Graduate Texts in Mathematics

John B. Conway

Functions of One Complex Variable II

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To The Memory of my Parents

Cecile Marie Boudreaux

and

Edward Daire Conway, Jr.

Preface

This is the sequel to my book *Functions of One Complex Variable I*, and probably a good opportunity to express my appreciation to the mathematical community for its reception of that work. In retrospect, writing that book was a crazy venture.

As a graduate student I had had one of the worst learning experiences of my career when I took complex analysis; a truly bad teacher. As a non-tenured assistant professor, the department allowed me to teach the graduate course in complex analysis. They thought I knew the material; I wanted to learn it. I adopted a standard text and shortly after beginning to prepare my lectures I became dissatisfied. All the books in print had virtues; but I was educated as a modern analyst, not a classical one, and they failed to satisfy me.

This set a pattern for me in learning new mathematics after I had become a mathematician. Some topics I found satisfactorily treated in some sources; some I read in many books and then recast in my own style. There is also the matter of philosophy and point of view. Going from a certain mathematical vantage point to another is thought by many as being independent of the path; certainly true if your only objective is getting there. But getting there is often half the fun and often there is twice the value in the journey if the path is properly chosen.

One thing led to another and I started to put notes together that formed chapters and these evolved into a book. This now impresses me as crazy partly because I would never advise any non-tenured faculty member to begin such a project; I have, in fact, discouraged some from doing it. On the other hand writing that book gave me immense satisfaction and its reception, which has exceeded my grandest expectations, makes that decision to write a book seem like the wisest I ever made. Perhaps I lucked out by being born when I was and finding myself without tenure in a time (and possibly a place) when junior faculty were given a lot of leeway and allowed to develop at a slower pace—something that someone with my background and temperament needed. It saddens me that such opportunities to develop are not so abundant today.

The topics in this volume are some of the parts of analytic function theory that I have found either useful for my work in operator theory or enjoyable in themselves; usually both. Many also fall into the category of topics that I have found difficult to dig out of the literature.

I have some difficulties with the presentation of certain topics in the literature. This last statement may reveal more about me than about the state of the literature, but certain notions have always disturbed me even though experts in classical function theory take them in stride. The best example of this is the concept of a multiple-valued function. I know there are ways to make the idea rigorous, but I usually find that with a little work it isn't necessary to even bring it up. Also the term multiple-valued function violates primordial instincts acquired in childhood where I was sternly taught that functions, by definition, cannot be multiple-valued.

The first volume was not written with the prospect of a second volume to follow. The reader will discover some topics that are redone here with more generality and originally could have been done at the same level of sophistication if the second volume had been envisioned at that time. But I have always thought that introductions should be kept unsophisticated. The first white wine would best be a Vouvray rather than a Chassagne-Montrachet.

This volume is divided into two parts. The first part, consisting of Chapters 13 through 17, requires only what was learned in the first twelve chapters that make up Volume I. The reader of this material will notice, however, that this is not strictly true. Some basic parts of analysis, such as the Cauchy-Schwarz Inequality, are used without apology. Sometimes results whose proofs require more sophisticated analysis are stated and their proofs are postponed to the second half. Occasionally a proof is given that requires a bit more than Volume I and its advanced calculus prerequisite. The rest of the book assumes a complete understanding of measure and integration theory and a rather strong background in functional analysis.

Chapter 13 gathers together a few ideas that are needed later. Chapter 14, "Conformal Equivalence for Simply Connected Regions," begins with a study of prime ends and uses this to discuss boundary values of Riemann maps from the disk to a simply connected region. There are more direct ways to get to boundary values, but I find the theory of prime ends rich in mathematics. The chapter concludes with the Area Theorem and a study of the set S of schlicht functions.

Chapter 15 studies conformal equivalence for finitely connected regions. I have avoided the usual extremal arguments and relied instead on the method of finding the mapping functions by solving systems of linear equations. Chapter 16 treats analytic covering maps. This is an elegant topic that deserves wider understanding. It is also important for a study of Hardy spaces of arbitrary regions, a topic I originally intended to include in this volume but one that will have to await the advent of an additional volume.

Chapter 17, the last in the first part, gives a relatively self contained treatment of de Branges's proof of the Bieberbach conjecture. I follow the approach given by Fitzgerald and Pommerenke [1985]. It is self contained except for some facts about Legendre polynomials, which are stated and explained but not proved. Special thanks are owed to Steve Wright and Dov Aharonov for sharing their unpublished notes on de Branges's proof of the Bieberbach conjecture.

Chapter 18 begins the material that assumes a knowledge of measure theory and functional analysis. More information about Banach spaces is used here than the reader usually sees in a course that supplements the standard measure and integration course given in the first year of graduate study in an American university. When necessary, a reference will be given to Conway [1990]. This chapter covers a variety of topics that are used in the remainder of the book. It starts with the basics of Bergman spaces, some material about distributions, and a discourse on the Cauchy transform and an application of this to get another proof of Runge's Theorem. It concludes with an introduction to Fourier series.

Chapter 19 contains a rather complete exposition of harmonic functions on the plane. It covers about all you can do without discussing capacity, which is taken up in Chapter 21. The material on harmonic functions from Chapter 10 in Volume I is assumed, though there is a built-in review.

Chapter 20 is a rather standard treatment of Hardy spaces on the disk, though there are a few surprising nuggets here even for some experts.

Chapter 21 discusses some topics from potential theory in the plane. It explores logarithmic capacity and its relationship with harmonic measure and removable singularities for various spaces of harmonic and analytic functions. The fine topology and thinness are discussed and Wiener's criterion for regularity of boundary points in the solution of the Dirichlet problem is proved.

This book has taken a long time to write. I've received a lot of assistance along the way. Parts of this book were first presented in a pubescent stage to a seminar I presented at Indiana University in 1981-82. In the seminar were Greg Adams, Kevin Clancey, Sandy Grabiner, Paul McGuire, Marc Raphael, and Bhushan Wadhwa, who made many suggestions as the year progressed. With such an audience, how could the material help but improve. Parts were also used in a course and a summer seminar at the University of Tennessee in 1992, where Jim Dudziak, Michael Gilbert, Beth Long, Jeff Nichols, and Jeff vanEeuwen pointed out several corrections and improvements. Nathan Feldman was also part of that seminar and besides corrections gave me several good exercises. Toward the end of the writing process I mailed the penultimate draft to some friends who read several chapters. Here Paul McGuire, Bill Ross, and Liming Yang were of great help. Finally, special thanks go to David Minda for a very careful reading of several chapters with many suggestions for additional references and exercises.

On the technical side, Stephanie Stacy and Shona Wolfenbarger worked diligently to convert the manuscript to TEX. Jinshui Qin drew the figures in the book. My son, Bligh, gave me help with the index and the bibliography.

In the final analysis the responsibility for the book is mine.

A list of corrections is also available from my WWW page (http: //www.math.utk.edu/~conway/).

Thanks to R. B. Burckel.

I would appreciate any further corrections or comments you wish to make.

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Contents of Volume I

Contents of Volume II

Chapter 13

Return to Basics

In this chapter a few results of a somewhat elementary nature are collected. These will be used quite often in the remainder of this volume.

§1 Regions and Curves

In this first section a few definitions and facts about regions and curves in the plane are given. Some of these may be familiar to the reader. Indeed, some will be recollections from the first volume.

Begin by recalling that a region is an open connected set and a simply connected region is one for which every closed curve is contractible to a point (see 4.6.14). In Theorem 8.2.2 numerous statements equivalent to simple connectedness were given. We begin by recalling one of these equivalent statements and giving another. Do not forget that $\mathbb{C}_\infty$ denotes the extended complex numbers and $\partial_\infty G$ denotes the boundary of the set G in $\mathbb{C}_\infty$. That is, $\partial_\infty G = \partial G$ when G is bounded and $\partial_\infty G = \partial G \cup \{\infty\}$ when G is unbounded.

It is often convenient to give results about subsets of the extended plane rather than about $\mathbb{C}$. If something was proved in the first volume for a subset of $\mathbb{C}$, but it holds for subsets of $\mathbb{C}_\infty$ with little change in the proof, we will not hesitate to quote the appropriate reference from the first twelve chapters as though the result for $\mathbb{C}_\infty$ was proved there.

1.1 Proposition. *If G is a region in $\mathbb{C}_\infty$, the following statements are equivalent.*

(a) *G is simply connected.*

(b) *$\mathbb{C}_\infty \setminus G$ is connected*

(c) *$\partial_\infty G$ is connected.*

Proof. The equivalence of (a) and (b) has already been established in (8.2.2). In fact, the equivalence of (a) and (b) was established without assuming that G is connected. That is, it was only assumed that G was a simply connected open set; an open set with every component simply connected. The reader must also pay attention to the fact that the connectedness of G will not be used when it is shown that (c) implies (b). This will be used when it is shown that (b) implies (c).

So assume (c) and let us prove (b). Let F be a component of $\mathbb{C}_\infty \setminus G$; so F is closed. It follows that $F \cap \text{cl } G \neq \emptyset$ (cl denotes the closure operation in $\mathbb{C}$ while cl_∞ denotes the closure in the extended plane.) Indeed, if it were the case that $F \cap \text{cl } G = \emptyset$, then for every z in F there is an $\varepsilon > 0$ such that $B(z;\varepsilon) \cap G = \emptyset$. Thus $F \cup B(z;\varepsilon) \subseteq \mathbb{C}_\infty \setminus G$. But $F \cup B(z;\varepsilon)$ is connected. Since F is a component of $\mathbb{C}_\infty \setminus G$, $B(z;\varepsilon) \subseteq F$. Since z was an arbitrary point, this implies that F is an open set, giving a contradiction. Therefore $F \cap \text{cl } G \neq \emptyset$.

Let $z_0 \in F \cap \text{cl } G$; so $z_0 \in \partial_\infty G$. By (c) $\partial_\infty G$ is connected, so $F \cup \partial_\infty G$ is a connected set that is disjoint from G. Therefore $\partial_\infty G \subseteq F$ since F is a component of $\mathbb{C}_\infty \setminus G$. What we have just shown is that every component of $\mathbb{C}_\infty \setminus G$ must contain $\partial_\infty G$. Hence there can be only one component and so $\mathbb{C}_\infty \setminus G$ is connected.

Now assume that condition (b) holds. So far we have not used the fact that G is connected; now we will. Let $U = \mathbb{C}_\infty \setminus \text{cl}_\infty G$. Now $\mathbb{C}_\infty \setminus U = \text{cl}_\infty G$ and $\text{cl}_\infty G$ is connected. Since we already have that (a) and (b) are equivalent (even for non-connected open sets), U is simply connected. Thus $\mathbb{C}_\infty \setminus \partial_\infty G = G \cup U$ is the union of two disjoint simply connected sets and hence must be simply connected. Since (a) implies (b), $\partial_\infty G = \mathbb{C}_\infty \setminus (G \cup U)$ is connected. □

1.2 Corollary. *If G is a region in $\mathbb{C}$, then the map $F \to F \cap \partial_\infty G$ defines a bijection between the components of $\mathbb{C}_\infty \setminus G$ and the components of $\partial_\infty G$.*

Proof. If F is a component of $\mathbb{C}_\infty \setminus G$, then an argument that appeared in the preceding proof shows that $F \cap \partial_\infty G \neq \emptyset$. Also, since $\partial_\infty G \subseteq \mathbb{C}_\infty \setminus G$, any component C of $\partial_\infty G$ that meets F must be contained in F. It must be shown that two distinct components of $\partial_\infty G$ cannot be contained in F.

To this end, let $G_1 = \mathbb{C}_\infty \setminus F$. Since G_1 is the union of G and the components of $C_\infty \setminus G$ that are distinct from F, G_1 is connected. Since $\mathbb{C}_\infty \setminus G_1 = F$, a connected set, G_1 is simply connected. By the preceding proposition, $\partial_\infty G_1$ is connected. Now $\partial_\infty G_1 \subseteq \partial_\infty G$. In fact for any point z in $\partial_\infty G_1$, $\emptyset \neq B(z;\varepsilon) \cap (\mathbb{C}_\infty \setminus G_1) \subseteq B(z;\varepsilon) \cap (\mathbb{C}_\infty \setminus G)$. Also if $B(z;\varepsilon) \cap G = \emptyset$, then $B(z;\varepsilon) \subseteq \mathbb{C}_\infty \setminus G$ and $B(z;\varepsilon) \cap F \neq \emptyset$; thus $z \in \text{int } F$, contradicting the fact that $z \in \partial_\infty G_1$. Thus $\partial_\infty G_1 \subseteq \partial_\infty G$. Therefore any component of $\partial_\infty G$ that meets F must contain $\partial_\infty G_1$. Hence there can be only one such component of $\partial_\infty G$. That is, $F \cap \partial_\infty G$ is a component of $\partial_\infty G$.

This establishes that the map $F \to F \cap \partial_\infty G$ defines a map from the components of $\mathbb{C}_\infty \setminus G$ to the components of $\partial_\infty G$. The proof that this correspondence is a bijection is left to the reader. □

Recall that a *simple closed curve* in $\mathbb{C}$ is a path $\gamma : [a,b] \to \mathbb{C}$ such that $\gamma(t) = \gamma(s)$ if and only if $t = s$ or $|s - t| = b - a$. Equivalently, a simple closed curve is the homeomorphic image of $\partial\mathbb{D}$. Another term for a simple closed curve is a *Jordan curve*. The Jordan Curve Theorem is given here, but a proof is beyond the purpose of this book. See Whyburn [1964].

1.3 Jordan Curve Theorem. *If γ is a simple closed curve in $\mathbb{C}$, then $\mathbb{C} \setminus \gamma$ has two components, each of which has γ as its boundary.*

Clearly one of the two components of $\mathbb{C} \setminus \gamma$ is bounded and the other is unbounded. Call the bounded component of $\mathbb{C} \setminus \gamma$ the *inside* of γ and call the unbounded component of $\mathbb{C} \setminus \gamma$ the *outside* of γ. Denote these two sets by ins γ and out γ, respectively.

Note that if γ is a rectifiable Jordan curve, so that the winding number $n(\gamma; a)$ is defined for all a in $\mathbb{C} \setminus \gamma$, then $n(\gamma; a) \equiv \pm 1$ for a in ins γ while $n(\gamma; a) \equiv 0$ for a in out γ. Say γ is *positively oriented* if $n(\gamma; a) = 1$ for all a in ins γ. A curve γ is *smooth* if γ is a continuously differentiable function and $\gamma'(t) \neq 0$ for all t. Say that γ is a *loop* if γ is a positively oriented smooth Jordan curve.

Here is a corollary of the Jordan Curve Theorem

1.4 Corollary. *If γ is a Jordan curve,* ins γ *and* $(\text{out}\,\gamma) \cup \{\infty\}$ *are simply connected regions.*

Proof. In fact, $\mathbb{C}_\infty \setminus \text{ins}\ \gamma = \text{cl}_\infty(\text{out}\ \gamma)$ and this is connected by the Jordan Curve Theorem. Thus ins γ is simply connected by Proposition 1.1. Similarly, out $\gamma \cup \{\infty\}$ is simply connected. □

A *positive Jordan system* is a collection $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ of pairwise disjoint rectifiable Jordan curves such that for all points a not on any γ_j, $n(\Gamma; a) \equiv \sum_{j=1}^{m} n(\gamma_j; a) = 0$ or 1. Let out $\Gamma \equiv \{a \in \mathbb{C} : n(\Gamma; a) = 0\}$ = the *outside* of Γ and let ins $\Gamma \equiv \{a \in \mathbb{C} : n(\Gamma; a) = 1\}$ = the *inside* of Γ. Thus $\mathbb{C} \setminus \Gamma = \text{out}\ \Gamma \cup \text{ins}\ \Gamma$. Say that Γ is smooth if each curve γ_j in Γ is smooth.

Note that it is not assumed that ins Γ is connected and if Γ has more than one curve, out Γ is never connected. The boundary of an annulus is an example of a positive Jordan system if the curves on the boundary are given appropriate orientation. The boundary of the union of two disjoint closed annuli is also a positive Jordan system, as is the boundary of the union of two disjoint closed disks.

If X is any set in the plane and A and B are two non-empty sets, say that X *separates A from B* if A and B are contained in distinct components of the complement of X. The proof of the next result can be found on page 34 of Whyburn [1964].

1.5 Separation Theorem. *If K is a compact subset of the open set U, $a \in K$, and $b \in \mathbb{C}_\infty \setminus U$, then there is a Jordan curve γ in U such that γ is disjoint from K and γ separates a from b.*

In the preceding theorem it is not possible to get that the point a lies in ins γ. Consider the situation where U is the open annulus ann$(0; 1, 3)$,

$K = \{z; |z| = 3/2\}$, $a = 3/2$, and $b = 0$.

1.6 Corollary. *The curve γ in the Separation Theorem can be chosen to be smooth.*

Proof. Let $\Omega = \text{ins}\,\gamma$ and for the moment assume that $a \in \Omega$. The other case is left to the reader. Let $K_0 = K \cap \Omega$. Since $\gamma \cap K = \emptyset$, it follows that K_0 is a compact subset of Ω that contains a. Since Ω is simply connected, there is a Riemann map $\tau : \mathbb{D} \to \Omega$. By a compactness argument there is a radius r, $0 < r < 1$, such that $\tau(r\mathbb{D}) \supseteq K_0$. Since U is open and $\gamma \subseteq U$, r can be chosen so that $\tau(r\partial\mathbb{D}) \subseteq U$. Let σ be a parameterization of the circle $r\partial\mathbb{D}$ and consider the curve $\tau \circ \sigma$. Clearly $\tau \circ \sigma$ separates a from b, is disjoint from K, and lies inside U. □

Note that the proof of the preceding corollary actually shows that γ can be chosen to be an analytic curve. That is, γ can be chosen such that it is the image of the unit circle under a mapping that is analytic in a neighborhood of the circle. (See §4 below.)

1.7 Proposition. *If K is a compact connected subset of the open set U and b is a point in the complement of U, then there is a loop γ in U that separates K and b.*

Proof. Let $a \in K$ and use (1.6) to get a loop γ that separates a and b. Let Ω be the component of the complement of γ that contains a. Since $K \cap \Omega \neq \emptyset$, $K \cap \gamma = \emptyset$, and K is connected, it must be that $K \subseteq \Omega$. □

The next result is used often. A proof of this proposition can be given starting from Proposition 8.1.1. Actually Proposition 8.1.1 was not completely proved there since the statement that the line segments obtained in the proof form a finite number of closed polygons was never proved in detail. The details of this argument are combinatorially complicated. Basing the argument on the Separation Theorem obviates these complications.

1.8 Proposition. *If E is a compact subset of an open set G, then there is a smooth positively oriented Jordan system Γ contained in G such that $E \subseteq \text{ins}\,\Gamma \subseteq G$.*

Proof. Now G can be written as the increasing union on open sets G_n such that each G_n is bounded and $\mathbb{C} \setminus G_n$ has only a finite number of components (7.1.2). Thus it suffices to assume that G is bounded and $\mathbb{C} \setminus G$ has only a finite number of components, say $K_0, K_1, \dots, K_n$ where K_0 is the unbounded component.

It is also sufficient to assume that G is connected. In fact if $U_1, U_2, \dots$ are the components of G, then $\{U_m\}$ is an open cover of E. Hence there is a finite subcover. Thus for some integer m there are compact subsets E_k of U_k, $1 \leq k \leq m$, such that $E = \bigcup_1^m E_k$. If the proposition is proved