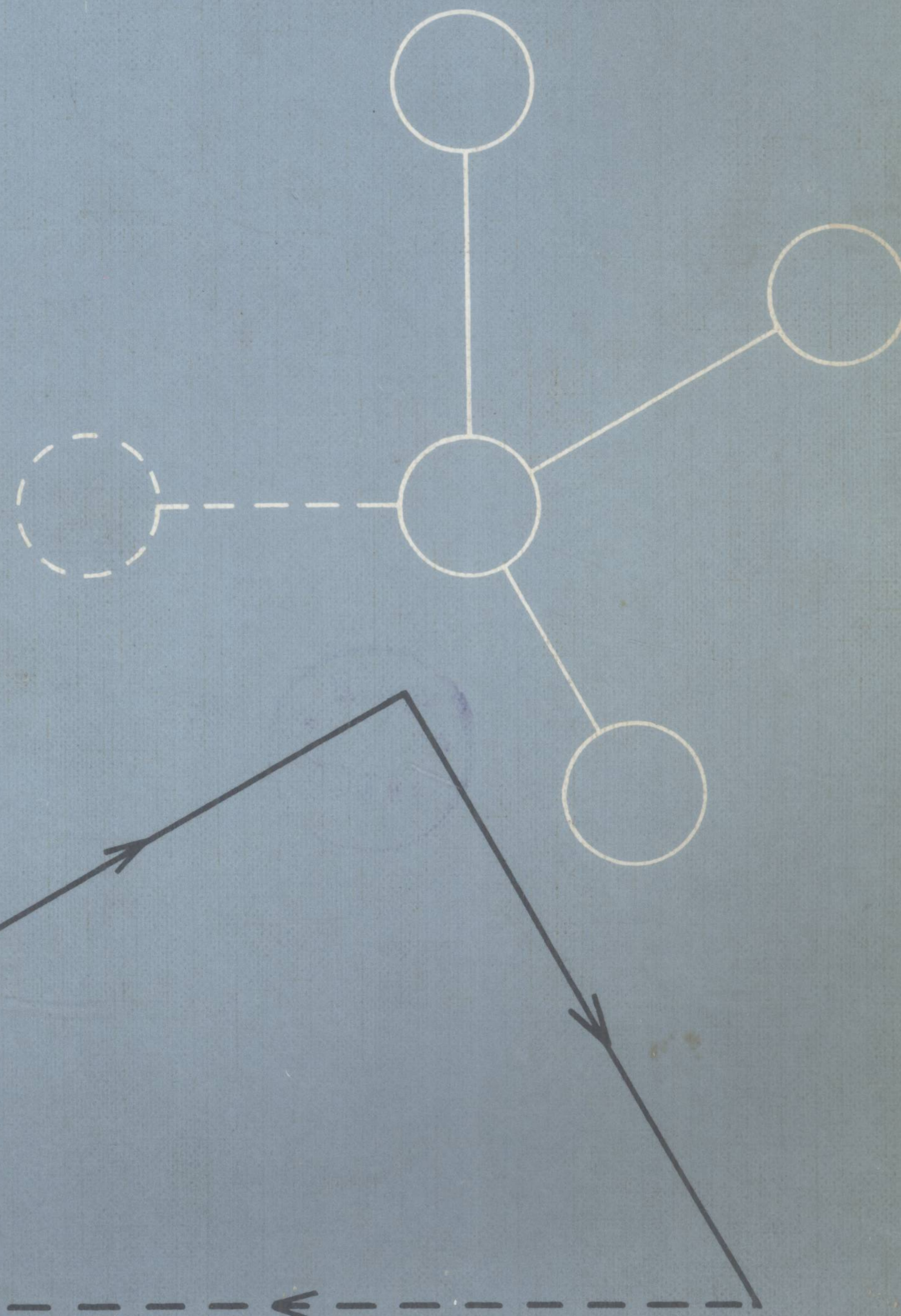
APPLIED MECHANICS

J.D. Walker



Applied Mechanics

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Applied Mechanics

To N. L. without whose generous help some time ago this book could never have been written

Preface

This book is intended to cover the requirements of students reading Applied Mechanics in the Technician Education Council (TEC) courses in Mechanical and Production Engineering, Ordinary National Certificate or Diploma courses in Engineering or Technology, or in the first year of the Higher National Diploma or Degree courses in Engineering.

To present the basic principles of applied mechanics in a manner acceptable to the student; to furnish the student with a liberal quantity of worked examples, including graphical solutions; to supply the student with a number of exercises for his own practice—these are the aims which the author has had predominantly in mind during the preparation of this work.

The introduction of SI units has helped to resolve many of the difficulties which arose from the different systems of units previously in use, particularly in the dynamics section of the subject. Perhaps there is a tinge of regret in realising that the controversies surrounding slugs and poundals will now become a thing of the past. Undoubtedly the student will benefit by the rational system now being accepted, and particularly by the fact that it will become standard practice.

Presumably all students taking a course of this type will have made a reasonable study of mathematics along lines suited to the requirements of this book. It would seem purposeless to claim that this presentation of applied mechanics has been made without recourse to mathematics. Any student who is not sufficiently at home with mathematics to be able to appreciate the development of formulae, and the solution of examples herein set down cannot claim to have reached the standard demanded by National Certificate requirements. Mathematics is one of the tools of the engineer: he is encouraged to use it in these pages.

It is likely that some students will come to a study of applied mechanics without being previously in a Mechanical Engineering Science class. For these students, matter is presented in this volume which has appeared previously in the author's book *Certificate Mechanical Engineering Science*. Topics such as vectors, relative velocity, force, and mass are among the examples of the need for this repetition. In other cases, a summary is provided of work with which the student should be familiar before proceeding with the more advanced work about to be presented.

Typical questions have been taken from examination papers set by the following organisations: the Union of Lancashire and Cheshire Institutes, the East Midlands Educational Institution, the Union of Educational Institutions, the Northern Counties Technical Examinations Council, the Yorkshire Council for Further Education, the Welsh Joint Education Committee, and the Institution of Mechanical Engineers. These have been acknowledged in the text and the author expresses his appreciation for the cooperation of these examining Boards.

The author would like to place on record his grateful acknowledgement of the valuable help given by the publishers and the general editor, and his indebtedness to many of his colleagues, particularly Mr A. R. Field for the considerable help with the revision, and to his wife for her patience and encouragement during a preparation period which only the wife of an author can appreciate.

J. D. WALKER

Units

The Système International d'Unités, abbreviated to SI, has been accepted by the International Organisation for Standardisation as the most rational system of measurement. It is being introduced all over the world and will ultimately be the only system of units in use not only in technical and business areas of work, but also in everyday life.

The system is based on six units, which are arbitrarily defined.

Quantity	Unit	Symbol
length	metre	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
temperature	kelvin	K
luminous intensity	candela	cd

Other units and combinations of units are, with limited exceptions, made up from the above.

Multiples of these units are expressed, again with limited exceptions, in powers of 1000, i.e. 10^3 10^6 10^{-3} 10^{-6}, and are given characteristic prefixes, of which the following are the more usual.

Prefix	Symbol	Multiply by
mega	M	1 000 000 $= 10^6$
kilo	k	1 000 $= 10^3$
milli	m	0·001 $= 10^{-3}$
micro	μ	0·000 001 $= 10^{-6}$

The following units are used in this book.

Length	m	metre
	km	kilometre
	mm	millimetre
	cm	centimetre
Area	m^2	square metre
	mm^2	square millimetre
Volume	m^3	cubic metre
	l	litre (1000 cm^3)
Mass	kg	kilogram
	g	gram
	t	tonne (1000 kg)
Density	kg/m^3	kilogram per cubic metre
Time	s	second
	min	minute
	h	hour
Velocity	km/h	kilometres per hour
	m/s	metres per second
Acceleration	m/s^2	metres per second per second
Temperature	K	kelvin
	°C	degree Celsius
Force	N	newton
	kN	kilonewton
Torque and moment	N m	newton metre
	kN m	kilonewton metre
Work and energy	J	joule
	kJ	kilojoule
Power	W	watt
	kW	kilowatt
Stress	N/m^2	newton per square metre
	N/mm^2	newton per square millimetre
Pressure	N/m^2	newton per square metre
		bar (10^5 N/m^2)
Frequency	Hz	hertz

Contents

Chapter 1

Velocity and Acceleration

It is usual to start a study of applied mechanics with the section dealing with motion, which is often called *dynamics*. Alternatively, a start could be made with the other main division of the subject, relating to stresses and strains, and forces and frameworks, generally referred to as *statics*, although the more specialised title 'strength of materials' is sometimes used to describe this section.

Whichever section is chosen as the starting-point, a knowledge of vectors is essential, since both velocities and forces are vector quantities. In general, there are two types of quantities: *scalar* and *vector*. A *scalar* quantity is one which can be defined purely by a number. For example, the number of nuts and bolts in a tray is a scalar quantity, since a number such as 50 would describe the quantity. Similarly, the length and diameter of the bolts are also scalar quantities, since again numbers such as 75 mm and 10 mm are sufficient to describe the length and the diameter. A *vector* quantity is one which is completely defined only when a direction is added to the number. The value of a force is completely given only when the direction 'downwards' is added to the magnitude '10 N'. The value of a velocity is completely given only when the direction, 'north-east', is added to the magnitude, '50 kilometres per hour'. Hence both forces and velocities are vector quantities.

A vector quantity can be represented by a line drawn to scale and in the stated direction, with an arrow indicating the sense of the line. This line is usually called a vector.

Addition and Subtraction of Vectors

Scalar quantities can be added together by the normal arithmetical rules of addition, and similarly one can be subtracted from another in the normal arithmetical manner. Addition and subtraction of vector quantities must, of course, take into account the direction of the vectors concerned. In fig.1.1 (i), lines A and B represent two vector quantities, such as two forces. We require to know their sum and their difference.

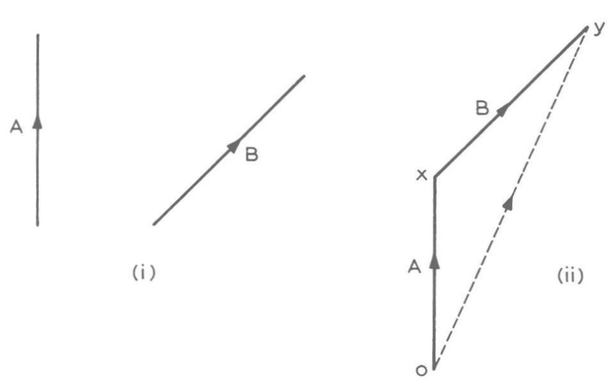

Fig.1.1 Sum of vector quantities

Sum of Two Vectors

1 Draw ***ox*** to represent in magnitude and direction the quantity A. Note that the bold italic type for the letters ***ox*** indicates that a vector is being represented, and therefore direction and sense from o to x is being taken into account.
2 From the point x at the end of ***ox***, draw the line ***xy*** to represent in magnitude and direction the quantity B.
3 Join oy. Then ***oy*** represents in magnitude and direction the sum of the vectors A and B.

Important. Note that the direction of the final vector is always from the start, o, to the finish, y.

Example 1.1 Find the sum of the two vector quantities (i) 30 units due east, (ii) 4 units due north (fig.1.2).

oa represents the 30 units due east quantity. From a we draw ***ab*** to represent 40 units due north. The vector ***ob*** (from start to finish) is the sum of the other two vectors, and by measurement is 50 units in the direction 53°8′ north of east.

Had there been more than two vector quantities to add together, the extra ones would have been added on, in turn, to the end of the previous vector.

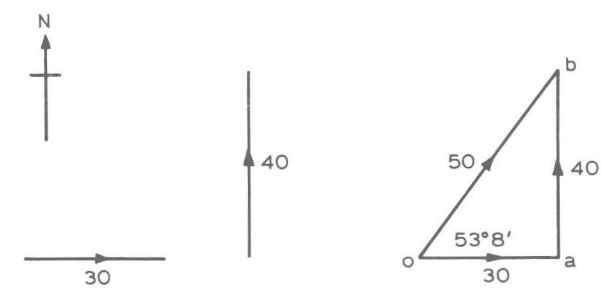

Fig.1.2

Subtraction of Two Vectors

The difference of two quantities (scalar or vector) can be written in the form of the addition of one of the quantities to the negative of the other.

Hence $$A - B = A + (-B)$$

Now if the line ***oa*** in fig.1.3 (i) represents a vector quantity A, then the line ***ao*** in fig.1.3 (ii) represents the vector $(-A)$. Making use of this fact, let us determine the difference of the two vector quantities represented by A and B in fig.1.1 (i), reproduced again in fig.1.4 (i).

Fig.1.3 Representation of negative vector

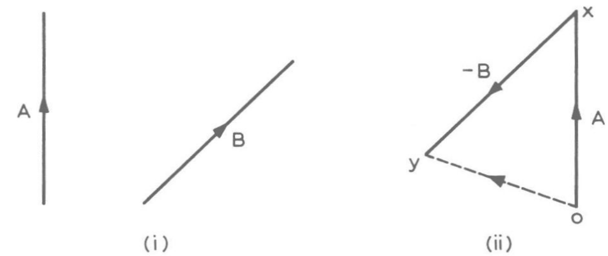

Fig.1.4 Subtraction of vector quantities

1 Draw ***ox*** to represent in magnitude and direction the quantity A.
2 From the point x at the end of ***ox***, draw the line ***xy*** to represent in magnitude and direction the quantity $-B$. This will mean that the direction of ***xy*** will be opposite to that of the given quantity B, but the magnitude represented by ***xy*** will be equal to that of B.
3 Join ***oy***. Then ***oy*** represents in magnitude and direction the sum of A and $(-B)$; that is, ***oy*** represents the difference of the two vectors.

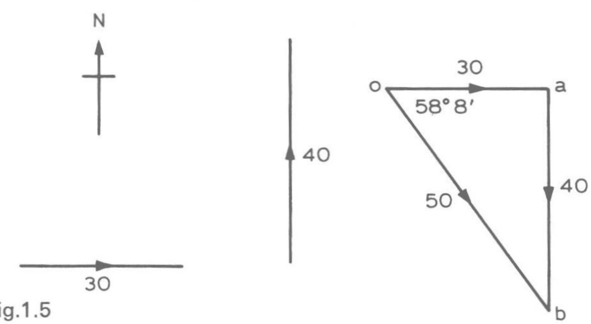

Fig.1.5

Example 1.2 Find the difference of the two vector quantities (i) 30 units due east, (ii) 40 units due north.

oa represents the 30 units due east quantity (fig.1.5). From a we draw ***ab*** in the southerly direction, opposite to the direction stated for the second vector. The length of ***ab*** represents the magnitude of 40 units. The vector ***ob*** (still from start to finish) is the difference of the other two vectors, and, by measurement, is 50 units in the direction 53°8′ south of east.

Hence (30 units east) − (40 units north) = 50 units 53° 8′ south of east.

Vector Notation

As an alternative to defining a vector quantity by means of 'compass points', such as north and north-east, it is quite usual to indicate the quantity in the following manner:

$$10_{90°}$$

This means that the magnitude of the vector is 10 units, and that its direction is 90° to the horizontal, measured in an anticlockwise direction.

Thus $10_{90°}$ and 10 units due north are the same,

$5_{0°}$ and 5 units due east are the same,

$6_{200°}$ and 6 units 20° south of west are the same.

Displacement

If a point moves from one position to another position, we say that it has been displaced. To indicate the amount of the displacement, we must state both the *magnitude*, sometimes called the distance, and the *direction* of the displacement. Thus displacement is a vector quantity. If the quantities mentioned in example 1.1 were displacements, suggesting that a point was displaced 30 m due east and then 40 m due north, its final displacement, being the sum of these two, would be 50 m 53°8′ north of east. This would indicate the resultant displacement of the point relative to the starting position. You will notice that the distance moved by the point is 30 + 40 = 70 m. This is another illustration of the difference between a scalar quantity, such as distance, and a vector quantity, such as displacement.

The unit of displacement is the *metre*.

Velocity

The *velocity* of a point is the rate of change of its displacement. If the velocity is uniform, that is if the point has equal displacements during equal intervals of time, then we can say that its velocity is equal to the displacement in unit time, or the displacement per second, since the second is the normal unit of time. Having uniform velocity, we should expect the point to have the same displacement in successive intervals of time.

If the velocity is varying, we can state its value at a given instant. We can say, for example, that the magnitude of the velocity of a point at a particular instant is 10 m/s, by which we mean that, if it maintained that velocity without change, then the point would move 10 m in the next second. In fact, the point may move 100 m or 2 m in the next second, if its varying velocity is increasing or decreasing.

Now velocity involves both magnitude and direction. The generally accepted concept of speed is, in fact, the magnitude portion of velocity. We say that the *speed* of a car is 60 km/h, but we say that the *velocity* of the car is 60 km/h due north.

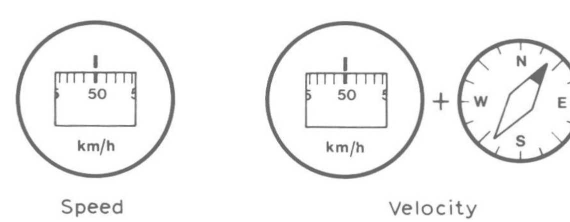

Fig.1.6 Speed and velocity. Speed deals with the magnitude or number part. Velocity has to do with both the magnitude and the direction of the motion.

Velocities are vector quantities, and their addition and subtraction, which play an important part in applied mechanics, are carried out as previously indicated.

Acceleration

Any change in the velocity of a moving point involves an *acceleration* (or a *retardation*, which is simply a negative acceleration).

Acceleration is the rate of change of velocity. If the velocity of a point is 10 m/s at one instant, and this velocity is increasing at the rate of 2 m/s each second, then at the end of 1 second the velocity will be 12 m/s, and at the end of the next second it will be 14 m/s. The rate of increase of 2 m/s each second (written 2 m/s^2) is the acceleration of the point. If the velocity were decreasing by 2 m/s each second, then we should refer to a retardation of 2 m/s^2.

In practice, accelerations change as much as velocities, but for the purpose of our work in applied mechanics we shall be concerned only with uniform acceleration, and with a rather special type of motion in which the varying acceleration follows a clearly defined mathematical pattern.

Accelerations, since they involve velocities, are also vector quantities.

Now because a change in velocity is produced if either the magnitude or the direction of the velocity is changed, it follows that two types of accelerations are involved in these changes. Figure 1.7 indicates the magnitude of the velocity of a car moving along a straight road; thus the direction of the velocity remains constant. The magnitude is increasing and the car is accelerating. The acceleration is in the direction of the velocity of the car, and is known as *linear tangential acceleration*, although at this stage it may not be very apparent as to why this name 'tangential' is used.

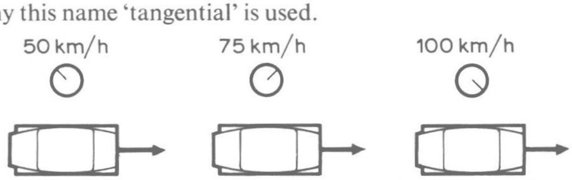

Fig.1.7 Acceleration due to change of magnitude of velocity. The direction of this car is constant. The magnitude of its velocity is changing from 50 to 100 km/h. Hence the car is accelerating.

Figure 1.8 indicates a car moving in a circular path whilst the speedometer remains constant, i.e. the speed of the car is constant. The direction of the velocity is constantly changing. It is always taken as tangential to the path of motion. The arrows are therefore vectors, indicating the velocity of the car at successive intervals. These velocities are 45 km/h NE, 45 km/h E, and 45 km/h SE. The velocities are changing, and hence acceleration is involved. This acceleration is known as *centripetal* or *radial acceleration*, and is perpendicular to the direction of motion, although we must wait until we come to study motion in a circular path before we fully appreciate this.

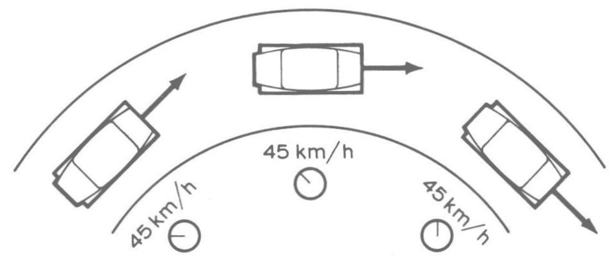

Fig.1.8 Acceleration due to change of direction of velocity. The magnitude of this car's velocity is constant at 45 km/h, its direction is, however, changing, as it moves in a curved path. There is, therefore, a change in the velocity of the car. In other words, the car is accelerating.

Relationship between Displacement, Velocity, Acceleration, and Time

Now let us establish some expressions which connect these four important quantities. The following symbols are usually adopted:

s = distance travelled in m
t = time in seconds
u = initial velocity in m
v = final velocity in m/s after t seconds
a = acceleration in m/s^2

Whilst the velocity increases uniformly from u m/s to v m/s, in time t seconds, the average velocity will be $\frac{u+v}{2}$ m/s.

Distance travelled = average velocity × time

i.e.
$$s = \frac{(u+v)t}{2} \quad \quad (1)$$

The initial velocity, u, will be changed to

$u + a$ at the end of 1 second
$u + 2a$ at the end of 2 seconds
$u + 3a$ at the end of 3 seconds
$u + at$ at the end of t seconds

Representing this by v, the final velocity, we have

$$v = u + at \quad \quad (2)$$

Taking u across to the other side, we have

$$v - u = at \quad \quad (3)$$

Now we have seen that

$$s = \frac{(u+v)t}{2}$$

which can easily be converted into

$$u + v = \frac{2s}{t}$$

or
$$v + u = \frac{2s}{t} \quad \quad (4)$$

Multiplying equations (3) and (4) together,

$$(v - u)(v + u) = \frac{2s}{t} \times at$$

The left-hand side is the difference of two squares, and on the right-hand side we can cancel out the t's. Hence we get

$$v^2 - u^2 = 2as \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Again, substitute the value of v as given in equation (2) into equation (1):

$$s = \frac{[u + (u + at)]t}{2}$$

$$s = \frac{(2u + at)t}{2}$$

$$s = ut + \tfrac{1}{2}at^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

These formulæ are extremely important. For convenience they are collected together here.

Uniform velocity ($a = 0$)

$$s = vt$$

Uniform acceleration $v = u + at$

$$s = \frac{(u + v)t}{2}$$

$$v^2 - u^2 = 2as$$

$$s = ut + \tfrac{1}{2}at^2$$

If the body is accelerating, a is positive.
If the body is retarding, a is negative.

Example 1.3 A car passes a certain point with a velocity of 10 m/s, and a point 1 km away with a velocity of 30 m/s, the acceleration being uniform. What is the average velocity of the car? How long did it take the car to travel the distance between the two points? What was the acceleration of the car?

Average velocity (since acceleration is uniform)

$$= \frac{u + v}{2}$$

$$= \frac{10 + 30}{2} \text{ m/s}$$

$$= 20 \text{ m/s}$$

$$\text{Time taken} = \frac{\text{distance (m)}}{\text{average velocity (m/s)}}$$

$$= \frac{1000}{20} \text{ s}$$

$$= 50 \text{ s}$$

$$\text{Increase in velocity} = (30 - 10) \text{ m/s} = 20 \text{ m/s}$$

This occurs uniformly in 50 seconds

$$\therefore \text{Acceleration} = \frac{\text{change in velocity}}{\text{time}}$$

$$= \frac{20}{50} \text{ m/s}^2$$

$$= 0{\cdot}4 \text{ m/s}^2$$

i.e. Average velocity is 20 m/s, the car takes 50 seconds to travel between the two points with an acceleration of 0·4 m/s².

Example 1.4 With what initial velocity must a body be travelling in order to come to rest in 100m as a result of a uniform retardation of 8 m/s²?

Acceleration $a = -8 \text{ m/s}^2$
Distance travelled $s = 100 \text{ m}$
Final velocity $v = 0 \text{ m/s}$
Initial velocity $u = ?$
Now $v^2 - u^2 = 2as$

$$0^2 - u^2 = 2 \times (-8) \times 100$$

$$u^2 = 1600$$

i.e. Initial velocity, u, *is 40 m/s.*

Example 1.5 Two signals are 1 km apart. A train passes the first with a speed of 108 km/h and passes the second signal 40 seconds later. During this period, the brakes are applied to give a uniform retardation. Determine the velocity of the train passing the second signal, and also the magnitude of the retardation.

$$\text{Initial velocity } u = \frac{108 \times 1000}{60 \times 60} = 30 \text{ m/s}$$

Distance travelled $s = 1000$ m
Time taken $t = 40$ seconds
Final velocity $v = ?$

$$s = \left(\frac{u + v}{2}\right)t$$

$$1000 = \left(\frac{30 + v}{2}\right)40$$

$$v = 20 \text{ m/s}$$

Retardation $s = ut + \tfrac{1}{2}at^2$

$$1000 = 30 \times 40 + \tfrac{1}{2} \times 40^2 a$$

$$a = -0{\cdot}25 \text{ m/s}^2$$

i.e. Retardation of train is 0·25 m/s², and the train passes the second signal at 20 m/s.

Velocity-time Diagrams

The velocity-time diagram is a graph in which the velocity of a point is plotted against a time base. Figure 1.9 illustrates three such diagrams. On the left-hand side is a velocity-time diagram for a point moving with uniform velocity; in the centre

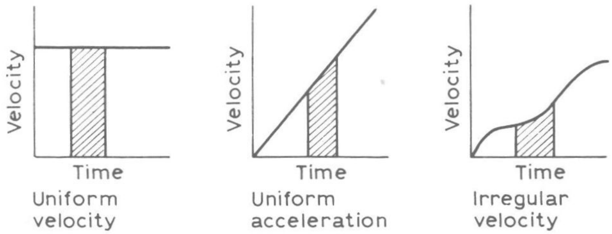

Fig.1.9 Velocity-time graph. The area under such a graph represents the distance travelled.

we have a diagram for a point moving with uniformly increasing velocity, i.e. with uniform acceleration; whilst the diagram on the right is for a point whose velocity is varying in an irregular manner.

In each case, the distance travelled by the point is numerically equal to the area under the corresponding portion of the graph. This important relationship applies to all forms of motion, however complicated they may be. Many problems dealing with varying velocities can be solved very neatly by using the velocity-time diagram. In the case of irregular variation of velocity, the area under the diagram can be found by one of the methods indicated in Appendix 2.

Example 1.6 A train travelling between two stations 3 km apart completes the journey in 5 minutes. During the first 30 seconds the train is moving with a constant acceleration; whilst a uniform retardation brings the train to rest in the last 20 seconds. For the remaining portion of the journey the train is moving with uniform speed. Calculate the value of (a) the uniform speed, (*b*) the acceleration, and (c) the retardation, both in units of m/s².

Illustrate your answer with a speed-time graph, and use this graph to obtain the distance travelled in the first and last minutes of the train's motion. NCTEC

Let a = acceleration, m/s²
f = retardation, m/s²
v = uniform velocity, m/s
then during first 30 seconds $v = at$ (since $u = 0$)

$$v = 30a \quad \ldots \quad (1)$$

During last 20 seconds $u = v - ft$

$$0 = v - 20f$$
$$v = 20f \quad \ldots \quad (2)$$

Equating (1) and (2), $30a = 20f$

$$\therefore f = \frac{3}{2}a \quad \ldots \quad (3)$$

Distance travelled whilst accelerating

$$s = ut + \tfrac{1}{2}at^2$$
$$= 0 + \tfrac{1}{2}a \times 30^2$$
$$s = 450a \text{ m}$$
$$= 15v \text{ m from (1)}$$

Distance travelled at uniform speed

$$s = vt$$
$$= [(5 \times 60) - (30 + 20)]v$$
$$= 250v \text{ m}$$

Distance travelled during retardation

$$s = vt - \tfrac{1}{2}ft^2 \text{ (negative due to retardation)}$$
$$= 20v - \tfrac{1}{2}f \times 20^2$$
$$= 20v - 200f$$
$$= 20v - 10v \text{ from (2)}$$
$$= 10v \text{ m}$$

Total distance travelled $= 15v + 250v + 10v$
$= 275v$
$= 3000$ m

$$\therefore v = \frac{3000}{275}$$
$$= 10{\cdot}9 \text{ m/s}$$

a) Uniform velocity = 10·9 m/s

b) Acceleration $a = \dfrac{v}{30}$ from (1)

$$= \frac{10{\cdot}9}{30}$$
$$= 0{\cdot}363 \text{ m/s}^2$$

c) Retardation $f = \dfrac{v}{20}$ from (2)

$$= \frac{10{\cdot}9}{20}$$
$$= 0{\cdot}545 \text{ m/s}^2. \text{ (Check with equation (3).)}$$

Figure 1.10 shows the velocity-time diagram with the shaded portions representing distances travelled during the first and last minutes.

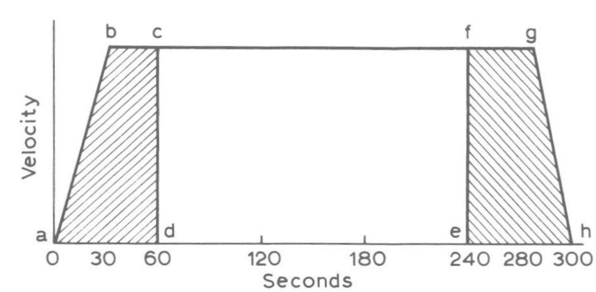

Fig.1.10

Distance travelled during the first minute

$$= \text{area of trapezium } abcd$$
$$= \tfrac{1}{2}(bc + ad) \times cd$$
$$= \tfrac{1}{2}(30 + 60) \times 10{\cdot}9$$
$$= 490 \text{m}$$

Distance travelled during last minute

$$= \text{area of trapezium } efgh$$
$$= \tfrac{1}{2}(fg + eh) \times fe$$
$$= \tfrac{1}{2}(40 + 60) \times 10{\cdot}9$$
$$= 545\,\text{m}$$

i.e. The uniform velocity is 10·9 m/s, the acceleration is 0·363 m/s², and the retardation is 0·545 m/s². During the first minute the train travels 490 m, and during the last minute 545 m.

Example 1.7 The maximum acceleration of a body is 4 m/s², and the maximum retardation is 8 m/s². What is the shortest time in which the body can move through a distance of 5 km from rest to rest?

The shortest time occurs when the body accelerates to its maximum speed and immediately retards. The velocity-time diagram is then a triangle, similar to fig.1.11.

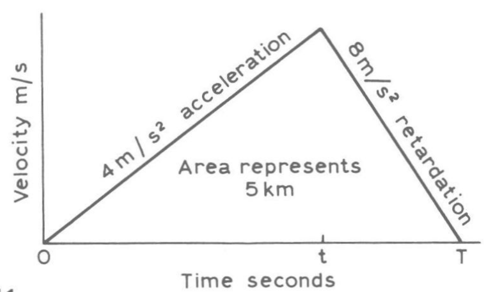

Fig.1.11

Let
T = total time in seconds
t = time to accelerate
$T - t$ = time to retard
v = maximum velocity, m/s

Initial velocity is zero and acceleration is uniform, hence maximum velocity is given by

$$v = at$$
$$= 4t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, for the retardation

$$v = 8(T - t)$$

Equating, $8(T - t) = 4t$

$$t = \frac{2}{3}T$$

Hence from (1) $v = 4 \times \frac{2}{3}T$

$$v = \frac{8}{3}T \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Area of triangle represents distance travelled:

$$\tfrac{1}{2}T.v = 5000$$
$$Tv = 10\,000$$

Substituting from (2),

$$T.\frac{8}{3}T = 10\,000$$
$$T^2 = 3750$$
$$T = 61 \text{ seconds}$$

i.e. The shortest time is 61 seconds.

Motion of Falling Bodies

When a body is allowed to fall freely to the ground, it moves with a uniform acceleration produced by the gravitational attraction of the earth. This acceleration varies from place to place on the earth's surface. Its value is approximately 9·81 m/s² at London, and this is the value which we shall normally use in our calculations. The small letter g is the symbol associated with the acceleration due to gravity.

The important fact in connection with falling bodies is that, since they are all subject to the same gravitational acceleration at the same place on the earth's surface, it follows that they will all travel through the same distance in the same interval of time. Whilst we may feel that a heavy body would fall quicker than a light body, and that it would travel faster, this is not, in fact, the case. Neglecting any wind resistance, the two bodies would move with the same velocity, and take the same time to fall through the same vertical height.

The equations for motion due to gravity follow the same pattern as those for linear motion, with the slight modification of introducing g as the acceleration, instead of a. We shall also have to be rather careful in connection with the sign of the various quantities involved. For example, a body thrown upwards will reach a certain height and then begin to fall. The direction, as well as the magnitude, of the velocity will change. To allow for this we consider all upward directions to be positive, both for displacements and velocities. Downward directions will be negative. This will include the gravitational acceleration, which is always down, and will therefore always be negative. Hence the motion equations applied to falling bodies become:

$$v = u - gt$$
$$v^2 - u^2 = -2gs$$
$$s = ut - \tfrac{1}{2}gt^2$$

Upward displacements and velocities are *positive*.
Downward displacements and velocities are *negative*.

Note that in all cases the value of s is the displacement of the body from the starting-point, and not necessarily the distance travelled from the starting-point. For example, the body may have travelled 50 m upwards and at the instant under consideration may have fallen 40 m from the highest point reached. The value of s will be + 10 m, indicating that the body is 10 m above the starting-point, although the body has actually travelled 50 + 40 = 90 m.

Example 1.8 A stone is allowed to fall from the edge of a cliff 122·5 m high. What will be its velocity after 3 seconds? How far will it have travelled at the end of 3 seconds? How long will it take the stone to reach the base of the cliff? What will be its striking velocity on reaching the base of the cliff?

Since the stone is just allowed to fall, its initial velocity is zero, i.e. $u = 0$.

Velocity after 3 seconds:

$$\begin{aligned} v &= u - gt \\ &= 0 - 9{\cdot}81 \times 3 \\ &= -29{\cdot}43\,\text{m/s} \end{aligned}$$

i.e. Its velocity is 29·43 m/s downwards.

Distance after 3 seconds:

$$\begin{aligned} s &= ut - \tfrac{1}{2}gt^2 \\ &= (0 \times 3) - (\tfrac{1}{2} \times 9{\cdot}81 \times 3^2) \\ &= -44{\cdot}15\,\text{m} \end{aligned}$$

i.e. The stone has travelled 44·15 m downwards from the point at which it was released.

Time taken to reach base:

In this case $u = 0\,\text{m/s}$

$s = -122{\cdot}5\,\text{m}$, i.e. 122·5 m down from the point of projection

$$\begin{aligned} s &= ut - \tfrac{1}{2}gt^2 \\ -122{\cdot}5 &= 0 - \tfrac{1}{2} \times 9{\cdot}81 \times t^2 \\ t^2 &= 25 \\ t &= 5\ \text{seconds} \end{aligned}$$

Striking velocity:

$$\begin{aligned} u &= 0\,\text{m/s} \\ s &= -122{\cdot}5\,\text{m} \\ v^2 - u^2 &= -2gs \\ v^2 - 0 &= -2 \times 9{\cdot}81 \times (-122{\cdot}5) \\ v^2 &= 2400 \\ v &= 49\,\text{m/s} \end{aligned}$$

i.e. After 3 seconds the velocity of the stone will be 29·43m/s. It *will have travelled 44·15 m in the first 3 seconds. It will reach the base of the cliff in 5 seconds, when its striking velocity will be 49 m/s.*

Example 1.9 A body is projected upwards with a velocity of 50 m/s from the top of a tower 100 m high. How long will it take to reach the ground? What will be the velocity with which the body strikes the ground?

$u = +50\,\text{m/s}$ (upwards)

$s = -100\,\text{m}$ (ground is 100 m below top of tower)

$$\begin{aligned} t &= ? \\ s &= ut - \tfrac{1}{2}gt^2 \\ -100 &= 50t - \tfrac{1}{2} \times 9{\cdot}81 \times t^2 \\ t^2 - 10{\cdot}2t - 20{\cdot}4 &= 0 \end{aligned}$$

which gives $t = 11{\cdot}9$ seconds

Striking velocity:

$$\begin{aligned} u &= +50\,\text{m/s} \\ s &= -100\,\text{m} \\ v &= ? \\ v^2 - u^2 &= -2gs \\ v^2 - 50^2 &= -2 \times 9{\cdot}81 \times (-100) \\ v^2 &= 2500 + 1962 \\ &= 4462 \\ v &= 66{\cdot}9\,\text{m/s} \end{aligned}$$

i.e. Body strikes the ground after 11·9 seconds, with a velocity of 66·9 m/s.

Example 1.10 State Newton's Laws of Motion. An object is dropped from an helicopter and strikes the ground 12 seconds later. Determine the height of the helicopter and the velocity with which the body strikes the ground.

If a second object had been projected upwards from the ground with a velocity of 200 m/s at the same instant as the first object was dropped from the helicopter, determine where and when they would meet. UEI

Reference is made to Newton's Laws of Motion in Appendix 3.

Height of helicopter:

$u = 0\,\text{m/s}$ (released from rest)

$t = 12$ seconds

$$\begin{aligned} s &= ? \\ s &= ut - \tfrac{1}{2}gt^2 \\ &= 0 \times 12 - \tfrac{1}{2} \times 9{\cdot}81 \times 12^2 \end{aligned}$$

$= -706\,\text{m}$ (negative sign indicating downward displacement from starting-point)

Striking velocity:

$$\begin{aligned} v &= u - gt \\ &= 0 - 9{\cdot}81 \times 12 \end{aligned}$$

$= -118\,\text{m/s}$ (velocity is downwards)

Considerable care is required with the last part of this question, particularly with regard to the signs.

s_1 = distance travelled by first object

s_2 = distance travelled by second object

Equation of motion of first object:

$$s_1 = ut - \tfrac{1}{2}gt^2$$

$s_1 = \tfrac{1}{2}gt^2$, since $u = 0$

Equation of second object:

$s_2 = 200t - \tfrac{1}{2}gt^2$ ($u = +200\,\text{m/s}$)

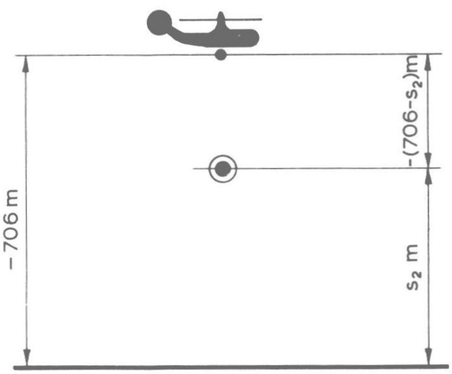

Fig.1.12

Now, referring to fig.1.12, we see that the ground is −706 m from the helicopter, so that the second object, having travelled s_2 from the ground, is $-(706 - s_2)$ from the helicopter. When this distance is equal to s_1, the two objects will meet.

$$-(706 - s_2) = s_1$$
$$-706 + s_2 = s_1$$
$$-706 + 200t - \tfrac{1}{2}gt^2 = -\tfrac{1}{2}gt^2$$
$$200t = 706$$
$$t = 3{\cdot}53 \text{ seconds}$$

Distance from ground to point of meeting:

$$s_2 = 200t - \tfrac{1}{2}gt^2$$
$$= 200 \times 3{\cdot}53 - \tfrac{1}{2} \times 9{\cdot}81 \times 3{\cdot}53^2$$
$$= 645 \text{ m from the ground}$$

i.e. The helicopter is 706 m high, and the object strikes the ground with a velocity of 118 m/s. A second object projected upwards from the ground with a velocity of 200 m/s would meet the first at a height of 645 m, 3·53 seconds after the instant of projection.

Angular Displacement, θ (theta)

So far, we have been thinking about motion in a straight line, such as the motion of a piston in a cylinder. The other important form of motion is *angular motion*, such as the motion of the crank around the crankshaft.

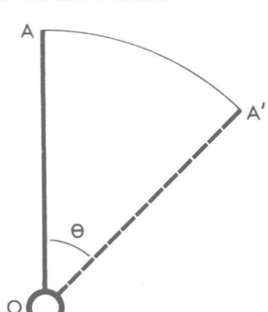

Fig.1.13 Angular displacement

The rod OA in fig.1.13 is moving in a clockwise direction about the fixed axis passing through O. The displacement of the rod can be expressed as an angle. Thus, when the rod has moved from the position OA to the position OA′, the displacement is θ radians.

The unit of angular displacement is the radian.

Angular Velocity, ω (omega)

Angular velocity is the rate of change of angular displacement. It is properly expressed in terms of radians per second, although for practical purposes the number of revolutions per second, or revolutions per minute, is often quoted.

Since 1 revolution is equal to 2π radians, it follows that

$$N \text{ rev/min} = \frac{2\pi N}{60} \text{ rad/s}$$
$$= \frac{\pi N}{30} \text{ rad/s}$$

$$\text{Conversely, } \omega \text{ rad/s} = \frac{30\omega}{\pi} \text{ rev/min}$$

As in the case of linear motion, the angular velocity either may be uniform or may vary.

Angular Acceleration, α (alpha)

Angular acceleration is the rate of change of angular velocity. For our purposes, we shall be concerned only with motion involving a constant angular acceleration or retardation. Angular retardation is a negative angular acceleration.

Relationship between Angular Displacement, Velocity, Acceleration, and Time

ω_1 = initial velocity in rad/s
ω_2 = final velocity in rad/s after t seconds
α = angular acceleration in rad/s²
θ = angle turned through in radians
t = time in seconds

The initial angular velocity, ω_1, will be increased by α rad/s at the end of each second. At the end of t seconds, its velocity will be $\omega_1 + \alpha t$.
This gives

$$\omega_2 = \omega_1 + \alpha t \quad . \; . \; . \; . \; . \; . \; (1)$$

or

$$\omega_2 - \omega_1 = \alpha t \quad . \; . \; . \; . \; . \; . \; . \; (2)$$

Since the velocity is increasing uniformly,

$$\text{average velocity} = \frac{\omega_1 + \omega_2}{2}$$

and since displacement = average velocity × time

$$\theta = \frac{\omega_1 + \omega_2}{2}t \quad . \; . \; . \; . \; . \; . \; (3)$$