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Singular Integrals and Related Topics

奇异积分和相关论题

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Preface

It is well-known that singular integrals is continuously regarded as a central role in harmonic analysis. There are many nice books related to singular integrals. In this book, there are at least two sides which differ from the other books. One of them is to establish more perfect theory of singular integrals. It includes not only the case of smooth kernels, but also the case of rough kernels. In the same way, we deal with some related operators, such as fractional integral operators and Littlewood-Paley operators. The other is to introduce more new theory on some oscillatory singular integrals with polynomial phases. This book is mainly provided to graduate students in analysis field. However, it is also beneficial to researchers in mathematics.

This book consists of five chapters. Let us now illustrate the choice of material in each chapter. Chapter 1 is devoted to the theory of the Hardy-Littlewood maximal operator as the basis of singular integrals and other related operators. It also includes the basic theory of the A_p weights. Chapter 2 is related to the theory of singular integrals. Since the theory of singular integrals with Calderón-Zygmund kernel has been introduced in many books, we will pay more attention to the singular integrals with homogeneous kernels. Specially, we will introduce more perfect theory of singular integrals with rough kernels, for instance the L^p boundedness of singular integrals with kernels in certain Hardy space on the unit sphere will be fully proved. In addition, the weighted L^p boundedness of singular integrals with rough kernels and their commutators will be also established. Chapter 3 is devoted to fractional integrals. In the same way, we will pay more attention to the case of rough kernels. It includes not only the $A(p,q)$ weight theory of fractional integrals with rough kernels, but also the theory of its commutators. Chapter 4 is to introduce a class of oscillatory singular integrals with polynomial phases. Note that this oscillatory singular integral is neither a Calderón-Zygmund operator nor a convolution operator. However there exists certain link between this oscillatory singular integral and the corresponding singular integral. Therefore, the criterion on the L^p boundedness of oscillatory singular integrals will become a crucial role in this chapter. It will discover an equivalent relation between the L^p boundedness of the oscillatory singular integral and that of the corresponding truncated singular integral. Chapter 5 is related to the Littlewood-Paley theory. In this chapter, we will establish two kinds of the weakest conditions on the kernel for the L^p boundedness of Marcinkiewicz integral operator with rough kernel. Finally, it is worth pointing out that as space is limited, the theory of singular integrals and related operators in this book is only worked on the Lebesgue spaces although there are many good results on other spaces such as Hardy spaces and BMO space.

It should be pointed out that many results in the later three chapters of this book reflect the research accomplishment by the authors of this book and their cooperators. We would like to acknowledge to Jiecheng Chen, Dashan Fan, Yongsheng Han, Yingsheng Jiang, Chin-Cheng Lin, Guozhen Lu, Yibiao Pan, Fernando Soria and Kozo Yabuta for their effective cooperates in the study of singular integrals. On this occasion, the authors deeply cherish the memory of Minde Cheng and Yongsheng Sun for their constant encourage. The first named author of this book, Shanzhen Lu, would like to express his thanks to his former students Wengu Chen, Yong Ding, Zunwei Fu, Yiqing Gui, Guoen Hu, Junfeng Li, Guoquan Li, Xiaochun Li, Yan Lin, Heping Liu, Mingju Liu, Zhixin Liu, Zongguang Liu, Bolin Ma, Huixia Mo, Lin Tang, Shuangping Tao, Huoxiong Wu, Qiang Wu, Xia Xia, Jingshi Xu, Qingying Xue, Dunyan Yan, Dachun Yang, Pu Zhang, and Yan Zhang for their cooperations and contributions to the study of harmonic analysis during the joint working period. Finally, Shanzhen Lu would like to express his deep gratitude to Guido Weiss for his constant encourage and help.

Shanzhen Lu
Yong Ding
Dunyan Yan

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Contents

Chapter 1

Hardy-Littlewood Maximal Operator

1.1 Hardy-Littlewood maximal operator

Let us begin with giving the definition of the Hardy-Littlewood maximal function, which plays a very important role in harmonic analysis.

Definition 1.1.1 (Hardy-Littlewood maximal function) *Suppose that f is a locally integrable on $\mathbf{R}^n$, i.e., $f \in L^1_{loc}(\mathbf{R}^n)$. Then for any $x \in \mathbf{R}^n$, the Hardy-Littlewood maximal function $Mf(x)$ of f is defined by*

$$Mf(x) = \sup_{r>0} \frac{1}{r^n} \int_{|y|\le r} |f(x-y)|dy. \tag{1.1.1}$$

Moreover, M is also called as the Hardy-Littlewood maximal operator.

Sometimes we need to use the following maximal functions. For $f \in L^1_{loc}(\mathbf{R}^n)$ and $x \in \mathbf{R}^n$,

$$M'f(x) = \sup_{r>0} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|dy, \tag{1.1.2}$$

where and below, $Q(x,r)$ denotes the cube with the center at x and with side r and its sides parallel to the coordinate axes. Moreover, $|E|$ denotes the Lebesgue measure of the set E. More general,

$$M''f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{1.1.3}$$

where the supremum is taken over all cubes or balls Q containing x.

For the Hardy-Littlewood maximal operator M, we would like to give the following some remarks.

Remark 1.1.1 *By (1.1.1)-(1.1.3), it is easy to see that there exist constants C_i $(i = 0, 1, 2, 3)$ depending only on the dimension n such that*

$$C_0 Mf(x) \le C_1 M'f(x) \le C_2 M''f(x) \le C_3 Mf(x) \tag{1.1.4}$$

for any $x \in \mathbb{R}^n$. That is, the Hardy-Littlewood maximal function Mf of f and the maximal functions $M'f$, $M''f$ are pointwise equivalent each other.

Remark 1.1.2 *For $f \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function $Mf(x)$ is a lower semi-continuous function on $\mathbb{R}^n$, and is then a measurable function on $\mathbb{R}^n$.*

By (1.1.4), we only need to show it for $M'f(x)$. In fact, it is sufficient to show that for any $\lambda \in \mathbb{R}$, the set $E = \{x \in \mathbb{R}^n : M'f(x) > \lambda\}$ is an open set. However, by the definition of $M'f(x)$ it suffices to show that E is open for all $\lambda > 0$. Equivalently, we only need to show that $E^c := \{x \in \mathbb{R}^n : M'f(x) \le \lambda\}$ is a closed set for all $\lambda > 0$.

Suppose that $\{x_k\} \subset E^c$ satisfying $x_k \to x$ as $k \to \infty$. We only need to show that for any $r > 0$

$$\frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)| dy \le \lambda. \tag{1.1.5}$$

Denote $Q_k = Q(x_k, r)$ and $f_k(y) = f(y)\chi_{Q(x,r)\triangle Q_k}(y)$ for all $k = 1, 2, \cdots$, where

$$Q(x,r)\triangle Q_k = \big(Q(x,r)\backslash Q_k\big) \bigcup \big(Q_k \backslash Q(x,r)\big).$$

Thus,

$$|f_k(y)| \le |f(y)| \quad \textit{for all } k \textit{ and } \lim_{k\to\infty} f_k(y) = 0.$$

Applying the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty} \frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f_k(y)| dy = 0. \tag{1.1.6}$$

On the other hand, it is clear that

$$\frac{1}{|Q(x,r)|}\int_{Q_k}|f(y)|dy = \frac{1}{|Q_k|}\int_{Q_k}|f(y)|dy \leq \lambda.$$

Hence

$$\begin{aligned}\frac{1}{|Q(x,r)|}\int_{Q(x,r)}|f(y)|dy &\leq \frac{1}{|Q(x,r)|}\int_{Q(x,r)\triangle Q_k}|f(y)|dy \\ &\quad+\frac{1}{|Q(x,r)|}\int_{Q_k}|f(y)|dy \\ &\leq \frac{1}{|Q(x,r)|}\int_{Q(x,r)}|f_k(y)|dy+\lambda.\end{aligned}$$

Let $k \to \infty$, by (1.1.6) we obtain (1.1.5).

Remark 1.1.3 *The Hardy-Littlewood maximal operator M is not a bounded operator from $L^1(\mathbb{R}^n)$ to itself.*

We only consider the case $n = 1$. Take $f(x) = \chi_{[0,1]}(x)$, then for any $x \geq 1$, we have

$$Mf(x) \geq \frac{1}{2x}\int_0^{2x}|f(y)|dy = \frac{1}{2x}.$$

Hence

$$\int_{\mathbb{R}} Mf(x)dx \geq \int_1^{\infty} Mf(x)dx \geq \int_1^{\infty}\frac{1}{2x}dx = \infty.$$

Although M is not a bounded operator on $L^1(\mathbb{R}^n)$, however, as its a replacement result we shall see that M is a bounded operator from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$, i.e., the weak $L^1(\mathbb{R}^n)$ space (see Definition 1.1.2 below).

Lemma 1.1.1 (Vitali type covering lemma) *Let E be a measurable subset of $\mathbb{R}^n$ and let $\mathcal{B}$ be a collection of balls B with bounded diameter $d(B)$ covering E in Vitali's sense, i.e. for any $x \in E$ there exist a ball $B_x \in \mathcal{B}$ such that $x \in B_x$. Then there exist a $\beta > 0$ depending only on n, and disjoint countable balls $B_1, B_2, \cdots, B_k, \cdots$ in $\mathcal{B}$ such that*

$$\sum_k |B_k| \geq \beta|E|.$$

In fact, it will be seen from the proof below that it suffices to take $\beta = 5^{-n}$.

Proof. Denote $\ell_0 = \sup\{d(B) : B \in \mathcal{B}\} < \infty$. Take $B_1 \in \mathcal{B}$ so that $d(B_1) \geq \frac{1}{2}\ell_0$. Again denote $\mathcal{B}_1 = \{B : B \in \mathcal{B}$ and $B \cap B_1 = \emptyset\}$ and $\ell_1 = \sup\{d(B) : B \in \mathcal{B}_1\}$, then we choose $B_2 \in \mathcal{B}_1$ such that $d(B_2) \geq \frac{1}{2}\ell_1$.

Suppose that $B_1, B_2, \cdots, B_k$ have been chosen from $\mathcal{B}$ according to the above way, then we denote

$$\mathcal{B}_k = \left\{B : B \in \mathcal{B} \text{ with } B \bigcap \left(\bigcup_{j=1}^{k} B_j\right) = \emptyset\right\}$$

and

$$\ell_k = \sup\{d(B) : B \in \mathcal{B}_k\}.$$

Next we choose $B_{k+1} \in \mathcal{B}_k$ such that $d(B_{k+1}) \geq \frac{1}{2}\ell_k$. Thus we may choose a sequence $B_1, B_2, \cdots,$ from $\mathcal{B}$ such that

(i) $B_1, B_2, \cdots, B_k, \cdots$ are disjoint;

(ii) $d(B_{k+1}) \geq \frac{1}{2}\sup\{d(B) : B \in \mathcal{B}_k\}$, and

$$\mathcal{B}_k = \left\{B : B \in \mathcal{B} \text{ and } B \bigcap \left(\bigcup_{j=1}^{k} B_j\right) = \emptyset\right\}$$

for $k = 1, 2, \cdots$.

If this process stops at some B_k, then it shows that $\mathcal{B}_k = \emptyset$. In this case, for any $x \in E$ there exists a ball $B_x \in \mathcal{B}$ such that $x \in B_x$ and $B_x \cap B_{k_0} \neq \emptyset$ with some $1 \leq k_0 \leq k$. Without loss of generality, we may assume that $B_x \cap B_j = \emptyset$ for $j = 1, 2, \cdots, k_0 - 1$. So, $d(B_{k_0}) \geq \frac{1}{2}d(B_x)$, and this implies $B_x \subset 5B_{k_0}$, where $5B_{k_0}$ expresses the five times extension of B_{k_0} with the same center. Thus, we have $E \subset \bigcup_{j=1}^{k} 5B_j$, and it leads to

$$|E| \leq \left|\bigcup_{j=1}^{k} 5B_j\right| \leq \sum_{j=1}^{k} |5B_j| \leq 5^n \sum_{j=1}^{k} |B_j|.$$

On the other hand, it is trivial when $\sum_{j=1}^{\infty} |B_k| = \infty$. So, we may assume that $\sum_{j=1}^{\infty} |B_k| < \infty$. Denote $B_k^* = 5B_k$. We will claim that

$$E \subset \bigcup_{k=1}^{\infty} B_k^*. \tag{1.1.7}$$

In fact, it suffices to prove that $B \subset \bigcup_{k=1}^{\infty} B_k^*$ for any $B \in \mathcal{B}$. Since $\sum_{j=1}^{\infty} |B_k| < \infty$, we have $d(B_k) \to 0$ as $k \to \infty$. Thus there exists k_0 such that $d(B_{k_0}) < \frac{1}{2}d(B)$. Of course, we may think that the index k_0 is the smallest with the above property. In this case, B_x must intersect with some B_j for $1 \le j \le k_0 - 1$. Otherwise, $d(B_{k_0}) \ge \frac{1}{2}d(B_x)$. As before, we get $B_x \subset 5B_j = B_j^*$ and (1.1.7) follows. Thus

$$|E| \le \left| \bigcup_{k=1}^{\infty} B_k^* \right| \le \sum_{k=1}^{\infty} |B_k^*| \le 5^n \sum_{k=1}^{\infty} |B_k|.$$

This completes the proof. ∎

Definition 1.1.2 (Weak L^p spaces) *Suppose that $1 \le p < \infty$ and f is a measurable function on $\mathbb{R}^n$. The function f is said to belong to the weak L^p spaces on $\mathbb{R}^n$, if there is a constant $C > 0$ such that*

$$\sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{1/p} \le C < \infty.$$

In other words, the weak $L^p(\mathbb{R}^n)$ is defined by

$$L^{p,\infty}(\mathbb{R}^n) = \{f : \ \|f\|_{p,\infty} < \infty\},$$

where

$$\|f\|_{p,\infty} := \sup_{\lambda>0} \lambda |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}|^{1/p}$$

denotes the seminorm of f in the weak $L^p(\mathbb{R}^n)$.

Remark 1.1.4 *It is easy to verify that for $1 \le p < \infty$, $L^p(\mathbb{R}^n) \subsetneqq L^{p,\infty}(\mathbb{R}^n)$.*

Definition 1.1.3 (Operator of type $(p,\ q)$) *Suppose that T is a sublinear operator and $1 \le p,\ q \le \infty$. T is said to be of weak type (p,q) if T is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$. That is, there exists a constant $C > 0$ such that for any $\lambda > 0$ and $f \in L^p(\mathbb{R}^n)$*

$$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \le \left(\frac{C}{\lambda}\|f\|_p\right)^q; \tag{1.1.8}$$

T is said to be of type (p,q) if T is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. That is, there exists a constant $C > 0$ such that for any $f \in L^p(\mathbb{R}^n)$

$$\|Tf\|_q \le C\|f\|_p, \tag{1.1.9}$$

where and below, $\|f\|_p = \|f\|_{L^p(\mathbb{R}^n)}$ denotes the L^p norm of $f(x)$.

When $p = q$ and the operator T satisfies (1.1.8) or (1.1.9), T is also said to be of weak type (p, p), respectively. Moreover, It is easy to see that an operator of type (p, q) is also of weak type (p, q), but its reverse is not hold generally.

Below we shall prove that the Hardy-Littlewood maximal operator M is of weak type $(1, 1)$ and type (p, p) for $1 < p \leq \infty$, respectively.

Theorem 1.1.1 *Let f be a measurable function on $\mathbb{R}^n$.*

(a) *If $f \in L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$, then $Mf(x) < \infty$ a.e. $x \in \mathbb{R}^n$.*

(b) *There exists a constant $C = C(n) > 0$ such that for any $\lambda > 0$ and $f \in L^1(\mathbb{R}^n)$*

$$|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}| \leq \frac{C}{\lambda}\|f\|_1.$$

(c) *There exists a constant $C = C(n, p) > 0$ such that for any $f \in L^p(\mathbb{R}^n)$ $1 < p \leq \infty$, $\|Mf\|_p \leq C\|f\|_p$.*

Proof. Obviously, the conclusion (a) is a direct result of the conclusions (b) and (c). Hence we only give the proof of (b) and (c).

Let us first consider (b). For any $\lambda > 0$, by Remark 1.1.1 the set

$$E_\lambda := \{x \in \mathbb{R}^n : Mf(x) > \lambda\}$$

is an open set, and is then a measurable set. By Definition 1.1.1, for any $x \in E_\lambda$, there exists a ball B_x with the center at x such that

$$\frac{1}{|B_x|}\int_{B_x} |f(y)|dy > \lambda.$$

Thus

$$|B_x| < \frac{1}{\lambda}\int_{B_x} |f(y)|dy \leq \frac{1}{\lambda}\|f\|_1 < \infty \text{ for all } x \in E_\lambda.$$

Therefore, if we denote $\mathcal{B} = \{B_x : x \in E_\lambda\}$, then $\mathcal{B}$ covers E_λ in Vitali's sense. By Lemma 1.1.1, we may choose disjoint countable balls $B_1, B_2, \cdots,$ $B_k, \cdots$ in $\mathcal{B}$ such that

$$\sum_k |B_k| \geq \beta|E_\lambda|.$$

Hence

$$\begin{aligned}\beta|E_\lambda| \leq \sum_k |B_k| &\leq \frac{1}{\lambda}\sum_k \int_{B_k} |f(y)|dy \\ &= \frac{1}{\lambda}\int_{\bigcup_k B_k} |f(y)|dy \\ &\leq \frac{1}{\lambda}\|f\|_1.\end{aligned}$$

Let us now turn to the proof of (c). Clearly, the conclusion (c) holds for $p = \infty$, we only consider the case $1 < p < \infty$. Let $f \in L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. For any $\lambda > 0$, write $f = f_1 + f_2$, where

$$f_1(x) = \begin{cases} f(x), & \text{for } |f(x)| \geq \lambda/2 \\ 0, & \text{for } |f(x)| < \lambda/2. \end{cases}$$

It is easy to see that $f_1 \in L^1(\mathbb{R}^n)$. Thus we have

$$|f(x)| \leq |f_1(x)| + \frac{\lambda}{2} \quad \text{and} \quad Mf(x) \leq Mf_1(x) + \frac{\lambda}{2}. \tag{1.1.10}$$

Hence, by (1.1.10) and the weak (1,1) boundedness of M (i.e. the conclusion (b)), we have

$$\begin{aligned} |E_\lambda| &= \left|\{x \in \mathbb{R}^n : Mf(x) > \lambda\}\right| \\ &\leq \left|\{x \in \mathbb{R}^n : Mf_1(x) > \lambda/2\}\right| \\ &\leq \frac{2\beta}{\lambda} \int_{\mathbb{R}^n} |f_1(x)|\, dx \\ &= \frac{2\beta}{\lambda} \int_{\{x \in \mathbb{R}^n:\ |f(x)| \geq \lambda/2\}} |f(x)|\, dx, \end{aligned}$$

where β is the constant in Lemma 1.1.1. Therefore

$$\begin{aligned} &\int_{\mathbb{R}^n} (Mf(x))^p dx \\ &= p \int_0^\infty \lambda^{p-1} |E_\lambda|\, d\lambda \\ &\leq p \int_0^\infty \lambda^{p-1} \left(\frac{2\beta}{\lambda} \int_{\{x \in \mathbb{R}^n:\ |f(x)| \geq \lambda/2\}} |f(x)|\, dx \right) d\lambda \\ &\leq 2\beta p \int_{\mathbb{R}^n} |f(x)| \left(\int_0^{2|f(x)|} \lambda^{p-2} d\lambda \right) dx \\ &= \frac{2\beta p}{p-1} \int_{\mathbb{R}^n} |f(x)|^p\, dx. \end{aligned}$$

Thus we finish the proof of Theorem 1.1.1.

Immediately, by the weak (1,1) boundedness of the Hardy-Littlewood maximal operator M we may get the Lebesgue differentiation theorem.

Theorem 1.1.2 (Lebesgue differentiation theorem) *Suppose that* $f \in L^1_{loc}(\mathbb{R}^n)$. *Then*

$$\lim_{r \to 0} \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy = f(x) \quad a.e. \ \ x \in \mathbb{R}^n,$$

where $B(x,r)$ denotes the ball with the center at x and radius r.

Proof. Since for any $R > 0$, $f\chi_{B(0,R)} \in L^1(\mathbb{R}^n)$, we may assume that $f \in L^1(\mathbb{R}^n)$. Denote

$$L_r(f)(x) = \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy,$$

and let

$$\Lambda(f)(x) = \overline{\lim_{r\to 0}} L_r(f)(x) - \underline{\lim_{r\to 0}} L_r(f)(x).$$

Then

$$\Lambda(f)(x) \le 2\sup_{r>0} |L_r(f)(x)| = 2Mf(x).$$

Let us first show that for any $\lambda > 0$

$$|E_\lambda(\Lambda f)| := |\{x \in \mathbb{R}^n : \Lambda(f)(x) > \lambda\}| = 0. \tag{1.1.11}$$

In fact, for any $\varepsilon > 0$ we may decompose $f = g + h$, where g is a continuous function with compact support set and $\|h\|_1 < \varepsilon$. Thus

$$\Lambda(f)(x) \le \Lambda(g)(x) + \Lambda(h)(x) = \Lambda(h)(x),$$

and it leads to

$$|E_\lambda(\Lambda f)| \le |E_\lambda(\Lambda h)| \le |E_{\lambda/2}(Mh)|.$$

By Theorem 1.1.1 (b), we have

$$|E_\lambda(\Lambda f)| \le \frac{2C}{\lambda}\|h\|_1 < \frac{2C\varepsilon}{\lambda}.$$

Thus, by the arbitrariness of ε we know (1.1.11) holds, and (1.1.11) shows that the limit $\lim_{r\to 0} L_r(f)(x)$ exists for a.e. $x \in \mathbb{R}^n$.

On the other hand, by the integral continuity, we have

$$\begin{aligned}
&\lim_{r\to 0} \|L_r(f) - f\|_1 \\
&= \lim_{r\to 0} \int_{\mathbb{R}^n} \left| \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy - f(x)\right| dx \\
&= \lim_{r\to 0} \int_{\mathbb{R}^n} \frac{1}{|B(0,r)|}\left|\int_{B(0,r)} [f(x-y) - f(x)]dy\right| dx \\
&\le \lim_{r\to 0} \frac{1}{|B(0,r)|}\int_{B(0,r)}\int_{\mathbb{R}^n} |f(x-y) - f(x)|dx\,dy = 0.
\end{aligned}$$

Hence there exists a subsequence $\{r_k\}$ satisfying $r_k \to 0$ as $k \to \infty$, such that

$$\lim_{k\to\infty} L_{r_k}(f)(x) = f(x) \quad \text{for a.e. } x \in \mathbf{R}^n.$$

Because $\lim_{r\to 0} L_r(f)(x)$ exists for a.e. $x \in \mathbf{R}^n$, thus

$$\lim_{r\to 0} \frac{1}{|B(x,r)|}\int_{B(x,r)} f(y)\,dy = f(x) \quad \text{a.e. } x \in \mathbf{R}^n,$$

which is the desired conclusion. ∎

Remark 1.1.5 *Clearly, by the equivalence of (1.1.1) - (1.1.3), it is easy to see that the conclusion of Theorem 1.1.2 still holds if we replace the ball $B(x,r)$ by cube $Q(x,r)$, even more generally, by a cube Q containing x.*

1.2 Calderón-Zygmund decomposition

Applying Lebesgue differentiation theorem, we may give a decomposition of $\mathbb{R}^n$, called as Calderón-Zygmund decomposition, which is extremely useful in harmonic analysis.

Theorem 1.2.1 (Calderón-Zygmund decomposition of $\mathbb{R}^n$) *Suppose that f is nonnegative integrable on $\mathbb{R}^n$. Then for any fixed $\lambda > 0$, there exists a sequence $\{Q_j\}$ of disjoint dyadic cubes (here by disjoint we mean that their interiors are disjoint) such that*

(1) $f(x) \le \lambda$ *for a.e.* $x \notin \bigcup_j Q_j$;

(2) $\left|\bigcup_j Q_j\right| \le \frac{1}{\lambda}\|f\|_1$;

(3) $\lambda < \frac{1}{|Q_j|}\int_{Q_j} f(x)\,dx \le 2^n\lambda$.

Proof. By $f \in L^1(\mathbb{R}^n)$, we may decompose $\mathbb{R}^n$ into a net of equal cubes with whose interiors are disjoint such that for every Q in the net

$$\frac{1}{|Q|}\int_Q f(x)\,dx \le \lambda.$$

Let Q' be any fixed cube in the net. We divide it into 2^n equal cubes, and denote Q'' is one of these cubes. Then there are the follwing two cases.
Case (i)

$$\frac{1}{|Q''|}\int_{Q''} f(x)\,dx > \lambda.$$

Case (ii)

$$\frac{1}{|Q''|}\int_{Q''} f(x)\,dx \le \lambda.$$

In the case (i) we have

$$\lambda < \frac{1}{|Q''|}\int_{Q''} f(x)\,dx \le \frac{1}{2^{-n}|Q'|}\int_{Q'} f(x)\,dx \le 2^n\lambda.$$

Hence, we do not sub-divide Q'' any further, and Q'' is chosen as one of the sequence $\{Q_j\}$.

For the case (ii) we continuously sub-divide Q'' into 2^n equal subcubes, and repeat this process until we are forced into the case (i). Thus we get a sequence $\{Q_j\}$ of cubes obtained from the case (i). By Theorem 1.1.2,

$$f(x) = \lim_{\substack{Q\ni x\\ |Q|\to 0}} \frac{1}{|Q|}\int_Q f(x)\,dx \le \lambda \quad \text{for a.e.} \quad x \notin \bigcup_j Q_j.$$

This proves the theorem.

Remark 1.2.1 *In place of* $\mathbb{R}^n$ *by a fixed cube* Q_0*, we may similarly discuss the Calderón-Zygmund decomposition on* Q_0 *for* $f \in L^1(Q_0)$ *and* $\lambda > 0$*. Moreover, we also may obtain the similar decomposition for* $f \in L^p(\mathbb{R}^n)$ $(p > 1)$.

An application of the Calderón-Zygmund decomposition on $\mathbb{R}^n$ is that it may be used to give the L^1 boundedness of the Hardy-Littlewood maximal operator M in some sense. More precisely, we have the following conclusion.

Theorem 1.2.2 *Suppose that* $f \in L^1(\mathbb{R}^n)$.

(i) If

$$\int_{\mathbb{R}^n} |f(x)| \log^+ |f(x)| dx < \infty,$$

then $Mf \in L^1_{loc}(\mathbb{R}^n)$.