Undergraduate Texts in Mathematics

Robert Bix

# CONICS AND CUBICS

## A Concrete Introduction to Algebraic Curves

Second Edition

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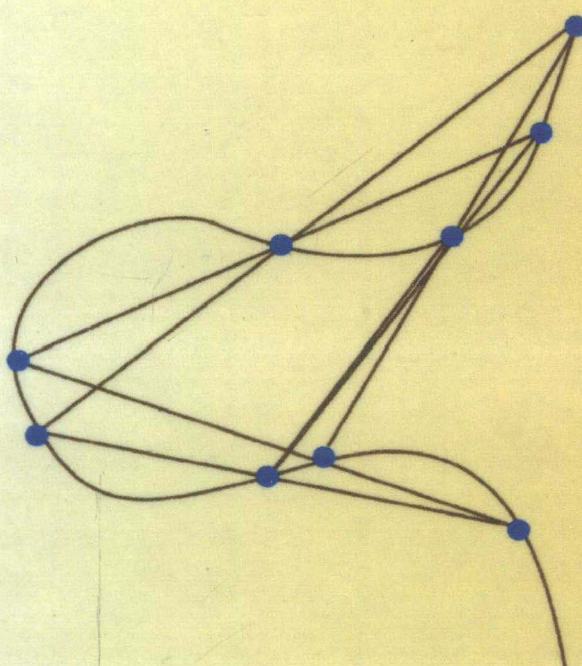

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# Preface

Algebraic curves are the graphs of polynomial equations in two variables, such as $y^3 + 5xy^2 = x^3 + 2xy$. This book introduces the study of algebraic curves by focusing on curves of degree at most 3—lines, conics, and cubics—over the real numbers. That keeps the results tangible and the proofs natural. The book is designed for a one-semester class for undergraduate mathematics majors. The only prerequisite is first-year calculus.

Algebraic geometry unites algebra, geometry, topology, and analysis, and it is one of the most exciting areas of modern mathematics. Unfortunately, the subject is not easily accessible, and most introductory courses require a prohibitive amount of mathematical machinery. We avoid this problem by basing proofs on high school algebra instead of linear algebra, abstract algebra, or complex analysis. This lets us emphasize the power of two fundamental ideas, homogeneous coordinates and intersection multiplicities.

Every line can be transformed into the $x$-axis, and every conic can be transformed into the parabola $y = x^2$. We use these two basic facts to analyze the intersections of lines and conics with curves of all degrees, and to deduce special cases of Bezout's Theorem and Noether's Theorem. These results give Pascal's Theorem and its corollaries about polygons inscribed in conics, Brianchon's Theorem and its corollaries about polygons circumscribed about conics, and Pappus' Theorem about hexagons inscribed in lines. We give a simple proof of Bezout's Theorem for curves of all degrees by combining the result for lines with induction on the degrees of the curves in one of the variables. We use Bezout's Theorem to classify cubics. We introduce elliptic curves by proving that a cubic becomes an abelian group when collinearity determines addition of points; this fact plays a key role in number theory, and it is the starting point of the 1995 proof of Fermat's Last Theorem.

The 2nd Edition differs from the 1st in Chapter IV by using power series to parametrize curves. We apply parametrizations in two ways: to derive the properties of intersection multiplicities employed in Chapters I–III and to extend the duality of curves and envelopes from conics to curves of higher degree.

The 2nd Edition also has a simpler proof of duality for conics in Theorem 7.3. There are new Exercises 5.7, 6.21–6.23, 7.17–7.23, 11.21, and 11.22 on conics, foci, and director circles.

A one-semester course can skip Sections 13 and 16, whose results are not needed in other sections. The more technical parts of Sections 14 and 15 can be covered lightly.

The exercises provide practice in using the results of the text, and they outline additional material. They can be homework problems when the book is used as a class text, and they are optional otherwise.

I am greatly indebted to Harry D'Souza for sharing his expertise, to Richard Alfaro for generating figures by computer, to Richard Belshoff for correcting errors, and to Renate McLaughlin, Kenneth Schilling, and my late brother Michael Bix for reviewing the manuscript. I am also grateful to the students at the University of Michigan-Flint who tried out the manuscript in classes.

Robert Bix
Flint, Michigan
November 2005

# Contents

CHAPTER I

# Intersections of Curves

## Introduction and History

### Introduction

An algebraic curve is the graph of a polynomial equation in two variables $x$ and $y$. Because we consider products of powers of both variables, the graphs can be intricate even for polynomials with low exponents. For example, Figure I.1 shows the graph of the equation

$$r^2 = \cos 2\theta$$

in polar coordinates. To convert this equation to rectangular coordinates and obtain a polynomial in two variables, we multiply both sides of the equation by $r^2$ and use the identity $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$. This gives

$$r^4 = r^2 \cos^2 \theta - r^2 \sin^2 \theta. \tag{1}$$

We use the usual substitutions $r^2 = x^2 + y^2$, $r \cos \theta = x$, and $r \sin \theta = y$ to rewrite (1) as

$$(x^2 + y^2)^2 = x^2 - y^2.$$

Multiplying this polynomial out and collecting its terms on the left gives

$$x^4 + 2x^2y^2 + y^4 - x^2 + y^2 = 0. \tag{2}$$

Thus Figure I.1 is the graph of a polynomial in two variables, and so it is an algebraic curve.

We add two powerful tools for studying algebraic curves to the familiar techniques of precalculus and calculus. The first is the idea that

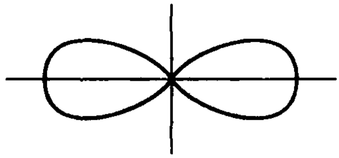

**Figure I.1**

curves can intersect repeatedly at a point. For example, it is natural to think that the curve in Figure I.1 intersects the $x$-axis twice at the origin because it passes through the origin twice. We develop algebraic techniques in Section 1 for computing the number of times that two algebraic curves intersect at the origin.

The second major tool for studying algebraïc curves is the system of homogeneous coordinates, which we introduce in Section 2. This is a bookkeeping device that lets us study the behavior of algebraic curves at infinity in the same way as in the Euclidean plane. Erasing the distinction between points of the Euclidean plane and those at infinity simplifies our work greatly by eliminating special cases.

We combine the ideas of Sections 1 and 2 in Section 3. We use homogeneous coordinates to determine the number of times that two algebraic curves intersect at any point in the Euclidean plane or at infinity. We also introduce transformations, which are linear changes of coordinates. We use transformations throughout our work to simplify the equations of curves.

We focus on the intersections of lines and other curves in Section 4. If a line $l$ is not contained in an algebraic curve $F$, we prove that the number of times that $l$ intersects $F$, counting multiplicities, is at most the degree of $F$. This introduces one of the main themes of our work: the geometric significance of the degree of a curve. We also characterize tangent lines in terms of intersection multiplicities.

## History

Greek mathematicians such as Euclid and Apollonius developed geometry to an extraordinary level in the third century B.C. Their algebra, however, was limited to verbal combinations of lengths, areas, and volumes. Algebraic symbols, which give algebraic work its power, arose only in the second half of the 1500s, most notably when François Vieta introduced the use of letters to represent unknowns and general coefficients.

Geometry and algebra were combined into analytic geometry in the first half of the 1600s by Pierre de Fermat and René Descartes. By asserting that any equation in two variables could be used to define a curve, they expanded the study of curves beyond those that could be constructed geometrically or mechanically.

Fermat found tangents and extreme points of graphs by using essentially the methods of present-day calculus. Calculus developed rapidly in the latter half of the 1600s, and its great power was demonstrated by Isaac Newton and Gottfried Leibniz. In particular, Newton used implicit differentiation to find tangents to curves, as we do after Theorem 4.10.

Apart from its role in calculus, analytic geometry developed gradually. Analytic geometers concentrated at first on giving analytic proofs of known results about lines and conics. Newton established analytic geometry as an important subject in its own right when he classified cubics, a task beyond the power of synthetic—that is, nonanalytic—geometry. We derive one of Newton's classifications of cubics in Chapter III.

While Fermat and Descartes were founding analytic geometry in the first half of the 1600s, Girard Desargues was developing a new branch of synthetic geometry called projective geometry. Renaissance artists and mathematicians had raised questions about drawing in perspective. These questions led Desargues to consider points at infinity and projections between planes, concepts we discuss at the start of Section 2. He used projections between planes to derive a remarkable number of theorems about lines and conics. His contemporary, Blaise Pascal, took up the projective study of conics, and their work was continued in the late 1600s by Philippe de la Hire.

Projective geometry languished in the 1700s as calculus and its applications dominated mathematics. Work on algebraic curves focused on their intersections, although multiple intersections were not analyzed systematically until the nineteenth century, as we discuss at the start of Chapter IV. We introduce intersection multiplicities in Section 1 so that we can automatically handle the special cases of theorems that arise from multiple intersections.

At the start of the 1800s, Gaspard Monge inspired a revival of synthetic geometry. His student Jean-Victor Poncelet championed synthetic projective geometry as a branch of mathematics in its own right. Mathematicians argued vigorously about the relative merits of synthetic and analytic geometry, although each subject actually drew strength from the other.

Analytic geometry was revolutionized when homogeneous coordinates were used to coordinatize the projective plane. Augustus Möbius introduced one system of homogeneous coordinates, barycentric coordinates, in 1827. He associated each point $P$ in the projective plane with the triples of signed weights to be placed at the vertices of a fixed triangle so that $P$ is the center of gravity. In 1830, Julius Plücker introduced the system of homogeneous coordinates that is currently used, which we introduce in Section 2.

Throughout the 1830s, Plücker used homogeneous coordinates to study curves. He obtained remarkable results, which we discuss in the

History for Chapter IV. Together with Riemann's work, which we discuss at the start of Chapter III, Plücker's results provided much of the inspiration for the subsequent development of algebraic geometry.

Möbius and Plücker also considered maps of the projective plane produced by invertible linear transformations of homogeneous coordinates. These are the transformations we discuss in Section 3. Much of nineteenth-century algebraic geometry was devoted to studying invariants, the algebraic combinations of coordinates of $n$-dimensional space that are preserved by invertible linear transformations. Founded by George Boole in 1841, invariant theory was developed in the latter half of the 1800s by such notable mathematicians as Arthur Cayley, James Sylvester, George Salmon, and Paul Gordan. Methods of abstract algebra came to dominate invariant theory when they were introduced by David Hilbert in the late 1800s and Emmy Noether in the early 1900s.

# §1. Intersections at the Origin

An important way to study a curve is to analyze its intersections with other curves. This analysis leads to the idea of two curves intersecting more than once at a point. We devote this section to studying multiple intersections at the origin, where the algebra is simplest.

A *polynomial* $f$ or $f(x, y)$ in two variables is a finite sum of terms of the form $ex^iy^j$, where the coefficient $e$ is a real number and the exponents $i$ and $j$ are nonnegative integers. We say that a term $ex^iy^j$ has *degree* $i+j$ and that the *degree* of a nonzero polynomial is the maximum of the degrees of the terms with nonzero coefficients. For example, the six terms of the polynomial

$$y^3 - 2x^3y + 7xy - 3x^2 + 7x + 5$$

have respective degrees 3, 4, 2, 2, 1, and 0, and the degree of the polynomial is 4. *We work over the real numbers exclusively until we introduce the complex numbers in Section* 10.

We define an *algebraic curve* formally to be a polynomial $f(x, y)$ in two variables, and we picture the algebraic curve as the graph of the equation $f(x, y) = 0$ in the plane. We abbreviate the term "algebraic curve" to "curve" because the only curves we consider are algebraic; that is, they are given by a polynomial equation in two variables. We refer both to the "curve $f(x, y)$" and to the "curve $f(x, y) = 0$," and we even rewrite the equation $f(x, y) = 0$ in algebraically equivalent forms. For example, we refer to the same curve as $y - x^2$, $y - x^2 = 0$, and $y = x^2$. Of course, we say that the curve $f(x, y)$ *contains* a point $(a, b)$ and that the point *lies*

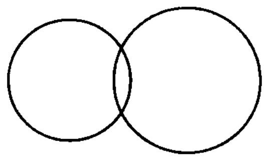

Figure 1.1

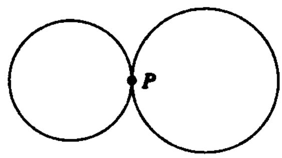


Figure 1.2

*on* the curve when $f(a, b) = 0$. When the polynomial $f(x, y)$ is nonzero, we refer to its degree as the *degree* of the curve $f(x, y) = 0$.

One reason we define a curve formally to be a polynomial rather than its graph is to keep track of repeated factors. We imagine that the points of the graph that belong to repeated factors are themselves repeated. For example, we think of the curve

$$(y - x^2)^2(y - x)^3$$

as two copies of the parabola $y = x^2$ and three copies of the line $y = x$. This idea helps the geometry reflect the algebra.

We turn now to the idea that curves can intersect more than once at a point. As we noted in the chapter introduction, it is natural to think that the curve in Figure I.1 intersects the $x$-axis twice at the origin because the curve seems to pass through the origin twice.

For a different type of example, note that two circles with overlapping interiors intersect at two points (Figure 1.1). As the circles move apart, their two points of intersection draw closer together until they coalesce into a single point $P$ (Figure 1.2). Accordingly, it seems natural to think that the circles in Figure 1.2 intersect twice at $P$.

Similarly, any line of positive slope through the origin intersects the graph of $y = x^3$ in three points (Figure 1.3). As the line rotates about the origin toward the $x$-axis, the three points of intersection move together at the origin, and they all coincide at the origin when the line reaches the $x$-axis. Accordingly, it is natural to think that the curve $y = x^3$ intersects the $x$-axis three times at the origin.

Let $O$ be the origin $(0, 0)$. We assign a value $I_O(f, g)$ to every pair of polynomials $f$ and $g$. We call this value the *intersection multiplicity* of $f$ and $g$ at $O$, and we think of it as the number of times that the curves $f(x, y) = 0$ and $g(x, y) = 0$ intersect at the origin.

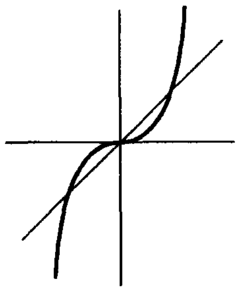

**Figure 1.3**

What properties should the assignment of the values $I_O(f,g)$ have? The proof of Theorem 1.7 will show that we need to allow for the possibility that curves intersect infinitely many times at the origin. We expect the following result, where the symbol $\infty$ denotes infinity:

**Property 1.1**
$I_O(f,g)$ is a nonnegative integer or $\infty$. □

The order in which we consider two curves should not affect the number of times they intersect at the origin. This suggests the next property:

**Property 1.2**

$$I_O(f,g) = I_O(g,f). \qquad \square$$

If either of two curves fails to contain the origin, they do not intersect there, and their intersection multiplicity at the origin should be zero. On the other hand, if both curves contain the origin, they do intersect there, and their intersection multiplicity should be at least 1. Thus, we expect the following property to hold:

**Property 1.3**
$I_O(f,g) \geq 1$ if and only if $f$ and $g$ both contain the origin. □

Of course, we consider $\infty$ to be greater than every integer, so that Property 1.3 allows for the possibility that $I_O(f,g) = \infty$ when $f$ and $g$ both contain the origin.

The $y$- and $x$-axes seem to intersect as simply as possible at the origin, and so we expect them to intersect only once there. Since the axes have equations $x = 0$ and $y = 0$, we anticipate the following property:

**Property 1.4**

$$I_O(x,y) = 1. \qquad \square$$

Let $f$, $g$, and $h$ be three polynomials in two variables, and let $(a, b)$ be a point. The equations

$$f(a,b) = 0 \qquad \text{and} \qquad g(a,b) = 0 \tag{1}$$

imply the equations

$$f(a,b) = 0 \qquad \text{and} \qquad g(a,b) + f(a,b)h(a,b) = 0. \tag{2}$$

Conversely, the equations in (2) imply the equations in (1). In short, $f$ and $g$ intersect at $(a, b)$ if and only if $f$ and $g + fh$ intersect there. Generalizing this to multiple intersections at the origin suggests the following:

**Property 1.5**

$$I_O(f,g) = I_O(f, g + fh). \qquad \square$$

One reason to expect that Property 1.5 holds for multiple as well as single intersections is the discussion accompanying Figures 1.1–1.3, which suggests that we can think of a multiple intersection of two curves as the coalescence of single intersections.

The equations $f(a, b) = 0$ and $g(a, b)h(a, b) = 0$ hold if and only if either $f(a, b) = 0 = g(a, b)$ or $f(a, b) = 0 = h(a, b)$. Thus, $f$ and $gh$ intersect at a point if and only if either $f$ and $g$ intersect there or $f$ and $h$ intersect there. That is, we get the points where $f$ and $gh$ intersect by combining the intersections of $f$ and $g$ with the intersections of $f$ and $h$. As above, we expect this property to extend to multiple intersections because we think of a multiple intersection as the coalescence of single intersections. Thus, we expect the following:

**Property 1.6**

$$I_O(f, gh) = I_O(f,g) + I_O(f,h). \qquad \square$$

The value of $I_O(f, g)$ does not depend on the order of $f$ and $g$ (by Property 1.2). Thus, Property 1.5 states that the intersection multiplicity of two curves at the origin remains unchanged when we add a multiple of either curve to the other. Likewise, Property 1.6 shows that we can break up a product of two polynomials in either position of $I_O(\_,\_)$.

Property 1.6 reinforces the idea that repeated factors in a polynomial correspond to repeated parts of the graph. For example, Properties 1.2, 1.4, and 1.6 show that

$$I_O(x^2, y) = 2I_O(x, y) = 2.$$

When we think of $x^2 = 0$ as two copies of the line $x = 0$, it makes sense that $x^2 = 0$ intersects the line $y = 0$ twice at the origin, because each of the two copies of $x = 0$ intersects $y = 0$ once.

We use the term *intersection properties* to refer to Properties 1.1–1.6 and further properties introduced in Sections 3, 11, and 12. We must prove that we can assign values $I_O(f,g)$ for all pairs of curves $f$ and $g$ so that Properties 1.1–1.6 hold. We postpone this proof until Chapter IV so that we can proceed with our main task, using intersection properties to study curves. Of course, the results we obtain depend on our proving the intersection properties in Chapter IV.

In the rest of this section, we show how Properties 1.1–1.6 can be used to compute the intersection multiplicity of two curves at the origin. The discussion accompanying Figures 1.1–1.3 suggests that $I_O(f,g)$ measures how closely the curves $f$ and $g$ approach each other at the origin. When $f$ is a factor of $g$, the graph of $g = 0$ contains the graph of $f = 0$. Thus, we are led to expect the following result:

**Theorem 1.7**

*If $f$ and $g$ are polynomials such that $f$ is a factor of $g$ and the curve $f = 0$ contains the origin $O$, then $I_O(f,g)$ is $\infty$.*

**Proof**

Consider first the case where $g$ is the zero polynomial 0. (The theorem includes this case because the zero polynomial has every polynomial $f$ as a factor, since $0 = f \cdot 0$.) Since $I_O(f,0) \geq 1$ (by Property 1.3), it follows for every positive integer $n$ that

$$\begin{aligned} n \leq nI_O(f,0) &= I_O(f,0^n) \quad \text{(by Property 1.6)} \\ &= I_O(f,0). \end{aligned}$$

Because this holds for every positive integer $n$, $I_O(f,0)$ must be $\infty$.

In general, if $g$ is any polynomial that has $f$ as a factor, we can write $g = fh$ for a polynomial $h$. Then we have

$$\begin{aligned} I_O(f,g) &= I_O(f,fh) \\ &= I_O(f,fh - fh) \quad \text{(by Property 1.5)} \\ &= I_O(f,0) = \infty, \end{aligned}$$

by the previous paragraph. □

The proof of Theorem 1.7 shows why we needed to allow infinite intersection multiplicities in Property 1.1.

The following result shows that we can disregard factors that do not contain the origin when we compute intersection multiplicities at the origin:

**Theorem 1.8**

*If $f$, $g$, and $h$ are curves and $g$ does not contain the origin, we have*

$$I_O(f,gh) = I_O(f,h).$$

**Proof**

Properties 1.6, 1.3, and 1.1 show that

$$I_O(f, gh) = I_O(f, g) + I_O(f, h) = I_O(f, h),$$

since $I_O(f, g) = 0$ because $g$ does not contain the origin. □

To illustrate the use of the intersection properties, we find the number of times that $y - x^2$ and $y^3 + 2xy + x^6$ intersect at the origin. We use Property 1.5 to eliminate $y$ from the second polynomial by subtracting a suitable multiple of the first. To find this multiple, we use long division with respect to $y$ to divide the first polynomial into the second, as follows:

$$\begin{array}{r|lllll}
 & y^2 + & x^2y & + 2x & + & x^4 \\ \hline
y - x^2 & y^3 + & 2xy & & + & x^6 \\
 & y^3 - x^2y^2 & & & & \\ \hline
 & x^2y^2 + & 2xy & & & \\
 & x^2y^2 - & x^4y & & & \\ \hline
 & & (2x + x^4)y & & + & x^6 \\
 & & (2x + x^4)y & - 2x^3 & - & x^6 \\ \hline
 & & & 2x^3 & + & 2x^6.
\end{array}$$

Each step of the division eliminates the highest remaining power of $y$ until only a polynomial in $x$ is left: the three steps of the division eliminate the $y^3$, $y^2$, and $y$ terms. The division shows that

$$y^3 + 2xy + x^6 = (y - x^2)(y^2 + x^2y + 2x + x^4) + 2x^3 + 2x^6. \qquad (3)$$

Thus, we are left with the remainder $2x^3 + 2x^6$, which does not contain $y$, when we subtract a multiple of $y - x^2$ from $y^3 + 2xy + x^6$. It follows that

$$I_O(y - x^2, y^3 + 2xy + x^6) = I_O(y - x^2, 2x^3 + 2x^6)$$

(by (3) and Property 1.5)

$$= I_O(y - x^2, x^3(2 + 2x^3))$$

$$= I_O(y - x^2, x^3) \quad \text{(by Theorem 1.8)}$$

$$= 3I_O(y - x^2, x) \quad \text{(by Property 1.6)}$$

$$= 3I_O(y, x)$$

(by Properties 1.2 and 1.5, since $y - x^2$ differs from $y$ by a multiple of $x$)

$$= 3 \quad \text{(by Properties 1.2 and 1.4).}$$

Thus, $y = x^2$ intersects $y^3 + 2xy + x^6 = 0$ three times at the origin.

Of course, a polynomial $p(x)$ in one variable $x$ is a finite sum of terms of the form $ex^i$, where $e$ is a real number and $i$ is a nonnegative integer. By generalizing the previous paragraph, we can find the number of times that a curve of the form $y = p(x)$ intersects any curve $g(x, y) = 0$ at the origin. This is easy to do because we do not need long division to find the remainder when $g(x, y)$ is divided by $y - p(x)$ with respect to $y$. The next theorem shows that the remainder is $g(x, p(x))$, the result of substituting $p(x)$ for $y$ in $g(x, y)$. For example, we did not have to use long division above to find the remainder when $y^3 + 2xy + x^6$ is divided by $y - x^2$. All we needed to do was substitute $x^2$ for $y$ in $y^3 + 2xy + x^6$ to find that the remainder is $(x^2)^3 + 2x(x^2) + x^6 = 2x^3 + 2x^6$, as before.

**Theorem 1.9**
*Let $p(x)$ and $g(x, y)$ be polynomials.*

(i) *If we use long division with respect to $y$ to divide $g(x, y)$ by $y - p(x)$, the remainder is $g(x, p(x))$. This means that there is a polynomial $h(x, y)$ such that*

$$g(x, y) = (y - p(x))h(x, y) + g(x, p(x)). \tag{4}$$

(ii) *In particular, $y - p(x)$ is a factor of $g(x, y)$ if and only if $g(x, p(x))$ is the zero polynomial.*

**Proof**
(i) Let $h(x, y)$ be the quotient when we use long division with respect to $y$ to divide $y - p(x)$ into $g(x, y)$. The remainder is a polynomial $r(x)$ in $x$ because each step of the division eliminates the highest remaining power of $y$. We have

$$g(x, y) = (y - p(x))h(x, y) + r(x). \tag{5}$$

Substituting $p(x)$ for $y$ in (5) makes $y - p(x)$ zero and shows that

$$g(x, p(x)) = r(x).$$

Together with (5), this gives (4).

(ii) If $g(x, p(x))$ is the zero polynomial, (4) shows that $y - p(x)$ is a factor of $g(x, y)$. Conversely, if $y - p(x)$ is a factor of $g(x, y)$, we can write

$$g(x, y) = (y - p(x))k(x, y)$$

for a polynomial $k(x, y)$. Substituting $p(x)$ for $y$ shows that $g(x, p(x))$ is zero. □

We obtain a familiar result from Theorem 1.9 if we assume that $x$ does not appear in $p$ or $g$. Then $p$ is a real number $b$, and $g$ is a polynomial $g(y)$ in $y$. When we divide $g(y)$ by $y - b$, the quotient is a polynomial