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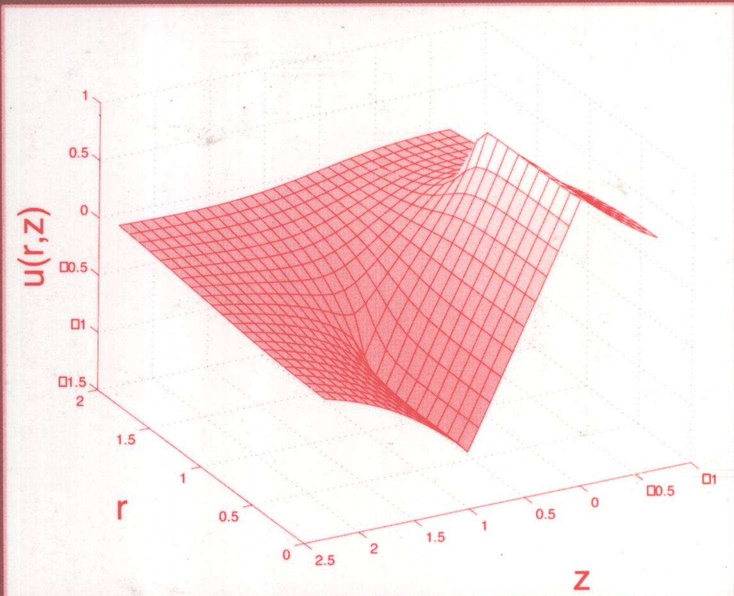
Mixed Boundary Value Problems



Dean G. Duffy

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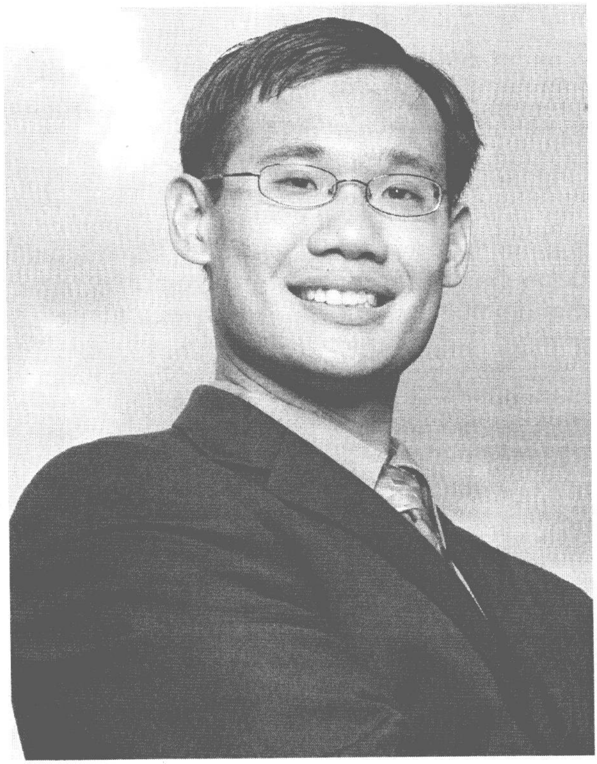

Dedicated to Dr. Stephen Teoh

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Author

Dean G. Duffy received his bachelor of science in geophysics from Case Institute of Technology (Cleveland, Ohio) and his doctorate of science in meteorology from the Massachusetts Institute of Technology (Cambridge, Massachusetts). He served in the United States Air Force from September 1975 to December 1979 as a numerical weather prediction officer. After his military service, he began a twenty-five year (1980 to 2005) association with NASA at the Goddard Space Flight Center (Greenbelt, Maryland) where he focused on numerical weather prediction, oceanic wave modeling and dynamical meteorology. He also wrote papers in the areas of Laplace transforms, antenna theory and mechanical engineering. In addition to his NASA duties he taught engineering mathematics, differential equations and calculus at the United States Naval Academy (Annapolis, Maryland) and the United States Military Academy (West Point, New York). Drawing from his teaching experience, he has written several books on transform methods, engineering mathematics and Green's functions. This present volume is his fourth book for Chapman & Hall/CRC Press.

Introduction

Purpose. This book was conceived while I was revising my engineering mathematics textbook. I noticed that in many engineering and scientific problems the nature of the boundary condition changes, say from a Dirichlet to a Neumann condition, along a particular boundary. Although these mixed boundary value problems appear in such diverse fields as elasticity and biomechanics, there are only two books (by Sneddon[1] and Fabrikant[2]) that address this problem and they are restricted to the potential equation. The purpose of this book is to give an updated treatment of this subject.

The solution of mixed boundary value problems requires considerable mathematical skill. Although the analytic solution begins using a conventional technique such as separation of variables or transform methods, the mixed boundary condition eventually leads to a system of equations, involving series or integrals, that must be solved. The solution of these equations often yields a Fredholm integral equation of the second kind. Because these integral equations usually have no closed form solution, numerical methods must be employed. Indeed, this book is just as much about solving integral equations as it involves mixed boundary value problems.

Prerequisites. The book assumes that the reader is familiar with the conventional methods of mathematical physics: generalized Fourier series, transform methods, Green's functions and conformal mapping.

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[1] Sneddon, I. N., 1966: *Mixed Boundary Value Problems in Potential Theory.* North Holland, 283 pp.

[2] Fabrikant, V. I., 1991: *Mixed Boundary Value Problems of Potential Theory and Their Applications in Engineering.* Kluwer Academic, 451 pp.

Audience. This book may be used as either a textbook or a reference book for anyone in the physical sciences, engineering, or applied mathematics.

Chapter Overview. The purpose of Chapter 1 is twofold. The first section provides examples of what constitutes a mixed boundary value problem and how their solution differs from commonly encountered boundary value problems. The second part provides the mathematical background on integral equations and special functions that the reader might not know.

Chapter 2 presents mixed boundary value problems in their historical context. Classic problems from mathematical physics are used to illustrate how mixed boundary value problems arose and some of the mathematical techniques that were developed to handle them.

Chapters 3 and 4 are the heart of the book. Most mixed boundary value problems are solved using separation of variables if the domain is of limited extent or transform methods if the domain is of infinite or semi-infinite extent. For example, transform methods lead to the problem of solving dual or triple Fourier or Bessel integral equations. We then have a separate section for each of these integral equations.

Chapters 5 through 7 are devoted to additional techniques that are sometimes used to solve mixed boundary value problems. Here each technique is presented according to the nature of the partial differential or the domain for which it is most commonly employed or some other special technique.

Numerical methods play an important role in this book. Most integral equations here require numerical solution. All of this is done using MATLAB and the appropriate code is included. MATLAB is also used to illustrate the solutions.

We have essentially ignored brute force numerical integration of mixed boundary value problems. In most instances conventional numerical methods are simply applied to these problems. Because the solution is usually discontinuous along the boundary that contains the mixed boundary condition, analytic techniques are particularly attractive.

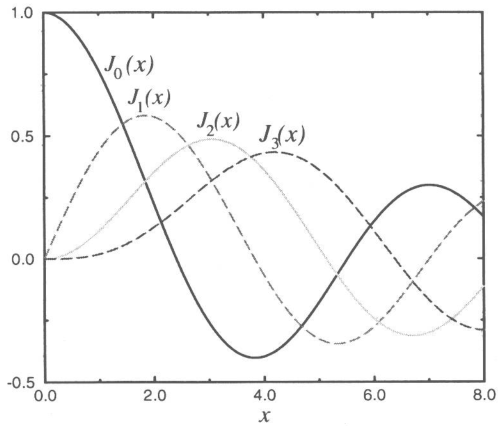
An important question in writing any book is what material to include or exclude. This is especially true here because many examples become very cumbersome because of the nature of governing equations. Consequently we include only those problems that highlight the mathematical techniques in a straightforward manner. The literature includes many more problems that involve mixed boundary value problems but are too complicated to be included here.

Features. Although this book should be viewed primarily as a source book on solving mixed boundary value problems, I have included problems for those who truly wish to master the material. As in my earlier books, I have included intermediate results so that the reader has confidence that he or she is on the right track.

List of Definitions

Function	Definition
$\delta(t-a)$	$= \begin{cases} \infty, & t = a, \\ 0, & t \neq a, \end{cases} \quad \int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$
$\Gamma(x)$	gamma function
$H(t-a)$	$= \begin{cases} 1, & t > a, \\ 0, & t < a. \end{cases}$
$H_n^{(1)}(x), H_n^{(2)}(x)$	Hankel functions of first and second kind and of order n
$I_n(x)$	modified Bessel function of the first kind and order n
$J_n(x)$	Bessel function of the first kind and order n
$K_n(x)$	modified Bessel function of the second kind and order n
$P_n(x)$	Legendre polynomial of order n
$\mathrm{sgn}(t-a)$	$= \begin{cases} -1, & t < a, \\ 1, & t > a. \end{cases}$
$Y_n(x)$	Bessel function of the second kind and order n

Contents



Chapter 1
Overview

In the solution of differential equations, an important class of problems involves satisfying boundary conditions either at end points or along a boundary. As undergraduates, we learn that there are three types of boundary conditions: 1) the solution has some particular value at the end point or along a boundary (Dirichlet condition), 2) the derivative of the solution equals a particular value at the end point or in the normal direction along a boundary (Neumann condition), or 3) a linear combination of Dirichlet and Neumann conditions, commonly called a "Robin condition." In the case of partial differential equations, the nature of the boundary condition can change along a particular boundary, say from a Dirichlet condition to a Neumann condition. The purpose of this book is to show how to solve these *mixed boundary value* problems.

1.1 EXAMPLES OF MIXED BOUNDARY VALUE PROBLEMS

Before we plunge into the details of how to solve a mixed boundary value problem, let us examine the origins of these problems and the challenges to their solution.

• Example 1.1.1: Separation of variables

Mixed boundary value problems arise during the solution of Laplace's equation within a specified region. A simple example[1] is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < \pi, \quad 0 < y < \infty, \tag{1.1.1}$$

subject to the boundary conditions

$$u_x(0, y) = u(\pi, y) = 0, \qquad 0 < y < \infty, \tag{1.1.2}$$

$$\lim_{y\to\infty} u(x, y) \to 0, \qquad 0 < x < \pi, \tag{1.1.3}$$

and

$$\begin{cases} u(x, 0) = 1, & 0 \le x < c, \\ u_y(x, 0) = 0, & c < x \le \pi. \end{cases} \tag{1.1.4}$$

The interesting aspect of this problem is the boundary condition given by Equation 1.1.4. For x between 0 and c, it satisfies a Dirichlet condition which becomes a Neumann condition as x runs between c and π.

The problem posed by Equation 1.1.1 to Equation 1.1.4 is very similar to those solved in an elementary course on partial differential equations. For that reason, let us try and apply the method of separation variables to solve it. Assuming that $u(x, y) = X(x)Y(y)$, we obtain

$$\frac{X''}{X} = -\frac{Y''}{Y} = -k^2, \tag{1.1.5}$$

with

$$X'(0) = X(\pi) = 0, \qquad \text{and} \qquad \lim_{y\to\infty} Y(y) \to 0. \tag{1.1.6}$$

Particular solutions that satisfy Equation 1.1.5 and Equation 1.1.6 are

$$u_p(x, y) = B_n \exp\left[-\left(n - \tfrac{1}{2}\right) y\right] \cos\left[\left(n - \tfrac{1}{2}\right) x\right], \tag{1.1.7}$$

with $n = 1, 2, 3, \ldots$. Because the most general solution to our problem consists of a superposition of these particular solutions, we have that

$$u(x, y) = \sum_{n=1}^{\infty} \frac{A_n}{n - \frac{1}{2}} \exp\left[-\left(n - \tfrac{1}{2}\right) y\right] \cos\left[\left(n - \tfrac{1}{2}\right) x\right]. \tag{1.1.8}$$

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[1] See, for example, Mill, P. L., S. S. Lai, and M. P. Duduković, 1985: Solution methods for problems with discontinuous boundary conditions in heat conduction and diffusion with reaction. *Indust. Eng. Chem. Fund.*, **24**, 64–77.

Substituting this general solution into the boundary condition given by Equation 1.1.4, we obtain

$$\sum_{n=1}^{\infty} \frac{A_n}{n-\frac{1}{2}} \cos\left[\left(n-\tfrac{1}{2}\right)x\right] = 1, \qquad 0 \le x < c, \tag{1.1.9}$$

and

$$\sum_{n=1}^{\infty} A_n \cos\left[\left(n-\tfrac{1}{2}\right)x\right] = 0, \qquad c < x \le \pi. \tag{1.1.10}$$

Both Equations 1.1.9 and 1.1.10 have the form of a Fourier series except that there are two of them! Clearly the challenge raised by the boundary condition along $y = 0$ is the solution of this *dual Fourier cosine series* given by Equation 1.1.9 and Equation 1.1.10. This solution of these dual Fourier series will be addressed in Chapter 3.

- **Example 1.1.2: Transform methods**

In the previous problem, we saw that we could apply the classic method of separation of variables to solve mixed boundary value problems where the nature of the boundary condition changes along a boundary of finite length. How do we solve problems when the boundary becomes infinite or semi-infinite in length? The answer is transform methods.

Let us solve Laplace's equation[2]

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 < x < \infty, \quad 0 < y < h, \tag{1.1.11}$$

subject to the boundary conditions

$$u_x(0,y) = 0 \quad \text{and} \quad \lim_{x\to\infty} u(x,y) \to 0, \qquad 0 < y < h, \tag{1.1.12}$$

$$\begin{cases} u_y(x,0) = 1/h, & 0 \le x < 1, \\ u(x,0) = 0, & 1 < x < \infty, \end{cases} \tag{1.1.13}$$

and

$$u(x,h) = 0, \qquad 0 \le x < \infty. \tag{1.1.14}$$

The interesting aspect of this problem is the boundary condition given by Equation 1.1.13. It changes from a Neumann condition to a Dirichlet condition along the boundary $x = 1$.

To solve this boundary value problem, let us introduce the Fourier cosine transform

$$u(x,y) = \frac{2}{\pi}\int_0^\infty U(k,y)\cos(kx)\,dk, \tag{1.1.15}$$

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[2] See Chen, H., and J. C. M. Li, 2000: Anodic metal matrix removal rate in electrolytic in-process dressing. I: Two-dimensional modeling. *J. Appl. Phys.*, **87**, 3151–3158.