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FUNDAMENTAL RELATIONS AND DOUBLE RINGEL-HALL ALGEBRAS

陈江荣/著

首都经济贸易大学出版社
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Fundamental Relations and Double Ringel - Hall Algebras

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前　言

Ringel 定义了 finitary 环上的 Hall 代数（现在经常被称为 Ringel-Hall 代数），并证明了 Ringel-Hall 代数满足所谓的基本关系。在遗传代数的 Ringel-Hall 代数中定义扭乘法，基本关系就变成了量子 Serre 关系。特别地，一个表示有限型遗传代数 A 的 Ringel-Hall 代数与相应于 A 的复半单李代数 $\mathfrak{g}$ 的量子包络代数 $U_v(\mathfrak{g})$ 的正部分 U^+ 同构。后来，Green 把 Ringel 的结果推广到了任意有限维遗传代数的情形。因此，Ringel-Hall 代数方法提供了实现量子包络代数的很好的模型。最近，遗传代数 A 的 double Ringel-Hall 代数 $\mathcal{D}(A)$ 本身被证明同构于 Borcherds 意义下的广义 Kac-Moody 代数的量子包络代数，Ringel-Hall 代数方法也提供了实现可对称化的 Kac-Moody 代数的模型。

本书分为三个部分。Ringel 对基本关系的证明基于 $\mathrm{Ext}_A^1(S_i,S_j)=0$ 或 $\mathrm{Ext}_A^1(S_j,S_i)=0$ 的假设。本书第一部分推广了 Ringel 的结果，证明了没有上述假设条件，基本关系仍然成立；进一步证明了 Ringel-Hall 代数满足高阶基本关系。通过定义扭 Ringel-Hall 代数，基本关系和高阶基本关系刚好给出了量子 Serre 关系与高阶量子 Serre 关系。由此说明，量子 Serre 关系具有“范”性。另外，作为高阶基本关系的一个应用，我们证明了含有两个点的循环箭图上的合成半群代数（定义见§3.3）与其 generic 合成子代数（在 $q=0$ 时）同构。

第二部分研究了 tame 型 Ringel-Hall 代数的一类子代数（由合成代数与一个管上的模生成），证明了这些子代数具有 Hopf 代数结构。当 Q 是一个 $\widetilde{A}$ 型非循环箭图且管状分支取为一个非齐次管或一个次数为 1 的齐次管时，我们给出了这类子代数的生成元和生成关系，并证明了这类子代数与循环箭图的 Ringel-Hall 代数同构。因此，我们可以用非循环箭图上的 Ringel-Hall 代数结构去研究循环箭图上的 Ringel-Hall 代数。

第三部分研究了含有两个点的循环箭图 Δ_2 的 double Ringel-Hall 代数

$\mathcal{D}(\Delta_2)$的有限维表示。从两个不同角度给出了不可约$\mathcal{D}(\Delta_2)$-权模的刻画,并建立了$\mathcal{D}(\Delta_2)$-权模与有无限个变元的多项式代数$\mathbb{C}[z_l^{\pm}|l\geq 1]$-模之间的对应关系。

本书的完成得到了我的导师北京师范大学邓邦明教授的精心指导,在此由衷地对我的恩师表示感谢。同时感谢首都经济贸易大学给予的资助和出版社编辑老师们提出的修改意见。

摘　要

通过推广 Hall 与 Steinitz 的工作,Ringel 于 1990 年引入了 finitary 代数的 Hall 代数。后经 Ringel,Green,Lusztig 等人的发展,Ringel-Hall 代数成为量子群和 Kac-Moody 李代数的一个最佳实现模型。Ringel-Hall 代数方法因此成为量子群研究中的一个重要工具。特别地,代数表示论的方法和技巧可以用来研究量子群和李代数的结构和表示。

本书主要研究了 Ringel-Hall 代数的基本关系以及仿射型 Ringel-Hall 代数的结构和表示。主要工作分为以下三个部分:

第一,Ringel 的一个重要发现是,在 Ringel-Hall 代数中,两个不同构的单模 S_i 与 S_j 满足所谓的基本关系。Ringel 的证明是基于 $\mathrm{Ext}_A^1(S_i,S_j)=0$ 或 $\mathrm{Ext}_A^1(S_j,S_i)=0$ 的假设。我们推广了 Ringel 的结果,首先证明了没有上述假设条件,基本关系仍然成立;进一步证明了 Ringel-Hall 代数满足高阶基本关系。通过定义扭 Ringel-Hall 代数,基本关系和高阶基本关系刚好给出了量子 Serre 关系与高阶量子 Serre 关系。由此说明,量子 Serre 关系具有"范"性。另外,作为高阶基本关系的一个应用,我们证明了含有两个点的循环箭图上的合成半群代数(定义见 §3.3)与其 generic 合成子代数(在 $q=0$ 时)P_i 同构。

第二,本书研究了 tame 型 Ringel-Hall 代数的一类子代数(由合成代数与一个管上的模生成),证明了这些子代数具有 Hopf 代数结构。当 Q 是一个 $\widetilde{A}$ 型非循环箭图且管状分支取为一个非齐次管或一个次数为 1 的齐次管时,我们给出了这类子代数的生成元和生成关系并证明了这类子代数与循环箭图的 Ringel-Hall 代数同构。因此,我们可以用非循环箭图上的 Ringel-Hall 代数结构去研究循环箭图上的 Ringel-Hall 代数。

第三,本书的第三部分研究了含有两个点的循环箭图 Δ_2 的 double Ringel-Hall 代数$\mathcal{D}(\Delta_2)$的有限维表示。参考文献[4]中,作者利用仿射量子群

的 Drinfeld 实现刻画了量子群 $U_v(\widehat{\mathfrak{sl}}_2)$的有限维不可约表示，基于这项工作，我们构造了有限维不可约$\mathcal{D}(\Delta_2)$－权模。确切地说，我们从两个不同角度给出了不可约$\mathcal{D}(\Delta_2)$－权模的刻画，并建立了$\mathcal{D}(\Delta_2)$－权模与有无限个变元的多项式代数$\mathbb{C}[z_l^{\pm}\mid l\geqslant 1]$－模之间的对应关系。

关键词:Ringel-Hall 代数，基本关系，高阶基本关系，仿射箭图，权模，不可约模。

Abstract

By generalizing the definition of Hall algebra of the ring of p-adic integers studied by Steinitz [48], and later by Hall [20], Ringel [38] introduced the basic notion of Ringel-Hall algebra of a finitary algebra. Later, by the work of Ringel, Green, Lusztig, and many others, Ringel-Hall algebras provide a nice framework for the realization of quantized enveloping algebras and Kac-Moody Lie algebras. Ringel-Hall algebra approach thus becomes an important tool in the study of quantum groups. In particular, the machinery of representation theory of algebras can be used to investigate the structure and representations of quantum groups and Lie algebras.

In this book, we study the fundamental relations in Ringel-Hall algebras, the structure of Ringel-Hall algebras of affine type and their representations as well. The book consists of the following three parts.

(1) Ringel has made the following remarkable discovery, namely, in a Ringel-Hall algebra, two non-isomorphic simple modules S_i and S_j satisfy the so-called fundamental relations. However, Ringel's proof was based on the assumption $\mathrm{Ext}_A^1(S_i, S_j) = 0$ or $\mathrm{Ext}_A^1(S_j, S_i) = 0$. In this book, we generalize Ringel's result. We first show that fundamental relations hold without the above assumption; then we show that Ringel-Hall algebras satisfy higher order fundamental relations, too. By twisting the multiplication, these relations become quantum Serre relations and higher order quantum Serre relations, respectively. As an application of higher order fundamental relations, we prove that for the cyclic quiver of two vertices, the composition monoid algebra is isomorphic to the generic composition algebra specialized at $q = 0$.

(2) In the second part, we study certain subalgebras of Ringel-Hall algebras

of tame type, which are generated by the composition algebra and those modules from a single tube $\mathfrak{T}$, and prove that these subalgebras inherit Hopf algebra structures. In case Q is an acyclic quiver of type $\tilde{A}$ and $\mathfrak{T}$ is a non-homogeneous tube or a homogeneous tube of degree 1, we present the generators and the generating relations for such subalgebras. Moreover, it is shown that such subalgebras are isomorphic to the Ringel-Hall algebra of a cyclic quiver. Therefore, the structure of the Ringel-Hall algebra of an acyclic quiver can be applied to study the structure of the Ringel-Hall algebra of a cyclic quiver.

(3) In the final part, we study finite dimensional representations of the double Ringel-Hall algebra $\mathcal{D}(\Delta_2)$ of the cyclic quiver Δ_2 with two vertices. In reference [4], the authors classified finite dimensional irreducible representations of $U_v(\widehat{\mathfrak{sl}_2})$ by making use of the Drinfeld presentation of affine quantum groups. Based on their work, we construct finite dimensional irreducible weight $\mathcal{D}(\Delta_2)$-modules. More precisely, we study irreducible $\mathcal{D}(\Delta_2)$-weight modules in two ways, and successfully establish a connection between finite dimensional weight $\mathcal{D}(\Delta_2)$-modules and finite dimensional $\mathbb{C}[z_l^{\pm} \mid l \geqslant 1]$-modules, where $\mathbb{C}[z_l^{\pm} \mid l \geqslant 1]$ is the polynomial ring with infinitely many variables.

Key words: Ringel-Hall algebra, fundamental relation, higher order fundamental relation, affine quiver, weight module, irreducible module.

Contents

1 Introduction

1.1 Background

In [16] Gabriel introduced the notion of representations of quivers and discovered a remarkable connection between the indecomposable representations of Dynkin quivers and the positive roots of the corresponding finite dimensional complex simple Lie algebras. Soon after, the representations of tame (valued) quivers were studied in [14,35,13]. In [28], Kac proved that for an arbitrary quiver Q without loops, the dimension vectors of indecomposable representations of Q over an algebraically closed field correspond bijectively to the positive roots of the corresponding Kac-Moody algebra.

In 1990s, Ringel [38,40] defined Hall algebras (nowadays called Ringel-Hall algebras in the literature) of finitary rings in order to deal with possible filtrations of modules with fixed factors. A remarkable observation of Ringel is that Ringel-Hall algebras satisfy the so-called fundamental relations, which are similar to the quantum Serre relations—the defining relations for quantized enveloping algebras. Later, it was shown in [42] that by twisting the multiplication in Ringel-Hall algebras of hereditary algebras, fundamental relations become the quantum Serre relations themselves. In particular, Ringel [38] showed that the twisted Ringel-Hall algebra of a representation-finite hereditary algebra A is isomorphic to the positive part U^+ of the quantized enveloping algebra $U_v(\mathfrak{g})$ of the complex semisimple Lie algebra $\mathfrak{g}$ associated with A. Later on, Green [18] extended Ringel's result to arbitrary finite dimensional hereditary algebras. More precisely, he showed that the generic composition algebra of an arbitrary hereditary algebra is isomorphic to the positive part of the associated quantized enveloping algebra. Hence, Ringel-Hall algebra approach provides a nice framework for the realization of quantized enveloping algebras. Recently, it was shown that the whole double Ringel-Hall algebra $\mathcal{D}(A)$ of the hereditary algebra A is isomorphic to the quantized enveloping algebra of a generalized Kac-Moody algebra in the sense of Borcherds [3] (see [44, 21, 47, 9, 56]).

We remark that Ringel-Hall algebra approach also provides a realization of symmetrizable Kac-Moody algebras (see [39, 36]). Also, inspired by Ringel's work, Lusztig [31] gave a geometric realization of quantum groups in terms of representation varieties of quivers.

1.2 Main results

Let A be a finitary algebra (e. g., finitely generated algebra) over a finite field $\mathbb{F}_q$. By A-**mod** we denote the category of finite dimensional left A-modules. The integral Ringel-Hall algebra $\mathfrak{h}(A)$ of A is by definition the free abelian group with basis $u_{[M]}$, indexed by isoclasses $[M]$ of finite dimensional A-modules M. The multiplication is given by

$$u_{[M]}u_{[N]} = \sum_{[L]} F^L_{M,N} u_{[L]},$$

where $F^L_{M,N}$ is the number of submodules X of L such that $X \cong N$ and $L/X \cong M$.

Let $S_i, i \in I$, be a complete set of simple A-modules in A-**mod**. For simplicity, we write $u_i = u_{[S_i]}$ for each $i \in I$. For each $i \in I$, the endomorphism algebra $D_i := \mathrm{End}_A(S_i)$ of $\mathrm{S_i}$ is a finite field extension of $\mathbb{F}_q$, and let $q_i = |\mathrm{D_i}|$.

Choose $i \neq j$ in I. Suppose $\mathrm{Ext}^1_A(S_i, S_i) = 0$. Ringel showed in [38] that under the assumption $\mathrm{Ext}^1_A(S_i, S_j) = 0$, we have in $\mathfrak{h}(A)$,

$$\sum_{r=0}^{n} (-1)^r q_i^{\frac{r(r-1)}{2}} \begin{bmatrix} n \\ r \end{bmatrix}_{q_i} u_i^r u_j u_i^{n-r} = 0,$$

where $n = 1 + \dim_{\mathrm{End}_A(S_i)} \mathrm{Ext}^1_A(S_j, S_i)$, and under the assumption $\mathrm{Ext}^1_A(S_j, S_i) = 0$, we have

$$\sum_{r=0}^{m}(-1)^r q_i^{\frac{r(r-1)}{2}}\left[\!\left[\frac{m}{r}\right]\!\right]_{q_i} u_i^{m-r}u_ju_i^r = 0,$$

where $m = 1 + \dim \operatorname{Ext}_A^1(S_i, S_j)_{\operatorname{End}_A(S_i)}$. These are called the fundamental relations.

In this thesis, we first show that, to obtain the fundamental relations, the assumption $\operatorname{Ext}_A^1(S_i,S_j)=0$ or $\operatorname{Ext}_A^1(S_j,S_i)=0$ is indeed not necessary.

For $i \neq j \in I$, we consider the D_i-D_j-bimodule $\operatorname{Ext}_A^1(S_j,S_i)$ and define

$$c'_{i,j} = -\dim_{D_i}\operatorname{Ext}_A^1(S_j,S_i),\ c''_{i,j} = -\dim \operatorname{Ext}_A^1(S_i,S_j)_{D_i}.$$

Set $c_{i,j} = c'_{i,j} + c''_{i,j}$. We have the following theorem.

Theorem 1.2.1. *Let A be a finitary algebra over a finite field $\mathbb{F}_q$. Let $i, j \in I$ with $i \neq j$ and suppose $c_{i,i} = 2$, i. e., $\operatorname{Ext}_A^1(S_i,S_i)=0$. Then we have in $\mathfrak{h}(A)$,*

$$\sum_{r=0}^{n}(-1)^r q_i^{\frac{(r+c''_{i,j})(r+c''_{i,j}-1)}{2}}\left[\!\left[\begin{matrix}n\\r\end{matrix}\right]\!\right]_{q_i} u_i^r u_j u_i^{n-r} = 0, \qquad (1.2.1.1)$$

where $n = 1 - c_{i,j}$.

The fundamental relations obtained in [38] are exactly the formula (1.2.1.1) for the cases $c''_{i,j}=0$ and $c'_{i,j}=0$. In some sense, the theorem means that the fundamental relations are "universal".

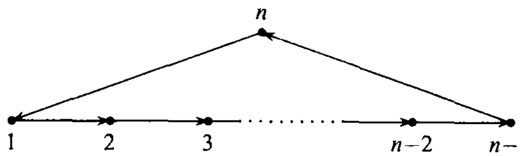
Then, by twisting the multiplication of the Ringel-Hall algebra of an arbitrary finitary algebra, the quantum Serre relations are also obtained.

We further show that Ringel-Hall algebras satisfy the higher order fundamental relations which give rise to the higher order quantum Serre relations in the twisted version.

Theorem 1.2.2. *Let A be a finitary algebra over a finite field $\mathbb{F}_q$. Let $i, j \in I$ with $i \neq j$. Suppose $c_{i,i} = c_{j,j} = 2$, i. e., $\operatorname{Ext}_A^1(S_i,S_i)=0=\operatorname{Ext}_A^1(S_j,S_j)$. Then for $n \geqslant 1$ and $m \geqslant 1 - nc_{i,j}$,*

$$\sum_{r=0}^{m}(-1)^r q_i^{\frac{(r-m-nc'_{i,j})(r-m-nc'_{i,j}+1)}{2}}\left[\!\left[\begin{matrix}m\\r\end{matrix}\right]\!\right]_{q_i} u_i^r u_j^n u_i^{m-r} = 0. \qquad (1.2.2.1)$$

This theorem has the following nice application. Let Δ_n be the cyclic quiver of n vertices, i. e.,



We use the notation $\mathfrak{h}_q(\Delta_2)$ to denote the generic Ringel-Hall algebra of Δ_2 over the polynomial ring $\mathbb{Q}[q]$. Let $\mathfrak{c}_q(\Delta_2)$ be its subalgebra generated by simple modules. For each $n \geqslant 1$, setting $m = 2n+1$ and $q = 0$, Theorem 1.2.2 implies that in the composition subalgebra $\mathfrak{c}_0(\Delta_2)$ (as $\mathbb{Q}$-algebra), there hold

$$u_1^{n+1}u_2^n u_1^n = u_1^n u_2^n u_1^{n+1},$$

$$u_2^{n+1}u_1^n u_2^n = u_2^n u_1^n u_2^{n+1}.$$

On the other hand, following [37], we have the monoid of generic extensions $\mathcal{M} = \mathcal{M}(\Delta_2)$ which has as elements the isomorphism classes of nilpotent representations of Δ_2. The multiplication in $\mathcal{M}$ is giving by taking generic extensions. By $\mathcal{M}_C$ we denote the submonoid generated by the simple representations.

Combining the above results with [34, Theorem2.2.15] gives the following theorem.

Theorem 1.2.3. *We have $\mathbb{Q}$-algebras isomorphism*: $\mathbb{Q}\mathcal{M}_C \xrightarrow{\sim} \mathfrak{c}_0(\Delta_2)$.

Some relations between the Ringel-Hall algebra of a finitary algebra and Ringel-Hall algebras of its factor algebras are also studied. More precisely, if B is a factor algebra of A, then the Ringel-Hall algebra $\mathfrak{h}(B)$ of B is a factor algebra of $\mathfrak{h}(A)$. Also, the Lie algebra associated with B is a factor algebra of the Lie algebra associated with A.

Second, we study certain subalgebras of double Ringel-Hall algebra of tame type. Let Q be an extended Dynkin quiver and $A = \mathbb{F}_q Q$ be the path algebra. Following [50], we have the reduced double Ringel-Hall algebra $\mathcal{D}(A)$ over $\mathbb{C}$ which admits a Hopf algebra structure.

It is well known that (see [13]) the Auslander-Reiten quiver of A has a separating tubular family, and those indecomposable modules from a same tube $\mathfrak{T}$ form an abelian exact uniserial subcategory, closed under extensions. This allows us to define the Ringel-Hall algebra $\mathfrak{h}_{\mathcal{T}}$ of $\mathcal{T}$, which is a subalgebra of $\mathfrak{h}(A)$ generated by $\{u_i, u_{[M]} \mid i \in I, M \in \mathcal{T}\}$. Now let $\mathcal{T}$ be an arbitrary tube, define $\mathcal{D}_{\mathcal{T}}(A)$ to be the subalgebra of $\mathcal{D}(A)$ generated by $\{u_i^{\pm}, K_i^{\pm 1}, u_{[M]}^{\pm} \mid i \in I, M \in \mathcal{T}\}$. Our first result in this direction is the following.

Proposition 1.2.4. *$\mathcal{D}_{\mathcal{T}}(A)$ is a Hopf subalgebra of $\mathcal{D}(A)$.*

This implies particularly that we can get infinitely many Hopf subalgebras of $\mathcal{D}(A)$.

Now we restrict our attention to the tame quivers of type $\tilde{A}$. A tube is said to be of degree 1 if it arises from an irreducible monic polynomial over $\mathbb{F}_q$ of degree 1. We have the following results.

Theorem 1.2.5. *Let A be the path algebra of an acyclic quiver of type $\tilde{A}$ over $\mathbb{F}_q$. Assume $\mathcal{T}$ is a non-homogeneous tube or a homogeneous tube of degree 1 of A. Then the algebra $\mathcal{D}_{\mathcal{T}}(A)$ is generated by the generators $u_i^+, u_i^-, K_i, K_i^{-1}$ $(i = 1, 2, \ldots, n)$ and $z_s^{\pm}$ $(s \geqslant 1)$ with the generating relations given in Lemma 4.3.3, Lemma 4.3.4 and (4.3.4.1), (4.3.4.2).*

Corollary 1.2.6. *Let A be the path algebra of an acyclic quiver (with n vertices) of type $\tilde{A}$ over $\mathbb{F}_q$. Assume $\mathcal{T}$ is a non-homogeneous tube or a homogeneous tube of degree 1 of A. Then $\mathcal{D}_{\mathcal{T}}(A)$ is isomorphic to $\mathcal{D}(\mathbb{F}_q \Delta_n)$, where Δ_n is the cyclic quiver with n vertices.*

Therefore, we may use the structure of $\mathcal{D}_{\mathcal{T}}(A)$ to study the double Ringel-Hall algebra of a cyclic quiver. For example, the BGP-reflection functors for acyclic quivers can be used to study the Lusztig's symmetries of $\mathcal{D}(\mathbb{F}_q \Delta_n)$ though these functors can not be directly defined for the cyclic case.

The final part of the thesis deals with the construction of finite dimensional representa-tions of the double Ringel-Hall of the cyclic quiver Δ_2. In [4], the authors classified finite dimensional irreducible $U_v(\widehat{\mathfrak{sl}_2})$-modules. Namely, every finite dimensional irreducible representation of $U_v(\widehat{\mathfrak{sl}_2})$ on which the center acts trivially is a tensor product of evaluation representations $V_{n_1}(a_1) \otimes \cdots \otimes V_{n_r}(a_r)$

with the v-strings $S_{n_1}(a_1), \ldots, S_{n_r}(a_r)$ in general position (see [4, Theorem 4.8]). Here, $V_n(a)$ ($a \in \mathbb{C}^\times$) is the so-called evaluation representation obtained by pulling back the classical $n+1$-dimensional irreducible representation V_n of $U_v(\widehat{\mathfrak{sl}}_2)$ via the evaluation homomorphism $\mathrm{ev}_a : U_v(\widehat{\mathfrak{sl}}_2) \to U_v(\mathfrak{sl}_2)$ (see [4, Proposition4.1]). Their construction relies on the second realization of quantum affine algebras given by Drinfeld ([15]).

Motivated by the idea in [4], we present two constructions of finite dimensional weight $\mathcal{D}(\Delta_2)$-modules. Our work is based on the structure theorem of $\mathcal{D}(\Delta_2)$ essentially due to [45].

On the one hand, the restriction of a weight $\mathcal{D}(\Delta_2)$-module V is again a weight $U_v(\widehat{\mathfrak{sl}}_2)$-module, we denote this module by $\overline{V}$. Further, let $\overline{V}$ be of type I, i.e., it has a decomposition

$$\begin{aligned}\overline{V} = & V_{m_1}(a_{11})^{s_{11}} \oplus \cdots \oplus V_{m_1}(a_{1t_1})^{s_{1t_1}} \\ & \oplus V_{m_2}(a_{21})^{s_{21}} \oplus \cdots \oplus V_{m_2}(a_{2t_2})^{s_{2t_2}} \\ & \cdots\cdots \\ & \oplus V_{m_r}(a_{r1})^{s_{r1}} \oplus \cdots \oplus V_{m_r}(a_{rt_r})^{s_{rt_r}},\end{aligned}$$

where $s_{ij} \geqslant 1$, $m_1 > m_2 > \cdots > m_r \geqslant 0$, and for each fixed i, a_{ij} are pairwise distinct complex numbers.

Let W be a finite dimensional $\mathbb{C}[z_l^{\pm} \mid l \geqslant 1]$-module. Then for each $m \geqslant 0$ and $a \in \mathbb{C}^\times$, $V_m(a) \otimes_{\mathbb{C}} W$ becomes naturally a module over $\mathcal{D}(\Delta_2)$. We denote this module by $V_m(a, W)$.

Theorem 1.2.7. *Let V be afinite dimensional irreducible weight $\mathcal{D}(\Delta_2)$-module with $\overline{V}$ being of type I. Then $V \cong V_m(a, W)$ for some $\mathbb{C}[z_l^{\pm} \mid l \geqslant 1]$-module W. Moreover, for every finite dimensional $\mathbb{C}[z_l^{\pm} \mid l \geqslant 1]$-module W,*

(1) *The $\mathcal{D}(\Delta_2)$-module $V_m(a, W)$ is irreducible if and only if W is an irreducible $\mathbb{C}[z_l^{\pm} \mid l \geqslant 1]$-module.*

(2) *The $\mathcal{D}(\Delta_2)$-module $V_m(a, W)$ is indecomposable if and only if W is an indecomposable $\mathbb{C}[z_l^{\pm} \mid l \geqslant 1]$-module.*