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symposia on theoretical physics

Lectures presented at the
1965 Summer School
of the Institute
of Mathematical Sciences
Madras, India

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Introduction

Volume 5 of this series contains the lectures delivered at the Second Matscience Summer School conducted in Bangalore in August, 1965.

Theoretical physics today occupies a privileged position in fundamental sciences for the obvious reason that it is closely related to experimental physics on the one hand and draws its strength from mathematical methods on the other. Studies in theoretical physics at Matscience lay more emphasis on the mathematical aspects taking care, of course, not to get away from the world of observation and reality. In the second Matscience summer school, we had a fair share of these mathematical methods presented in the lectures of Grossmann on nested hilbert spaces, and Ranganathan and Teplitz on the homology theory of Feynman integrals.

The lecture course by Oakes on weak interactions was the fifth series on a subject dealt with earlier at our Institute by Marshak, Takeda, Fujii, and Meyer, demonstrating the importance of a subject in which the theoretical physicist has experienced as many triumphs as failures. One of the interesting features of the summer school was the lecture course on relativity by two representatives of the Paris School, Kichenassamy and Baktavatsalou.

Our Institute supports and fosters the mathematician's attitude to other disciplines so well expressed by Marshall Stone in the second volume of this series— "I try to help everyone, but I have also my own concerns." Professor Unni initiated a series of lectures in the field of mathematics which will be a regular feature of future scientific meetings at Matscience. With this motivation, the lectures of Kelley and Arens, delivered later during the year, have also been included in this volume.

Alladi Ramakrishnan

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Lectures on Nested Hilbert Spaces

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1. INTRODUCTION

In this paper, we shall be concerned with a class of vector spaces that are convenient tools for the investigation of some questions in scattering and quantum field theory.

In a space of this kind, calculations are carried out in essentially the same manner as in a Hilbert space, with the qualification that products are defined only if certain initial and final subsets of a partially ordered set have a non-empty intersection. On the other hand, the gain in generality simplifies the language, e.g., in the study of fields (which need not always be smeared), of unphysical sheets and scattering states. Proofs and references can be found in other papers by the author.†

2. ALGEBRAIC INDUCTIVE LIMIT

Let I be a partially ordered set. Assume that it satisfies the following condition:

(I_1): Any two elements in I have at least one common successor. That is, given any two elements q and r in I, there exists at least one s in I such that $s \geqslant r$ and $s \geqslant q$. This will be written as

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$s \geqslant r, q$. For every $r \in I$, let V_r be a vector space. Assume that the spaces V_r are mutually disjoint. For every $r \in I$ and every s such that $s \geqslant r$, let E_{sr} be a linear mapping from V_r into V_s. Assume the following:

(Ns_1): E_{sr} is injective, i.e., satisfies the condition that if $E_{sr} f_r = 0$, then $f_r = 0$. Assume also that the family E_{sr} satisfies the conditions:

(Ind_1): For every $r \in I$, $E_{rr} = 1$ (Identity in V_r).
(Ind_2): If $s \geqslant r \geqslant p$, then $E_{sp} = E_{sr} E_{rp}$.

The condition (Ns_1) above is not necessary for the definition of algebraic inductive limit. It will, however, always be satisfied in the situations with which we are concerned.

Now consider the union of all the spaces V_r. In this set, define a relation $f_r \sim f_q$ which means: There exists a common successor $s \geqslant r, q$ such that

$$E_{sr} f_r = E_{sq} f_q \tag{1}$$

It is easy to verify that this is an equivalence relation. The union $\bigcup_{r \in I} V_r$ can be decomposed into classes with respect to this relation.

If (1) holds for some $s \geqslant r, q$, then it holds for every common successor of r and of q. Indeed, let $E_{sr} f_r = E_{sq} f_q$ and $w \geqslant r, q$. Let z be any common successor of s and of w. Then, $E_{zr} f_r = E_{zs} E_{sr} f_r = E_{zs} E_{sq} f_q = E_{zq} f_q$. Consequently, $E_{zw} E_{wr} f_r = E_{zw} E_{wq} f_q$. Since E_{zw} is injective, it follows that $E_{wr} f_r = E_{wq} f_q$. If $f_r \sim f_q$ and $g_r \sim g_q$, then $f_r + g_r \sim f_q + g_q$. If $f_r \sim f_q$ and φ is any complex number, then $\varphi f_r \sim \varphi f_q$. Therefore, the set of classes forms a vector space. This vector space will be denoted by V_I and called the algebraic inductive limit of the spaces V_r with respect to E_{sr} and I. In symbols,

$$V_I = \varinjlim \,[V_r; E_{sr}; I] \tag{2}$$

For every $r \in I$, there exists a natural embedding E_{Ir} of V_r into V_I. If f_r belongs to V_r and f is the class of f_r, we will write

$$f = E_{Ir} f_r \tag{3}$$

and call f_r the representative of f in V_r. For every $f \in V_I$, we will denote by $J(f)$ the set of all $r \in I$ such that f has a representative in V_r. That is, $r \in J(f)$ means that $f \in E_{Ir} V_r$. It follows directly that $J(f)$ is a final subset of I. This means that if $r \in J(f)$ and

$s \geqslant r$, then $s \in J(f)$. If f_r and f_s are representatives of f and $s \geqslant r$, then

$$E_{sr} f_r = f_s \tag{4}$$

If f and g are any two vectors in V_I, then

$$J(f+g) \supseteq J(f) \cap J(g) \tag{5}$$

3. DEFINITION OF NESTED HILBERT SPACE

Let H_r $(r \in I)$ be a family of Hilbert spaces. For the moment, the spaces H_r are considered to be mutually disjoint. Assume that the index set I satisfies the following conditions:

(I_1): Any two elements in I have at least one common successor (see Section 2).

(I_2): In I, there is defined an order-reversing involution, i.e., a one-to-one mapping $r \longleftrightarrow \bar{r}$ of I onto itself such that $\bar{\bar{r}} = r$ $(r \in I)$ and that $\bar{r} \geqslant \bar{s}$ if and only if $r \leqslant s$.

(I_3): There exists an element $o \in I$ such that $\bar{o} = o$.

Elements of the Hilbert space H_r will be denoted by f_r, g_r, etc.; a bounded operator from H_r into H_s is denoted by A_{sr}. The scalar product of $f_r \in H_r$ and $g_r \in H_r$ is denoted by (f_r, g_r); it is linear in g_r and antilinear in f_r. The norm of $f_r \in H_r$ is denoted by $\|f_r\|$. The adjoint of the bounded operator A_{sr} will be denoted by $(A_{sr})^*_{rs}$. It is the bounded operator from H_s into H_r, defined by

$$(g_r, (A_{sr})^*_{rs} f_s) = (A_{sr} g_r, f_s)$$

for every $f_r \in H_r$ and every $f_s \in H_s$. On the left-hand side of this equation, the scalar product is in H_r; on the right-hand side, the scalar product is in H_s.

We will now continue the definition of nested Hilbert space. For every $r \in I$ and every $s \geqslant r$, let E_{sr} be a linear mapping from H_r into H_s. Assume that the conditions (Ns_1), (Ind_1), and (Ind_2) of Section 2 are satisfied. Assume further that E_{sr} satisfies the following conditions.

(Ns_2): E_{sr} is bounded (i.e., there exists a constant C such that $\|E_{sr} f_r\| \leqslant C \|f_r\|$ for every $f_r \in H_r$).

(Ns_3): The range of E_{sr} is dense in H_s. This means that if $(g_s, E_{sr} f_r) = 0$ for every $f_r \in H_r$, then $g_s = 0$.

If a linear mapping E_{sr} satisfies the conditions (Ns_1), (Ns_2), and (Ns_3) then its adjoint $(E_{sr})^*_{rs}$ also satisfies these conditions.

Definition. Let I be a set that satifies the conditions (I_1), (I_2), and (I_3). Let H_r $(r \in I)$ be a family of Hilbert spaces and E_{sr} $(r \in I, s \geqslant r)$ be a family of linear mappings which satisfy the conditions (Ns_1), (Ns_2), (Ns_3), (Ind_1), and (Ind_2). The algebraic inductive limit

$$H_I = \varinjlim [H_r; E_{sr}; I]$$

will be called a nested Hilbert space if the following conditions are satisfied:

(NH_1): If we let r and q be any two elements of I, then there exists a $p \leqslant r, q$ such that

$$E_{Ip}H_p = E_{Iq}H_q \cap E_{Ir}H_r \tag{6}$$

(NH_2): For every $r \in I$, there exists a unitary mapping $u_{\bar{r}r}$ from H_r onto $H_{\bar{r}}$ such that

$$u_{oo} = 1 \tag{7a}$$

$$E_{\bar{r}\bar{s}} = u_{\bar{r}r}\,(E_{sr})^*_{rs}u_{s\bar{s}} \tag{7b}$$

$$(r \in I, s \geqslant r)$$

4. SCALAR PRODUCT

Let H_I be a nested Hilbert space and f be any vector in H_I. Consider the final subset $J(f) \subseteq I$ defined at the end of Section 2. It follows from (NH_1) that $J(f)$ has the following property: If $r \in J(f)$ and $q \in J(f)$, then there exists at least one $p \leqslant r, q$ (a common predecessor of r and of q) such that $p \in J(f)$. Namely, one can take the p defined by (6).

Let $\bar{J}(f)$ be the set of all $r \in I$ such that $\bar{r}$ belongs to $J(f)$. Therefore, $\bar{J}(f)$ is an initial subset of I, which contains at least one common successor of any two of its elements. Let f and g be two vectors in H_I, such that the intersection $\bar{J}(f) \cap J(g)$ is not empty. We shall show that the number $(u_{r\bar{r}}f_{\bar{r}}, g_r)$ is independent of the choice of $r \in \bar{J}(f) \cap J(g)$. That is, if q is any other element of $\bar{J}(f) \cap J(g)$, then

$$(u_{r\bar{r}}f_{\bar{r}}, g_r) = (u_{q\bar{q}}f_{\bar{q}}, g_q) \tag{8}$$

To prove (8), notice first that there exist elements p and z in $\bar{J}(f) \cap J(g)$, such that $p \leqslant r$, $q \leqslant z$. Then, $\bar{z} \leqslant \bar{r}$, $\bar{q} \leqslant \bar{p}$ and so, by (7a), (7b), and (4),

$$(f_{\bar{r}}, u_{\bar{r}r} g_r) = (E_{\bar{r}\bar{z}} f_{\bar{z}}, u_{\bar{r}r} E_{rp} g_p) = ((E_{zr})^*_{\bar{r}z} u_{z\bar{z}} f_{\bar{z}}, E_{rp} g_p)$$
$$= (u_{z\bar{z}} f_{\bar{z}}, E_{zp} g_p) = (u_{z\bar{z}} f_{\bar{z}}, g_z)$$

Similarly,

$$(f_{\bar{q}}, u_{\bar{q}q} g_q) = (u_{\bar{z}z} f_{\bar{z}}, g_z)$$

which proves (8).

The elements p and z have to lie in $\bar{J}(f) \cap J(g)$ because otherwise the representatives $f_{\bar{z}}$ and g_p would not exist. The number (8) will be denoted by $\langle f|g\rangle$ and called the scalar product of f and of g. Therefore, the scalar product $\langle f|g\rangle$ is defined if and only if the vectors f and g satisfy the condition that $\bar{J}(f) \cap J(g)$ is not empty. It is easy to verify that $\langle g|f\rangle$ is defined if and only if $\langle f|g\rangle$ is, and that $\langle g|f\rangle$ is the complex conjugate of $\langle f|g\rangle$. It is trivial that

$$\langle g|\varphi f\rangle = \varphi\langle g|f\rangle = \langle \varphi^* g|f\rangle$$

for every complex number. It is a little more difficult to verify the following: If f, g, h are such that $\langle f|h\rangle$ and $\langle g|h\rangle$ are defined, then $\langle f+g|h\rangle$ is also defined. Namely, if $\bar{J}(f) \cap J(h)$ and $\bar{J}(g) \cap J(h)$ are not empty, then one still has to worry about the possibility that $\bar{J}(f) \cap J(g) \cap J(h)$ might be empty. Since we know only that $J(g+f) \supseteq J(g) \cap J(f)$ [see equation (5)], it is not immediately clear that $\bar{J}(f+g) \cap J(h)$ is not empty. These problems would not arise if I were totally ordered.

To see that no trouble occurs, one needs the following auxiliary fact:

If $p \leqslant r, q$ is such that

$$E_{Ip} H_p = H_{Ir} H_r \cap E_{Iq} H_q \tag{9}$$

then

$$E_{I\bar{p}} H_{\bar{p}} = E_{I\bar{r}} H_{\bar{r}} + E_{I\bar{q}} H_{\bar{q}} \tag{10}$$

The last equation means that the vector subspace $E_{I\bar{p}} H_{\bar{p}} \subseteq H_I$ consists precisely of the vectors $\nu \in H_I$ which can be written as a sum $\nu = \nu' + \nu''$, where $\nu' \in E_{I\bar{r}} H_{\bar{r}}$ and $\nu'' \in E_{I\bar{q}} H_{\bar{q}}$. This decomposition is not unique, since the intersection $E_{I\bar{r}} H_{\bar{r}} \cap E_{I\bar{q}} H_{\bar{q}}$ contains non-zero vectors. We shall not prove here that (10) follows from (9).

With the help of (10), it is easy to see that $\langle f+g|h\rangle$ is defined. Let $r \in \bar{J}(f) \cap J(h)$ and $q \in \bar{J}(g) \cap J(h)$. Let $p \leqslant r, q$ be such that $E_{Ip} H_p = E_{Ir} H_r \cap E_{Iq} H_q$. Then $p \in J(h)$. On the other hand, $\bar{q} \in J(g)$ and $\bar{r} \in J(f)$. This means that $g \in E_{I\bar{q}} H_{\bar{q}}$ and $f \in E_{I\bar{r}} H_{\bar{r}}$. Consequently, $f + g \in E_{I\bar{q}} H_{I\bar{q}} + E_{I\bar{r}} H_{\bar{r}} = E_{I\bar{p}} H_{\bar{p}}$, which shows that $\bar{p} \in J(f+g)$. This means that $p \in J(f+g) \cap J(h)$ so that the scalar product $\langle f+g|h\rangle$ is defined.

5. NESTED HILBERT SPACE ASSOCIATED TO AN ORTHONORMAL BASIS

Let H_o be a separable Hilbert space and $\{h_o^{(k)}\}$ $(k = 1, 2, \ldots)$ an orthonormal basis of H_o. Let I be the set of all sequences of strictly positive numbers. Therefore, $r \in I$ is a sequence $\{r(k)\}$ of numbers $r(k) > 0$. Consider in I the natural partial order; $r \geqslant q$ means $r(k) \geqslant q(k)$ for every k. Define in I an order-reversing involution by $\bar{r}(k) = 1/r(k)$. It is easy to verify that this set I satisfies the conditions (I_1), (I_2), and (I_3).

Let V be the vector space of all finite linear combinations of the vectors $h_o^{(k)}$. Let $f = \sum_k c_k h_o^{(k)}$ and $g = \sum_k d_k h_o^{(k)}$ belong to V. Given any $r \in I$, consider the scalar product

$$(f, g)_r = \sum_k c_k^* r^{-2}(k) d_k$$

Denote by H_r the completion of V with respect to the norm defined by this scalar product. It is a Hilbert space. We now have associated a Hilbert space H_r to every $r \in I$. If $s \geqslant r$ (in I) denote by E_{sr} the natural embedding of H_r into H_s.

It can be verified that every such E_{sr} satisfies the conditions (Ns_1), (Ns_2), and (Ns_3) and that the family $\{E_{sr}\}$ satisfies the conditions (Ind_1) and (Ind_2). Consider the algebraic inductive limit

$$H_I = \underrightarrow{\lim} \, [H_r; E_{sr}; I]$$

It can be shown that H_I is a nested Hilbert space, i.e., that the conditions (NH_1) and (NH_2) are satisfied. The family $u_{\bar{r}r}$ is defined as follows:

Denote by $h^{(k)}$ $(k = 1, 2, \ldots)$ the vectors

$$h^{(k)} = E_{Io} h_o^{(k)} \tag{11}$$

in H_I. It is easy to see that $J(h^{(k)}) = I$, i.e., that $h^{(k)}$ has a representative $h_r^{(k)}$ for every $r \in I$. Consider in H_r the vectors $e_r^{(r;k)}$ defined by

$$e_r^{(r;k)} = r(k) h_r^{(k)} \tag{12}$$