - \mathbf{L}\mathbf{M} - \mathbf{M}\mathbf{L}^T \overset{\text{def}}{=} \overset{\nabla}{\mathbf{M}} .$$

The definition of $\frac{D}{Dt}\mathbf{M}$ in Equation (40) is therefore seen to be the upper convected rate, here denoted by $\overset{\nabla}{\mathbf{M}}$.

## 1.10 Lower convected invariant derivatives

Other body-fixed frames besides the frame with base vectors $\mathbf{g}_j$ can be defined. Reciprocal base vectors $\mathbf{g}^i$ are defined by

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j , \tag{44}$$

and we may express the tensor $\mathbf{M}$ as

$$\mathbf{M}(\xi,t) = M_{ij}(\boldsymbol{\xi},t)\, \mathbf{g}^i(\boldsymbol{\xi},t) \otimes \mathbf{g}^j(\boldsymbol{\xi},t) . \tag{45}$$

As an alternative to Equation (40), we can define the rate of tensor $\mathbf{M}$ by

$$\frac{D}{Dt}\mathbf{M} \overset{\text{def}}{=} \left[\frac{\partial}{\partial t}M_{ij}(\boldsymbol{\xi},t)\right]\mathbf{g}^i(\boldsymbol{\xi},t) \otimes \mathbf{g}^j(\boldsymbol{\xi},t) . \tag{46}$$

This rate differs from the upper convected rate defined in Equation (40) since the vectors $\mathbf{g}^i$ deform in time differently than $\mathbf{g}_i$. In fact, the rate defined in Equation (46) is the lower convected rate $\overset{\Delta}{\mathbf{M}}$, given by

$$\overset{\Lambda}{\mathbf{M}} = \frac{d\mathbf{M}}{dt} + \mathbf{L}^T\mathbf{M} + \mathbf{ML} \tag{47}$$

Useful in the transformation of equations (46) to (47) is relation

$$\frac{d\mathbf{g}^i}{dt} \overset{\text{def}}{=} \frac{\partial}{\partial t}\mathbf{g}^i(\xi, t) = -\mathbf{L}^T\mathbf{g}^i, \tag{48}$$

which follows from Equation (42) and the material derivative of Equation (44).

## 1.11 Corotational invariant derivatives

The corotational rate $\overset{\circ}{\mathbf{M}}$, defined by

$$\overset{\circ}{\mathbf{M}} \overset{\text{def}}{=} \frac{1}{2}(\overset{\vee}{\mathbf{M}} + \overset{\Lambda}{\mathbf{M}}) = \frac{d\mathbf{M}}{dt} - \mathbf{WM} + \mathbf{MW},$$

$$\mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T), \tag{49}$$

corresponds to the time derivative of $\mathbf{M}$ with respect to a basis which shares the rotation of the neighborhood of the material point, but not its deformation.

## 1.12 Other invariant derivatives

Another invariant rate is

$$\overline{\mathbf{M}} \overset{\text{def}}{=} \frac{1}{2}(\overset{\Lambda}{\mathbf{M}} - \overset{\vee}{\mathbf{M}}) = \mathbf{DM} + \mathbf{MD}.$$

Hence, the four rates $\overset{\vee}{\mathbf{M}}$, $\overset{\Lambda}{\mathbf{M}}$, $\overset{\circ}{\mathbf{M}}$, and $\overline{\mathbf{M}}$ are included as special cases in the general rate

$$\frac{D}{Dt}\mathbf{M} \overset{\text{def}}{=} \frac{d\mathbf{M}}{dt} - \mathbf{WM} + \mathbf{MW} - a(\mathbf{DM} + \mathbf{MD}) \tag{50}$$

where $a$ is constant. This rate satisfies invariance, expressed by (36), and preserves symmetry (in the sense that the rate of a



symmetric tensor is necessarily symmetric) for all values of $a$. The choices $a = 1$, $-1$, and $0$ correspond to $\overset{\vee}{\mathbf{M}}$, $\overset{\Lambda}{\mathbf{M}}$, and $\overset{\circ}{\mathbf{M}}$, respectively.

Some properly invariant rates, such as

$$\frac{D}{Dt}\mathbf{M} \overset{\text{def}}{=} \left[\frac{\partial}{\partial t}M_{ij}\right]\mathbf{g}^i \otimes \mathbf{g}_j = \frac{d\mathbf{M}}{dt} - \mathbf{LM} + \mathbf{ML}$$

and

$$\frac{D}{Dt}\mathbf{M} \overset{\text{def}}{=} \left[\frac{\partial}{\partial t}M^i_{\ j}\right]\mathbf{g}_i \otimes \mathbf{g}^j = \frac{d\mathbf{M}}{dt} + \mathbf{L}^T\mathbf{M} - \mathbf{ML}^T,$$

have the disadvantage that the rate of a symmetric tensor is not in general symmetric.

## 1.13 List of Maxwell models

$$\lambda\overset{\vee}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\eta\mathbf{D} \quad \text{(upper convected Maxwell model, UCM)} \tag{51}$$

$$\lambda\overset{\Delta}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\eta\mathbf{D} \quad \text{(lower convected Maxwell model, LCM)} \tag{52}$$

$$\lambda\overset{\circ}{\boldsymbol{\tau}} + \boldsymbol{\tau} = 2\eta\mathbf{D} \quad \text{(corotational Maxwell Model, COM)} \tag{53}$$

$$\lambda\frac{D\boldsymbol{\tau}}{Dt} + \boldsymbol{\tau} = 2\eta\mathbf{D} \quad \text{(interpolated Maxwell model)} \tag{54}$$

where $\frac{D\tau}{Dt}$ is given by (50).

## 1.14 Invariant derivatives of vectors

We may find time derivatives of vectors which are invariant under superposed rigid motions by Oldroyd's method. We may decompose the vector $\mathbf{a}$ in the basis $\mathbf{g}_i$, or its dual $\mathbf{g}^i$:

$$\mathbf{a} = a^i\mathbf{g}_i = a_i\mathbf{g}^i.$$