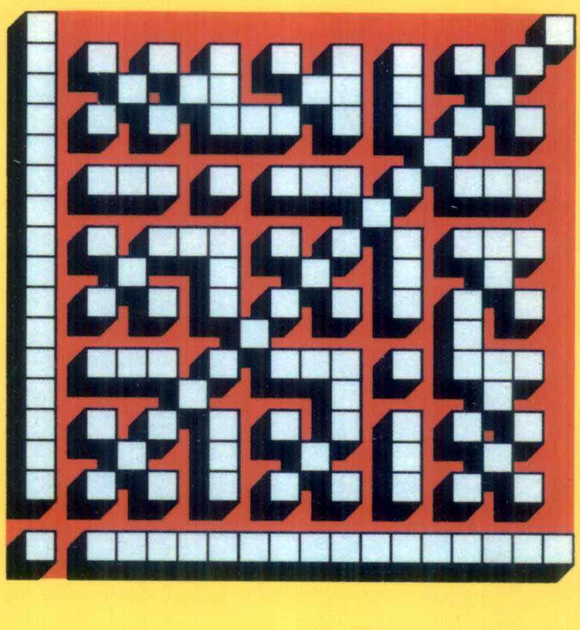
Undergraduate Texts in Mathematics

Tom M. Apostol

Introduction to Analytic Number Theory

解析数论导论



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Tom M. Apostol

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Preface

This is the first volume of a two-volume textbook[1] which evolved from a course (Mathematics 160) offered at the California Institute of Technology during the last 25 years. It provides an introduction to analytic number theory suitable for undergraduates with some background in advanced calculus, but with no previous knowledge of number theory. Actually, a great deal of the book requires no calculus at all and could profitably be studied by sophisticated high school students.

Number theory is such a vast and rich field that a one-year course cannot do justice to all its parts. The choice of topics included here is intended to provide some variety and some depth. Problems which have fascinated generations of professional and amateur mathematicians are discussed together with some of the techniques for solving them.

One of the goals of this course has been to nurture the intrinsic interest that many young mathematics students seem to have in number theory and to open some doors for them to the current periodical literature. It has been gratifying to note that many of the students who have taken this course during the past 25 years have become professional mathematicians, and some have made notable contributions of their own to number theory. To all of them this book is dedicated.

[1] The second volume is scheduled to appear in the Springer-Verlag Series Graduate Texts in Mathematics under the title **Modular Functions and Dirichlet Series in Number Theory**.

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Historical Introduction

The theory of numbers is that branch of mathematics which deals with properties of the whole numbers,

$$1, 2, 3, 4, 5, \ldots$$

also called the *counting numbers*, or *positive integers*.

The positive integers are undoubtedly man's first mathematical creation. It is hardly possible to imagine human beings without the ability to count, at least within a limited range. Historical record shows that as early as 3500 BC the ancient Sumerians kept a calendar, so they must have developed some form of arithmetic.

By 2500 BC the Sumerians had developed a number system using 60 as a base. This was passed on to the Babylonians, who became highly skilled calculators. Babylonian clay tablets containing elaborate mathematical tables have been found, dating back to 2000 BC.

When ancient civilizations reached a level which provided leisure time to ponder about things, some people began to speculate about the nature and properties of numbers. This curiosity developed into a sort of number-mysticism or numerology, and even today numbers such as 3, 7, 11, and 13 are considered omens of good or bad luck.

Numbers were used for keeping records and for commercial transactions for over 2000 years before anyone thought of studying numbers themselves in a systematic way. The first scientific approach to the study of integers, that is, the true origin of the theory of numbers, is generally attributed to the Greeks. Around 600 BC Pythagoras and his disciples made rather thorough studies of the integers. They were the first to classify integers in various ways:

Even numbers: 2, 4, 6, 8, 10, 12, 14, 16, . . .
Odd numbers: 1, 3, 5, 7, 9, 11, 13, 15, . . .
Prime numbers: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, . . .
Composite numbers: 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, . . .

A *prime number* is a number greater than 1 whose only divisors are 1 and the number itself. Numbers that are not prime are called *composite*, except that the number 1 is considered neither prime nor composite.

The Pythagoreans also linked numbers with geometry. They introduced the idea of *polygonal numbers*: triangular numbers, square numbers, pentagonal numbers, etc. The reason for this geometrical nomenclature is clear when the numbers are represented by dots arranged in the form of triangles, squares, pentagons, etc., as shown in Figure I.1.

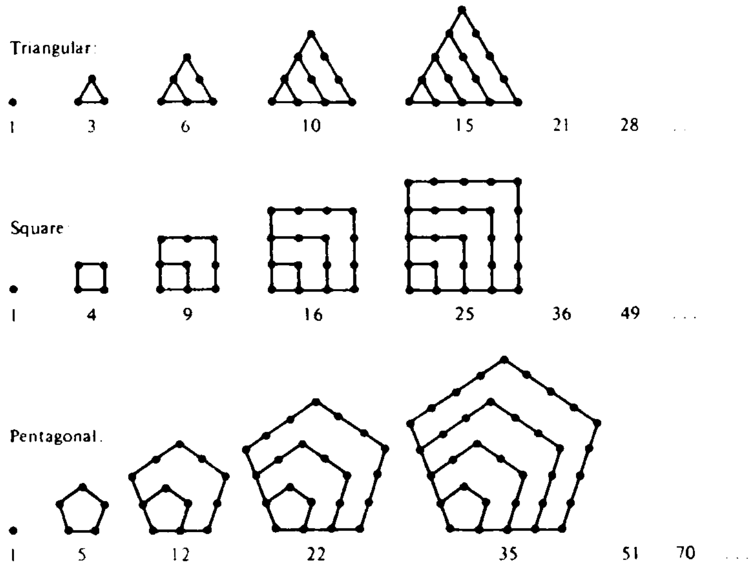

Figure I.1

Another link with geometry came from the famous Theorem of Pythagoras which states that in any right triangle the square of the length of the hypotenuse is the sum of the squares of the lengths of the two legs (see Figure I.2). The Pythagoreans were interested in right triangles whose sides are integers, as in Figure I.3. Such triangles are now called *Pythagorean triangles*. The corresponding triple of numbers (x, y, z) representing the lengths of the sides is called a *Pythagorean triple*.

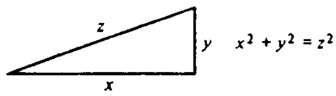

Figure 1.2

A Babylonian tablet has been found, dating from about 1700 BC, which contains an extensive list of Pythagorean triples, some of the numbers being quite large. The Pythagoreans were the first to give a method for determining infinitely many triples. In modern notation it can be described as follows: Let n be any odd number greater than 1, and let

$$x = n, \qquad y = \tfrac{1}{2}(n^2 - 1), \qquad z = \tfrac{1}{2}(n^2 + 1).$$

The resulting triple (x, y, z) will always be a Pythagorean triple with $z = y + 1$. Here are some examples:

x	3	5	7	9	11	13	15	17	19
y	4	12	24	40	60	84	112	144	180
z	5	13	25	41	61	85	113	145	181

There are other Pythagorean triples besides these; for example:

x	8	12	16	20
y	15	35	63	99
z	17	37	65	101

In these examples we have $z = y + 2$. Plato (430–349 BC) found a method for determining all these triples; in modern notation they are given by the formulas

$$x = 4n, \qquad y = 4n^2 - 1, \qquad z = 4n^2 + 1.$$

Around 300 BC an important event occurred in the history of mathematics. The appearance of Euclid's *Elements*, a collection of 13 books, transformed mathematics from numerology into a deductive science. Euclid was the first to present mathematical facts along with rigorous proofs of these facts.

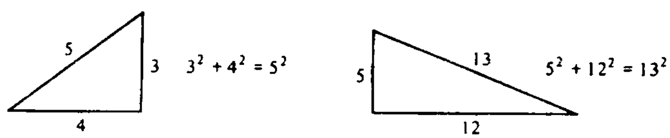

Figure 1.3

Three of the thirteen books were devoted to the theory of numbers (Books VII, IX, and X). In Book IX Euclid proved that there are infinitely many primes. His proof is still taught in the classroom today. In Book X he gave a method for obtaining all Pythagorean triples although he gave no proof that his method did, indeed, give them all. The method can be summarized by the formulas

$$x = t(a^2 - b^2), \qquad y = 2tab, \qquad z = t(a^2 + b^2),$$

where t, a, and b, are arbitrary positive integers such that $a > b$, a and b have no prime factors in common, and one of a or b is odd, the other even.

Euclid also made an important contribution to another problem posed by the Pythagoreans—that of finding all perfect numbers. The number 6 was called a perfect number because $6 = 1 + 2 + 3$, the sum of all its proper divisors (that is, the sum of all divisors less than 6). Another example of a perfect number is 28 because $28 = 1 + 2 + 4 + 7 + 14$, and 1, 2, 4, 7, and 14 are the divisors of 28 less than 28. The Greeks referred to the proper divisors of a number as its "parts." They called 6 and 28 perfect numbers because in each case the number is equal to the sum of all its parts.

In Book IX, Euclid found all *even* perfect numbers. He proved that an even number is perfect if it has the form

$$2^{p-1}(2^p - 1),$$

where both p and $2^p - 1$ are primes.

Two thousand years later, Euler proved the converse of Euclid's theorem. That is, every even perfect number must be of Euclid's type. For example, for 6 and 28 we have

$$6 = 2^{2-1}(2^2 - 1) = 2 \cdot 3 \qquad \text{and } 28 = 2^{3-1}(2^3 - 1) = 4 \cdot 7.$$

The first five even perfect numbers are

6, 28, 496, 8128 and 33,550,336.

Perfect numbers are very rare indeed. At the present time (1983) only 29 perfect numbers are known. They correspond to the following values of p in Euclid's formula:

2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11,213, 19,937, 21,701, 23,209, 44,497, 86,243, 132,049

Numbers of the form $2^p - 1$, where p is prime, are now called *Mersenne numbers* and are denoted by M_p in honor of Mersenne, who studied them in 1644. It is known that M_p is prime for the 29 primes listed above and composite for all values of $p < 44{,}497$. For the following primes,

$$p = 137, 139, 149, 199, 227, 257$$

although M_p is composite, no prime factor of M_p is known.