V. K. Stokes

Theories of Fluids with Microstructure

An Introduction

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Vijay Kumar Stokes

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With 44 Figures

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पूज्यचरणेभ्यः श्रीमत्पितृभ्यः
सादरं समर्पितः

To my parents

Preface

This book provides an introduction to theories of fluids with microstructure, a subject that is still evolving, and information on which is mainly available in technical journals. Several approaches to such theories, employing different levels of mathematics, are now available. This book presents the subject in a connected manner, using a common notation and a uniform level of mathematics. The only prerequisite for understanding this material is an exposure to fluid mechanics using Cartesian tensors.

This introductory book developed from a course of semester-length lectures that were first given in the Department of Chemical Engineering at the University of Delaware and subsequently were given in the Department of Mechanical Engineering at the Indian Institute of Technology, Kanpur.

The encouragement of Professor A. B. Metzner and the warm hospitality of the Department of Chemical Engineering, University of Delaware, where the first set of notes for this book were prepared (1970-71), are acknowledged with deep appreciation. Two friends and colleagues, Dr. Raminder Singh and Dr. Thomas F. Balsa, made helpful suggestions for the improvement of this manuscript.

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Schenectady, New York, 1984 VIJAY K. STOKES

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Introduction

This book provides an introduction to the theories of fluids with microstructure. Flows of such fluids can exhibit many effects that are not possible in classical nonpolar Stokesian fluids. This material attempts to present a connected account of three different types of theories in a common notation. In keeping with the introductory nature of the presentation, only Cartesian tensors have been used. The level of presentation only assumes an exposure to fluid mechanics using Cartesian tensors. The notation is explained in the Appendix, where some of the results, which are required, have also been summarized.

In addition to the usual concepts of nonpolar fluid mechanics, there are two main physical concepts that go into building theories of fluids with microstructure: couple stresses and the concept of internal spin. Couple stresses are a consequence of assuming that the mechanical action of one part of a body on another, across a surface, is equivalent to a force and a moment distribution. In classical nonpolar mechanics, moment distributions are not considered, and the mechanical action is assumed to be equivalent to a force distribution only. The laws of motion can then be used for defining the stress tensor which, necessarily, turns out to be symmetric. Thus, in nonpolar mechanics, the state of stress at a point is defined by a symmetric second order tensor which is a point function that has six independent components. However, in polar mechanics the mechanical action is assumed to be equivalent to both a force and a moment distribution. The state of stress is then measured by a stress tensor and a couple stress tensor. In general, neither of these second order tensors is symmetric, so that the state of stress at a point is measured by eighteen independent components. Thus, the concept of couple stresses results from a study of the mechanical interactions taking place across a surface and, conceptually, is not related to the kinematics of motion.

On the other hand, the concept of microstructure is a kinematic one. For classical fluids without microstructure, all the kinematic parameters are assumed to be determined, once the velocity field is specified. Thus, if the velocity field is identically zero, then there is no motion, and the linear and angular momenta of all material elements must also be identically zero. However, even when the velocity field is zero, the angular momentum may be visualized as being nonzero by "magnifying" the continuum picture until the identities of "individual" particles can be "seen." A particle may not have any velocity of translation, so that its macroscopic velocity and thus its linear momentum is zero, but the particle may be spinning about an axis.

This spin would give rise to an angular momentum. If the same phenomenon is assumed to hold true at the continuum level, then angular momentum can exist even in the absence of linear momentum. This is not true in the classical theories of fluid mechanics. At the kinematic level, a specification of the velocity field is then not sufficient, and additional kinematic measures, independent of the velocity field, must be introduced to describe this internal spin. Such a fluid is said to have microstructure.

The concepts of couple stresses and microstructure are conceptually different. The first concept has its origins in the way mechanical interactions are modeled, while the second one is essentially a kinematic one, and arises out of an attempt to describe point particles having "structure." Whereas in a general theory of fluids with microstructure, couple stresses and internal spin may be present simultaneously, theories of fluids in which couple stresses are present, but microstructure is absent, are also possible. Similarly, microstructure may be considered in the absence of couple stresses. In this way the main consequences of each of these concepts may be studied before proceeding to the study of more general theories.

Several different approaches may be used for formulating such theories. For example, a statistical mechanics model, which assumes noncentral forces between particles, is known to give rise to couple stresses. Thus a continuum theory for fluids with microstructure may be obtained from such a model. The concept of couple stresses may also be introduced purely on the basis of a continuum argument. Microstructure can be introduced heuristically. Such theories may also be formulated by an averaging procedure in which the macroscopic variables are obtained by taking suitable averages over continuum domains. In this book an attempt has been made at unifying three different ways of developing such theories.

The simplest theory of couple stresses in fluids, in the absence of microstructure, is given in Chapter 3. The most important effect of couple stresses is to introduce a size-dependent effect that is not predicted by the classical nonpolar theories. For example, in pipe Poiseuille flow, even after all the variables have been nondimensionalized in the usual way, the velocity profile is a function of the pipe radius.

Chapter 4 describes a simple theory of fluids with microstructure in which couple stresses are absent. The inclusion of microstructure predicts phenomena such as stress relaxation and Bingham-plastic-like flow.

Finally, more general theories, in which both couple stresses and microstructure are accounted for in a systematic manner, are discussed in Chapters 5 and 6.

Several available approaches to formulating theories of fluids with microstructure have not been discussed. Neither has an attempt been made to compare all the theories that are now available. References to several excellent review articles, which compare different theories, are given below. In particular, the article by S. C. Cowin provides a detailed self-contained presentation of his theory of Polar fluids. The last two references give a fairly complete account of the polar fluid theories proposed by A. C. Eringen.

Pertinent references have been listed at the end of each chapter. Finally, a comprehensive bibliography, in chronological order, is given at the end of the book.

References

1. Ericksen, J. L. (1967). Continuum Theory of Liquid Crystals, *Appl. Mech. Rev.* **20**, 1029-1032.
2. Ariman, T., Turk, M. A., and Sylvester, N. D. (1973). Microcontinuum Fluid Mechanics — A Review, *Int. J. of Eng. Sci.* **11**, 905-930.
3. Ariman, T., Turk, M. A., and Sylvester, N. D. (1974). Review Article: Applications of Microcontinuum Fluid Mechanics, *Int. J. Eng. Sci.* **12**, 273-293.
4. Cowin, S. C. (1974). The Theory of Polar Fluids, "Advances in Applied Mechanics" (Chia-Shun Yih, Ed.), Vol. 14, Academic Press, New York, 279-347.
5. Eringen, A. C. (Editor), (1976). "Continuum Physics, Vol. IV: Polar and Non-Local Field Theories," Academic Press, New York.
6. Eringen, A. C. (1980). Theory of Anisotropic Micropolar Fluids, *Int. J. Eng. Sci.* **18**, 5-17.

Chapter 1

Kinematics of Flow

1.1 Introduction

This chapter examines some aspects of kinematic measures of motion. The discussion is limited to the classical model, in which microstructure is not considered, and where the motion is specified by the history of the particle velocity, so that there is no motion if the velocity field is identically zero. Theories in which there is motion even in the absence of a velocity distribution are considered in later chapters.

In the usual discussion on the kinematics of flow, an emphasis is placed on the rates of strain of material line elements and on the rates of shear strain between two orthogonal material line elements. The rates of rotation of line elements are also considered. However, the study of kinematics does not usually go beyond a consideration of the rate of deformation tensor, which determines the normal and shear strain rates, but does not measure all aspects of motion. Some higher order kinematic measures are introduced in this chapter.

Cartesian tensors have been used for developing the theory. In some instances a matrix representation for tensors has been used for convenience. A summary of the main results concerning Cartesian tensors, as well as a description of the notation, is given in the Appendix.

1.2 Velocity Gradient Tensor

The state of motion of a continuous body is assumed to be completely specified when the velocity distribution is known. Once the velocity $\mathbf{v}(\mathbf{x},t)$ of the particle at $\mathbf{x}$ is known at time t, a study of the deformations of line elements may be attempted. Since deformations of material elements can be very large over finite time intervals, a study of deformations over infinitesimal time intervals is appropriate. This leads naturally to a study of the rates of deformation rather than deformations.

Consider a mass of fluid that has the configuration B at time t and B' at time $t + \Delta t$. Let $\vec{PQ} = d\mathbf{x} = d\xi\mathbf{n}$ be a material line element in B which, at time t, is at the point $\mathbf{x}$, where $\mathbf{n}$ is a unit vector along PQ, as shown in Fig. 1.2.1. After time Δt, the same material line element, which will now be in configuration B', will move to $\vec{P'Q'} = d\mathbf{X} = d\Xi\mathbf{N}$, where the point P' has the position vector $\mathbf{X}$, and $\mathbf{N}$ is a unit vector along $P'Q'$. The general problem of kinematics is then to relate the line element $d\mathbf{X}$, at time $t + \Delta t$, to its original state $d\mathbf{x}$, at time t. Thus, the problem of studying short time deformations is to develop a relation of the form $d\mathbf{X} = \mathbf{f}(d\mathbf{x}, \Delta t)$. Let $\mathbf{v}$ and $\mathbf{v} + d\mathbf{v}$ be the velocities of the points P and Q, respectively, so that $\vec{PP'} = \mathbf{v}\,\Delta t$ and $\vec{QQ'} = (\mathbf{v} + d\mathbf{v})\Delta t$. Then, from the geometry shown in Fig. 1.2.1

$$d\mathbf{X} = d\mathbf{x} + d\mathbf{v}\,\Delta t \tag{1.2.1}$$

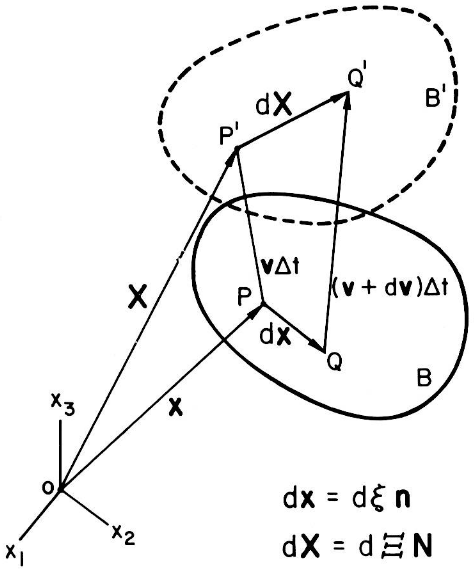

Fig. 1.2.1 Configurations of a body at two times.

Now $d\mathbf{v} = (dv_i)$ is the velocity difference $\mathbf{v}_Q - \mathbf{v}_P$ of the points P and Q at the same time t. Therefore, since at time t the velocity of the different fluid particles is a function of the coordinates x_i of $\mathbf{x}$,

$$dv_i = \frac{\partial v_i}{\partial x_r}\,dx_r = v_{i,r}\,dx_r$$

Thus

$$dv_i = g_{ri}\,dx_r \quad or \quad d\mathbf{v} = \mathbf{G}^T d\mathbf{x} \tag{1.2.2}$$

where $\mathbf{G} = (g_{ij}) = (v_{j,i})$ is called the velocity gradient tensor.

Using this result, Eq. (1.2.1) gives

$$dX_i = (\delta_{ri} + \Delta t\; g_{ri})\,dx_r \quad or \quad d\mathbf{X} = (\mathbf{I} + \Delta t\;\mathbf{G}^T)\,d\mathbf{x} \tag{1.2.3}$$

which shows that, for short intervals of time, the configurations of a material line element can be determined once the velocity gradient tensor $g_{ij} = v_{j,i}$ is known.

The next step is to develop expressions for rates of change of lengths and angles between line elements and the rates of rotation of line elements. Other aspects of deformation, such as the rate of twisting and the rate at which curvature is induced in line elements, will also be considered.

Important roles are played by the symmetric and skew-symmetric parts of **G**. The symmetric part **D**, with components $d_{ij} = \frac{1}{2}(g_{ij} + g_{ji}) = \frac{1}{2}(v_{j,i} + v_{i,j})$, is called the rate of deformation tensor and the skew-symmetric part **W**, with components $w_{ij} = \frac{1}{2}(g_{ij} - g_{ji}) = \frac{1}{2}(v_{j,i} - v_{i,j})$, is called the spin tensor.

1.3 Rate of Deformation Tensor

At time t, let $d\xi$ be the length of a material line element $d\mathbf{x}$ at the point $\mathbf{x}$, in the direct $\mathbf{n}$, so that $d\mathbf{x} = d\xi\mathbf{n}$. After time Δt the material line element becomes $d\mathbf{X}$, having a length $d\Xi$ in the direction $\mathbf{N}$ which, in general, is different from $\mathbf{n}$, so that $d\mathbf{X} = d\Xi\mathbf{N}$. Physically, the time rate of change of length of a material line element, per unit length, is of interest since it indicates the rate at which the line element is being stretched. More formally, the rate of normal strain, $\epsilon(\mathbf{x},t;\mathbf{n})$, at the point $\mathbf{x}$ at time t in the direction $\mathbf{n}$, is defined to be the time rate of change of the length of a material line element, per unit length, which is originally along the direction $\mathbf{n}$. Thus, for a material line element $d\mathbf{x} = d\xi\mathbf{n}$ at time t, which goes into $d\mathbf{X} = d\Xi\mathbf{N}$ at time $t + \Delta t$,

$$\epsilon(\mathbf{x},t;\mathbf{n}) = \frac{1}{d\xi}\frac{D}{Dt}(d\xi) = \lim_{\Delta t \to 0} \frac{1}{d\xi}\left[\frac{d\Xi - d\xi}{\Delta t}\right] \qquad (1.3.1)$$

where D/Dt denotes the material time derivative.

Now $dX_i = (\delta_{ri} + \Delta t\, g_{ri})dx_r = d\xi(\delta_{ri} + \Delta t\, g_{ri})n_r$ implies that $d\Xi^2 = dX_i dX_i = d\xi^2(\delta_{ri} + \Delta t\, g_{ri})(\delta_{si} + \Delta t\, g_{si})n_r n_s$, which can be written as

$$d\Xi = d\xi[1 + \Delta t(g_{rs} + g_{sr})n_r n_s + (\Delta t)^2 g_{ri}g_{si}n_r n_s]^{1/2}$$

or, since Δt is small,

$$d\Xi = d\xi[1 + \frac{\Delta t}{2}(g_{rs} + g_{sr})n_r n_s + 0(\Delta t^2)]$$

Use of the definition of ϵ given in Eq. (1.3.1) then results in

$$\epsilon(\mathbf{x},t;\mathbf{n}) = d_{rs}n_r n_s = \mathbf{n}^T\mathbf{D}\mathbf{n} \qquad (1.3.2)$$

which shows that normal strain rates are determined by the rate of deformation tensor.

Let $\mathbf{n}_1 = (l_i)$ and $\mathbf{n}_2 = (m_i)$ be two mutually orthogonal directions. Then, the rate of shear strain, $\gamma(\mathbf{x},t;\mathbf{n}_1,\mathbf{n}_2)$, between these two directions, is defined as the time rate of the decrease of the angle between two material line elements which are originally along $\mathbf{n}_1$ and $\mathbf{n}_2$, respectively. In order to obtain an expression for the shear strain rate, consider two material line elements which, at time t, are given by $d\mathbf{x} = d\xi_1\mathbf{n}_1$, that is by $dx_i = d\xi_1 l_i$, and $d\mathbf{y} = d\xi_2\mathbf{n}_2$, or $dy_i = d\xi_2 m_i$. After time Δt they will become, respectively, $dX_i = d\xi_1(\delta_{ri} + \Delta t\ g_{ri})l_r$ and $dY_i = d\xi_2(\delta_{si} + \Delta t\ g_{si})m_s$. By definition the angle between $d\mathbf{X}$ and $d\mathbf{Y}$ is $(\pi/2 - \gamma\Delta t)$, so that $dX_i dY_i = d\Xi_1 d\Xi_2 \cos(\pi/2 - \gamma\Delta t) = d\Xi_1 d\Xi_2 \sin(\gamma\Delta t)$, where $d\Xi_1$ and $d\Xi_2$ are, respectively, the lengths of $d\mathbf{X}$ and $d\mathbf{Y}$. Thus

$$d\Xi_1 d\Xi_2 \sin(\gamma\,\Delta t) = d\xi_1 d\xi_2 (\delta_{ri} + \Delta t\ g_{ri})(\delta_{si} + \Delta t\ g_{si})\, l_r m_s$$

$$= d\xi_1 d\xi_2 [(\Delta t)(g_{rs} + g_{sr})\, l_r m_s + (\Delta t)^2\, g_{ri} g_{si}\, l_r m_s]$$

as $l_r m_r = 0$. Since Δt is small, $\sin(\gamma\Delta t) \cong \gamma\Delta t$, so that

$$\gamma = \lim_{\Delta t \to 0} \frac{d\xi_1\ d\xi_2}{d\Xi_1\ d\Xi_2}\,[2d_{rs}\ l_r\ m_s + (\Delta t) g_{ri}\ g_{si}\ l_r\ m_s]$$

Now $d\Xi_1 = d\xi_1(1 + \Delta t\ d_{rs} l_r l_s)$, so that $d\xi_1/d\Xi_1 = [1 + \Delta t\ d_{rs} l_r l_s]^{-1}$, which, in the limit, gives $d\xi_1/d\Xi_1 = 1$. Finally

$$\gamma(\mathbf{x},t;\mathbf{n}_1,\mathbf{n}_2) = 2d_{rs} l_r m_s = 2\mathbf{n}_1^T \mathbf{D}\mathbf{n}_2 \tag{1.3.3}$$

Thus, the rate of deformation tensor gives all the information concerning normal and shear strain rates. By choosing the directions, along which these rates are being determined, along the coordinate axes, the following interpretation for $\mathbf{D}$ can be shown to be true: the diagonal components of $\mathbf{D}$ give the normal strain rate along the corresponding axis. For example, d_{11} is the rate of normal strain along the x_1 axis, and so on. On the other hand, the off-diagonal components give half the shear strain rate along the corresponding axes. For example, d_{12} is half the shear rate between the x_1 and x_2 axes, that is $d_{12} = \frac{1}{2}\gamma(\mathbf{x},t;\mathbf{i}_1,\mathbf{i}_2)$. Thus, the rate of deformation tensor is determined once the normal strain rates and shear strain rates are known, respectively, along and between any triple of mutually orthogonal directions.

1.4 Analysis of Strain Rates

The analysis of strain rates is concerned with the following three questions:

(*a*) At each point in a fluid, are there three mutually orthogonal directions such that the rates of shear strain between each pair are zero? Such a triple of mutually orthogonal directions, if they exist, are called the principal directions of strain rate.

(*b*) Along which directions does the rate of normal strain have extreme values? And

(*c*) Between which pair of orthogonal directions is the rate of shear strain a maximum?

Since **D** is a symmetric tensor, all its proper values are real and at least one set of three mutually orthogonal proper vectors exists. When the three proper values are distinct, there is only one such set of proper vectors. When two of the proper values are equal, there is one proper vector corresponding to the distinct proper value and every vector normal to it is a proper vector corresponding to the equal proper values. Finally, when the three proper values are equal, every direction is a proper vector of **D**. Every set of three mutually orthogonal proper vectors of **D** constitutes a set of principal axes of strain, for, if $\mathbf{n}_1$, $\mathbf{n}_2$ and $\mathbf{n}_3$ are a set of mutually orthogonal proper vectors, then, $\gamma(\mathbf{n}_1,\mathbf{n}_2) = 2\mathbf{n}_1^T\mathbf{D}\mathbf{n}_2 = 2\gamma_2\mathbf{n}_1^T\mathbf{n}_2 = 0$, since $\mathbf{D}\mathbf{n}_2 = \gamma_2\mathbf{n}_2$, and so on. Thus, at least one set of principal directions of strain rate always exists at each point.

The normal strain rate along the direction **n** is given by $\epsilon(\mathbf{n}) = \mathbf{n}^T\mathbf{D}\mathbf{n}$. Extreme values of $\epsilon(\mathbf{n})$ are to be found subject to the constraint $\mathbf{n}^T\mathbf{n} = 1$. These values are determined by $\mathbf{D}\mathbf{n} = \lambda\mathbf{n}$, so that the extreme values of the normal strain rate occur along the principal directions of strain rate.

The extreme values of the shear strain rate may be shown to occur between pairs of directions that bisect the principal axes of strain rate. The corresponding local extreme is given by the difference of the corresponding principal strain rates.

If the coordinate axes are chosen to be parallel to the principal axes of strain rate, the resulting coordinate system is called a principal coordinate system. In a principal coordinate system **D** has a diagonal representation, the off-diagonal elements being zero.

1.5 Spin Tensor

The spin tensor $w_{ij} = \frac{1}{2}(g_{ij} - g_{ji})$, which is the skew-symmetric part of the velocity gradient tensor, will now be shown to measure the rates of rotation of line elements in a certain average sense. With the skew-symmetric tensor w_{ij} can be associated the pseudovector $\omega_i = \frac{1}{2}e_{ijk}w_{jk}$, which gives the dual relation $w_{ij} = e_{ijk}\omega_k$. A use of $w_{ij} = \frac{1}{2}(v_{j,i} - v_{i,j})$ results in $\omega_i = \frac{1}{2}e_{ijk}v_{k,j}$ or $\boldsymbol{\omega} = \frac{1}{2}\nabla\times\mathbf{v}$. The pseudovector $\boldsymbol{\omega}$ is called the vorticity.

Let $\mathbf{n}_1 = (l_i)$, $\mathbf{n}_2 = (m_i)$ and $\mathbf{n}_3 = (n_i)$ be a set of right-handed orthonormal vectors. Consider a line element $d\mathbf{x} = d\xi_1 \mathbf{n}_1$, along $\mathbf{n}_1$, which is represented by OA in Fig. 1.5.1. After time Δt, the element $\vec{OA} = d\mathbf{x} = d\xi_1 \mathbf{n}_1$ will deform into $\vec{OB} = d\mathbf{X} = (\mathbf{I} + \Delta t\ \mathbf{G}^T) d\mathbf{x} = d\xi_1 (\mathbf{I} + \Delta t\ \mathbf{G}^T)\mathbf{n}_1$, so that the vector $\vec{AB} = \Delta t\ d\xi_1 \mathbf{G}^T \mathbf{n}_1$. If a perpendicu-

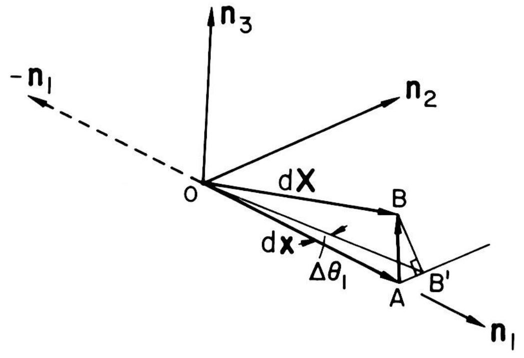

Fig. 1.5.1 Rotation of line elements about the **n** axis.

lar is dropped from B onto a line through A which is parallel to $\mathbf{n}_2$, meeting it at B', then $AB' = (\vec{AB})^T \mathbf{n}_2 = \Delta t\ d\xi_1 \mathbf{n}_1^T \mathbf{G} \mathbf{n}_2$. The rotation of $\vec{OA}$ about $\mathbf{n}_3$ is given by $\Delta\theta_1 = AB'/OA = \Delta t\ \mathbf{n}_1^T \mathbf{G} \mathbf{n}_2$. Similarly, the rotation $\Delta\theta_2$ of an element $d\mathbf{y} = d\xi \mathbf{n}_2$, along $\mathbf{n}_2$, about $\mathbf{n}_3$, is $\Delta\theta_2 = \Delta t\ \mathbf{n}_2^T \mathbf{G}(-\mathbf{n}_1) = -\Delta t\ \mathbf{n}_1^T \mathbf{G}^T \mathbf{n}_2$. Therefore, the average rotation of $d\mathbf{x}$ and $d\mathbf{y}$ about $\mathbf{n}_3$ is $(\Delta\theta_1 + \Delta\theta_2)/2 = \Delta t\ \mathbf{n}_1^T [\frac{1}{2}(\mathbf{G} - \mathbf{G}^T)] \mathbf{n}_2 = \Delta t\ \mathbf{n}_1^T \mathbf{W} \mathbf{n}_2$. Thus, the average rotation rate of two line elements along $\mathbf{n}_1$ and $\mathbf{n}_2$, about $\mathbf{n}_3$, is given by $\mathbf{n}_1^T \mathbf{W} \mathbf{n}_2 = w_{rs} l_r m_s$. This average rate would seem to depend on the choice of the elements being along $\mathbf{n}_1$ and $\mathbf{n}_2$. However, a use of $w_{rs} = e_{rst}\omega_t$, results in $w_{rs} l_r m_s = e_{rst} l_r m_s \omega_t = n_t \omega_t$, since $e_{rst} l_r m_s$ is the cross product $\mathbf{n}_3 = (n_i)$ of $\mathbf{n}_1$ and $\mathbf{n}_2$. The expression $n_t \omega_t = \mathbf{n}_3^T \boldsymbol{\omega}$ shows that this average rate of rotation is the same for every pair of orthogonal line elements in the plane of $\mathbf{n}_1$ and $\mathbf{n}_2$. Now, if $\Omega(\mathbf{n})$ is defined to be the average rate of counterclockwise rotation about $\mathbf{n}$, of any two perpendicular material line elements which lie in the plane normal to $\mathbf{n}$, then

$$\Omega(\mathbf{n}) = n_r \omega_r = \mathbf{n}^T \boldsymbol{\omega} \tag{1.5.1}$$

Thus $\boldsymbol{\omega}$, and therefore $\mathbf{W}$, measures the average rate of rotation of line elements.

Given $\boldsymbol{\omega}$, or $\mathbf{W}$, the maximum rate of rotation is $\Omega(\mathbf{n}) = |\boldsymbol{\omega}|$ and occurs for orthogonal line elements which lie in the plane perpendicular to $\boldsymbol{\omega}$. On the other hand $\Omega(\mathbf{n}) = 0$ for any $\mathbf{n}$ which is orthogonal to $\boldsymbol{\omega}$.